[Central Limit Theorem](#page-3-0)

Multiple Random Variables

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Based on Probability, Random Variables and Random Signal Principles, P.Z. Peebles,Jr. and B. Shi

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Central Limit Theorem

Definition

the central limit theorem says that the probability distribution function of the sum of large number of random variables approaches a Gauassian distribution. This theorem is known to apply some cases of statistically independent random variables.

Central Limit Theorem Uneqaul Distribution Case

Definition

the sum Y of N independent random variables $X_1, X_2, ..., X_N$ Let $Y = X_1 + X_2 + \cdots + X_N$, then

$$
\overline{Y}_N = \overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_N
$$

$$
\sigma_{Y_N}^2=\sigma_{X_1}^2+\sigma_{X_2}^2+\cdots+\sigma_{X_N}^2
$$

the probability distribution of Y asymptotically approaches to Gaussian distribution function as $N \rightarrow \infty$

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Sufficient Conditions Uneqaul Distribution Case

Definition

$$
\sigma_{X_i}^2 > B_1 > 0 \qquad i = 1, 2, ..., N
$$

$$
E[|X_i - \overline{X}_i|^3] < B_2 \qquad i = 1, 2, \ldots, N
$$

whre B_1 and B_2 are positive numbers these conditions guarantee that no one random variable in the sum dominates

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Distribution vs density functions Uneqaul Distribution Case

the central limit theorem guarantees

- only that the distribution of the sum of random variables become Gaussian
- the density of the sum of random variables is not always Gaussian
- the sum of continuous random variables :
	- under certain conditions on individual random variables the density of the sum is always Gaussian
- the sum of discrete random variables :
	- the density function may contain impulses and thus is not Gaussian.

Discrete Random Variable Examples distribution may contain impulses

the sum Y of N independent discrete random variables $Y = X_1 + X_2 + ... + X_N$

- **o** discrete random variable
- density function may contain impulses
- **•** therefore the density function is not Gaussian
- although the distribution approaches Gaussian
- when the possible discrete values of each random variable are $kb, k = 0, \pm 1, \pm 2, \dots$, where b is a constant
	- the envelope of the impulses in the density of the sum will be Gaussian
	- with the mean \varUpsilon_N and variance $\sigma_{Y_N}^2$

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Central Limit Theorem (1) Eqaul Distribution Case

Definition

the sum Y of N independent random variables $X_1, X_2, ..., X_N$ assume that $X_1, X_2, ..., X_N$ have the same distribution function. Let $Y_N = X_1 + X_2 + \cdots + X_N$ then $W_{\mathcal{N}} = (Y_{\mathcal{N}} - Y_{\mathcal{N}})/\sigma_{Y_{\mathcal{N}}}$ is the zero-mean, unit-variance random variable

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Central Limit Theorem (2) Eqaul Distribution Case

Definition

Let
$$
Y_N = X_1 + X_2 + \cdots + X_N
$$
 and $W_N = (Y_N - \overline{Y}_N)/\sigma_{Y_N}$

$$
W_N = (Y_N - \overline{Y}_N) / \sigma_{Y_N}
$$

= $\sum_{i=1}^N (X_i - \overline{X}_i) / \left[\sum_{i=1}^N \sigma_{\overline{X}_i}^2 \right]^{1/2}$
= $\frac{1}{\sqrt{N}\sigma_X} \sum_{i=1}^N (X_i - \overline{X}_i)$

where $X_i = \overline{X}$ and $\sigma_{X_i}^2 = \sigma_{\overline{X}}$

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Central Limit Theorem (3) Eqaul Distribution Case

$$
W_N = (Y_N - \overline{Y}_N) / \sigma_{Y_N}
$$

= $\left(X_i - \sum_{i=1}^N \overline{X}_i\right) / \left[\sum_{i=1}^N \sigma_{\overline{X}_i}^2\right]^{1/2}$
= $\sum_{i=1}^N (X_i - \overline{X}_i) / \left[\sum_{i=1}^N \sigma_{\overline{X}_i}^2\right]^{1/2}$
= $\sum_{i=1}^N (X_i - \overline{X}_i) / [N \sigma_X^2]^{1/2}$
= $\frac{1}{\sqrt{N} \sigma_X} \sum_{i=1}^N (X_i - \overline{X}_i)$

where
$$
\overline{X}_i = \overline{X}
$$
 and $\sigma_{X_i}^2 = \sigma_{\overline{X}}$
\n $\overline{Y}_N = \overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_N$ and $\sigma_{Y_N}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \cdots + \sigma_{X_N}^2$

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Characteristic Function(1) Eqaul Distribution Case

the characteristic function of W_N a zero mean, unit variance Gaussian random variable

$$
\Phi_{W_N}(\omega) = \exp(-\omega^2/2)
$$

 W_N is the density of the Gaussian random variable Fourier transforms are unique

$$
\Phi_{W_N}(\omega) = E[e^{j\omega W_N}]
$$

Characteristic Function(2) Eqaul Distribution Case

$$
W_N = \frac{1}{\sqrt{N}\sigma_X} \sum_{i=1}^N (X_i - \overline{X}_i)
$$

\n
$$
\Phi_{W_N}(\omega) = E[e^{j\omega W_N}] = E\left[\exp\left(\frac{j\omega}{\sqrt{N}\sigma_X} \sum_{i=1}^N (X_i - \overline{X})\right)\right]
$$

\n
$$
= E\left[\exp\left(\frac{j\omega}{\sqrt{N}\sigma_X} (X_1 - \overline{X})\right) \cdots \exp\left(\frac{j\omega}{\sqrt{N}\sigma_X} (X_N - \overline{X})\right)\right]
$$

\n
$$
= \left\{E\left[\exp\left(\frac{j\omega}{\sqrt{N}\sigma_X} (X_1 - \overline{X})\right)\right]\right\}^N
$$

$$
E[X_1]=E[X_2]=\cdots=E[X_N]
$$

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Characteristic Function (3) Eqaul Distribution Case

$$
\Phi_{W_N}(\omega) = \left\{ E \left[exp \left(\frac{j\omega}{\sqrt{N} \sigma_X} (X_1 - \overline{X}) \right) \right] \right\}^N
$$

$$
\ln[\Phi_{W_N}(\omega)] = N \ln \left\{ E \left[exp \left(\frac{j\omega}{\sqrt{N} \sigma_X} (X_1 - \overline{X}) \right) \right] \right\}
$$

$$
\begin{aligned} &E\left[\exp\left(\frac{j\omega}{\sqrt{N}\sigma_{X}}(X_{1}-\overline{X})\right)\right] \\ &=E\left[1+\frac{j\omega}{\sqrt{N}\sigma_{X}}(X_{1}-\overline{X})+\left(\frac{j\omega}{\sqrt{N}\sigma_{X}}\right)^{2}(X_{1}-\overline{X})^{2}+\frac{R_{N}}{N}\right] \\ &=1-\frac{\omega^{2}}{2N}+\frac{E[R_{N}]}{N} \end{aligned}
$$

where $E[R_N]$ approaches zero as $N \rightarrow \infty$

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Characteristic Function (4) Eqaul Distribution Case

$$
\ln[\Phi_{W_N}(\omega)] = N \ln \left[1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N} \right]
$$

\n
$$
\ln[1-z] = -\left[z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right], |z| < 1
$$

\n
$$
\ln[\Phi_{W_N}(\omega)] = -\frac{\omega^2}{2} + E[R_N] - \frac{N}{2} \left[\frac{\omega^2}{2N} + \frac{E[R_N]}{N}\right]^2 + \cdots
$$

\n
$$
\lim_{N \to \infty} \ln[\Phi_{W_N}(\omega)] = \ln \left[\lim_{N \to \infty} \Phi_{W_N}(\omega) \right] = -\frac{\omega^2}{2}
$$

\n
$$
\lim_{N \to \infty} \Phi_{W_N}(\omega) = e^{-\frac{\omega^2}{2}}
$$

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