

# Power Spectral Density - Continuous Time

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles, Jr. and B. Shi

# Outline

# Energy and average power in time domain

power spectral density for continuous time signals

## Energy, Average Power – deterministic, time domain

a deterministic signal  $x(t)$

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

the energy

$$E(T) = \int_{-T}^{+T} x^2(t) dt = \int_{-\infty}^{+\infty} x_T^2(t) dt$$

the average power

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt = \frac{1}{2T} \int_{-\infty}^{+\infty} x_T^2(t) dt$$

# Fourier transform

power spectral density for continuous time signals

Fourier Transform Pair  $x(t) \iff X(\omega)$

Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

a deterministic signal  $x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

# Fourier transform of $x_T(t)$

for continuous time signals

bounded duration, bounded variation

for a finite  $T$ ,  $x_T(t)$  is assumed to have bounded variation

$$\int_{-T}^{+T} |x(t)| dt < \infty$$

the Fourier transform of  $x_T(t)$

$$\begin{aligned} X_T(\omega) &= \int_{-\infty}^{+\infty} x_T(t) e^{-j\omega t} dt \\ &= \int_{-T}^{+T} x(t) e^{-j\omega t} dt \end{aligned}$$

# Fourier transforms of $x_T(t)$ and $X_T(t)$

power spectral density for continuous time signals

deterministic  $X_T(\omega)$  v.s. random  $X_T(\omega)$

a **deterministic** sample signal  $x_T(t)$

$$x_T(t) \iff X_T(\omega)$$

a **random process** signal  $X_T(t)$

$$X_T(t) \iff X_T(\omega)$$

# Parseval's theorem (I)

power spectral density for continuous time signals

for a deterministic  $x_T(t)$

a **deterministic** sample signal  $x_T(t)$

$$\int_{-\infty}^{+\infty} x_T(\tau)x_T^*(\tau)d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_T(\omega)X_T^*(\omega)d\omega$$

$$\int_{-\infty}^{+\infty} |x_T(\tau)|^2 d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$



# Parseval's theorem (II)

power spectral density for continuous time signals

for a deterministic  $x_T(t)$  v.s. a random  $X_T(t)$

- a **deterministic** signal  $x_T(t) \iff X_T(\omega)$

$$\int_{-\infty}^{+\infty} |x_T(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

- a **random** signal  $X_T(t) \iff X_T(\omega)$

$$\int_{-\infty}^{+\infty} E \left[ |X_T(t)|^2 \right] dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E \left[ |X_T(\omega)|^2 \right] d\omega$$

# Energy and average power in frequency domain

power spectral density for continuous time signals

## Energy, Average Power – Parseval's theorem applied

a deterministic signal  $x_T(t)$

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{otherwise} \end{cases} \quad x_T(t) \iff X_T(\omega)$$

the **energy** by Parseval's theorem

$$E(T) = \int_{-T}^{+T} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** by Parseval's theorem

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

# $E(T)$ and $P(T)$ in frequency domain – deterministic case

power spectral density for continuous time signals

deterministic  $x_T(t) \iff X_T(\omega)$

the **energy** for the **deterministic**  $X_T(\omega)$  in  $x_T(t) \iff X_T(\omega)$

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** for the **deterministic**  $X_T(\omega)$

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

the **power spectral density** for the **deterministic**  $X_T(\omega)$

$$\lim_{T \rightarrow \infty} \frac{|X_T(\omega)|^2}{2T}$$

# $E(T)$ and $P(T)$ in frequency domain – random case

power spectral density for continuous time signals

random  $X_T(t) \iff X_T(\omega)$

the **energy** for the **random**  $X_T(\omega)$  in  $X_T(t) \iff X_T(\omega)$

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E[|X_T(\omega)|^2] d\omega$$

the **average power** for the **random**  $X_T(\omega)$

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{E[|X_T(\omega)|^2]}{2T} d\omega$$

the **power spectral density** for the **random**  $X_T(\omega)$

$$\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

# Average power $P(T)$ – bounded duration $(-T, +T)$ power spectral density for continuous time signals

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

- not the average power in a **random process**  
only the power in **one** sample function
  - to obtain the **average power** over all possible realizations,  
replace  $x(t)$  by  $X(t)$   
take the **expected value** of  $x^2(t)$ , that is  $TE [X^2(t)]$
  - then, the **average power** is a **random variable**  
with respect to the **random process**  $X(t)$
- not the average power in an **entire** sample function
  - take  $T \rightarrow \infty$  to include all power in the **ensemble** member

# Average power $P_{XX}$ – unbounded duration $(-\infty, +\infty)$

power spectral density for continuous time signals

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

- replace  $x(t)$  by the **random variable**  $X(t)$
- take the **expected value** of  $x^2(t)$ , that is  $E[X^2(t)]$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt$$

- take  $T \rightarrow \infty$  to include all power

$$P_{XX} = \lim_{T \rightarrow \infty} P(T) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt$$

# Average power $P_{XX}$ – time average $A[\bullet]$

power spectral density for continuous time signals

## The time average

$$A_T[\bullet] = \frac{1}{2T} \int_{-T}^T [\bullet] dt \quad A[\bullet] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [\bullet] dt$$

## time average and sample average operations

$$\begin{aligned} P_{XX} &= \lim_{T \rightarrow \infty} P(T) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt \\ &= \lim_{T \rightarrow \infty} A_T[E[X^2(t)]] \\ &= A[E[X^2(t)]] \end{aligned}$$

# Measuring average power

power spectral density for continuous time signals

for deterministic and random signals

the **average power**  $P(T)$  for a deterministic signal  $x(t)$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

the **average power**  $P_{XX}$  for a random process  $X(t)$

$$\begin{aligned} P_{XX} &= \lim_{T \rightarrow \infty} P(T) \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt \\ &= A[E[X^2(t)]] \end{aligned}$$



# Power Spectral Density $S_{XX}(\omega)$

power spectral density for continuous time signals

the average power via power density

the average power  $P_{XX}$  for the random process  $X_T(\omega)$

$$\begin{aligned} P_{XX} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{S_{XX}(\omega)} d\omega \end{aligned}$$

the **Power Spectral Density (PSD)**  $S_{XX}(\omega)$

$$\boxed{S_{XX}(\omega)} = \boxed{\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}}$$

# Properties of Power Spectral Density

power spectral density for continuous time signals

- $S_{XX}(\omega) \geq 0$
- $S_{XX}(-\omega) = S_{XX}(\omega)$   $X(t)$  real
- $S_{XX}(\omega)$  real
- $\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A [E [X^2(t)]]$
- $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$
- $\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A [R_{XX}(t, t + \tau)]$
- $S_{XX}(\omega) = \int_{-\infty}^{+\infty} A [R_{XX}(t, t + \tau)] e^{-j\omega \tau} d\tau$

# Equations involving $S_{XX}(\omega)$

power spectral density for continuous time signals

the average power  $P_{XX}$  and the inverse Fourier transform of  $S_{XX}(\omega)$

the **average power** related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A [E [X^2(t)]]$$

the **autocorrelation** related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A [R_{XX}(t, t + \tau)]$$

# Average power related equation

power spectral density for continuous time signals

the average power  $P_{xx}$

the average power related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A [E [X^2(t)]]$$

- a random process  $X(t)$  in time domain
- a random process  $X(\omega)$  in frequency domain

$$X(t) = \lim_{T \rightarrow \infty} X_T(t) \quad X(\omega) = \lim_{T \rightarrow \infty} X_T(\omega)$$

- Parseval's theorem over  $X_T(t) \iff X_T(\omega)$

# Average power $P_{XX}$ in time / frequency domain

power spectral density for continuous time signals

## Average power $P_{XX}$ using $X_T(t)$ and $X_T(\omega)$

- Using a random process  $X_T(t)$  in time domain

$$\begin{aligned}P_{XX} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt \\&= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-\infty}^{+\infty} E[X_T^2(t)] dt \\&= \lim_{T \rightarrow \infty} A_T [E[X^2(t)]] = \boxed{A[E[X^2(t)]]}\end{aligned}$$

- Using a random process  $X_T(\omega)$  in frequency domain

$$\begin{aligned}P_{XX} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}} d\omega \\&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{S_{XX}(\omega)} d\omega\end{aligned}$$

# Autocorrelation related equation

power spectral density for continuous time signals

the Inverse Fourier transform of  $S_{XX}(\omega)$

the average power related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A[R_{XX}(t, t + \tau)]$$

- auto-correlation function

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] \Rightarrow R_{XX}(\tau)$$

- a random process  $X(t)$  in time domain
- a random process  $X(\omega)$  in frequency domain

# PSD and Auto-correlation

power spectral density for continuous time signals

## Fourier transform pairs

- $A[R_{XX}(t, t + \tau)] \iff S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t + \tau)] e^{-j\omega\tau} d\tau$$

$$A[R_{XX}(t, t + \tau)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega$$

- $R_{XX}(\tau) \iff S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

for a WSS  $X(t)$ ,  $A[R_{XX}(t, t + \tau)] = R_{XX}(\tau)$

# PSD and Auto-Correlation

power spectral density for continuous time signals

$S_{XX}(\omega)$  and  $R_{XX}(\tau)$

the power spectral density

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

the auto-correlation function

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$



# Fourier transform of a derivative function

power spectral density for continuous time signals

Fourier transform of  $\frac{d^n}{dt^n}x(t)$

$$x(t) \iff X(\omega)$$

$$\frac{d^n}{dt^n}x(t) \iff (j\omega)^n X(\omega)$$

$$Y(t) = \frac{d}{dt}X_T(t) \iff (j\omega)X_T(\omega) = Y(\omega)$$

# PSD and Auto-Correlation of a Derivative Function

power spectral density for continuous time signals

$S_{\dot{X}\dot{X}}(\omega)$  and  $R_{\dot{X}\dot{X}}(\tau)$

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$|X_T(\omega)|^2 = X_T(\omega)X_T^*(\omega)$$

$$Y(t) = \frac{d}{dt}X_T(t) = \dot{X}_T(t)$$

$$\begin{aligned} S_{YY}(\omega) &= \lim_{T \rightarrow \infty} \frac{E[|Y(\omega)|^2]}{2T} \\ &= \lim_{T \rightarrow \infty} \frac{\omega^2 E[|X(\omega)|^2]}{2T} \\ &= \omega^2 S_{XX}(\omega) \end{aligned}$$

# Fourier transforms of autocorrelation functions

power spectral density for continuous time signals

## Definition

Fourier transform of an autocorrelation functions

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$S_{\dot{X}\dot{X}}(\omega) = \int_{-\infty}^{+\infty} R_{\dot{X}\dot{X}}(\tau) e^{-j\omega\tau} d\tau$$

- auto-correlation function

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] \Rightarrow R_{XX}(\tau)$$

$$R_{\dot{X}\dot{X}}(t, t + \tau) = E[\dot{X}(t)\dot{X}(t + \tau)] \Rightarrow R_{\dot{X}\dot{X}}(\tau)$$

- a random process  $X(t)$  in time domain
- $\dot{X}(t) = \frac{d}{dt}X(t)$  : the derivative of  $X(t)$

# RMS Bandwidth

power spectral density for continuous time signals

## Definition

the standard deviation is

a measure of the spread in a density function.

the analogous quantity for the normalized power spectral density is  
a measure of its spread that we call the rms bandwidth  
(root-mean-square)

$$W_{rms}^2 = \frac{\int_{-\infty}^{+\infty} \omega^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

# RMS Bandwidth and Mean Frequency

power spectral density for continuous time signals

## Definition

the mean frequency  $\bar{\omega}_0$

$$\bar{\omega}_0 = \frac{\int_{-\infty}^{+\infty} \omega S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

the rms bandwidth

$$W_{rms}^2 = \frac{4 \int_{-\infty}^{+\infty} (\omega - \bar{\omega}_0)^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$



