

# Laurent Series and z-Transform Examples case 1.A

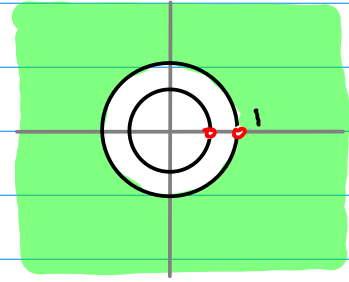
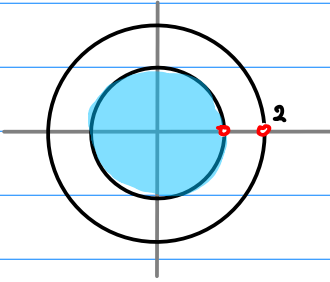
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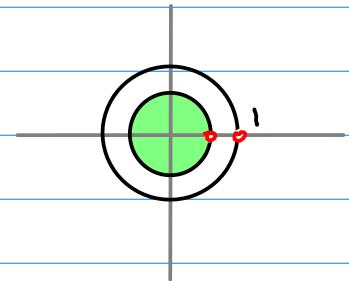
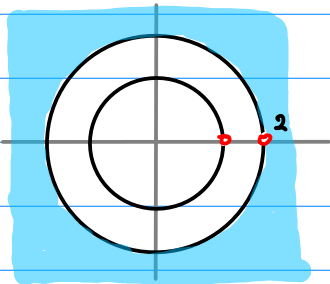
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1. A

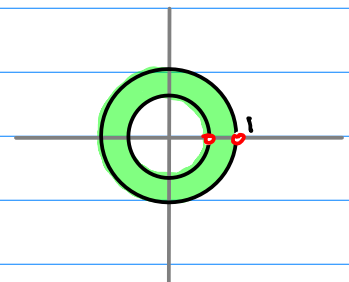
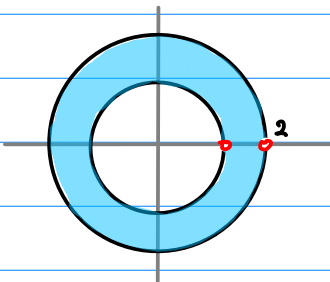
$$f(z) = \frac{-1}{(z-1)(z-2)} \rightarrow X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$



$$\sum_{n=0}^{\infty} \left[ \left(\frac{1}{2}\right)^{n+1} - 1 \right] z^n$$



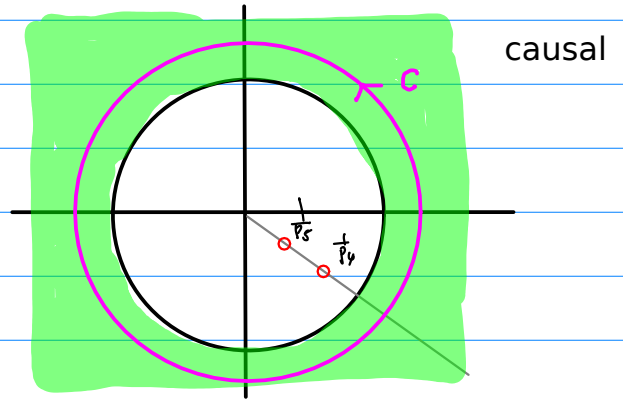
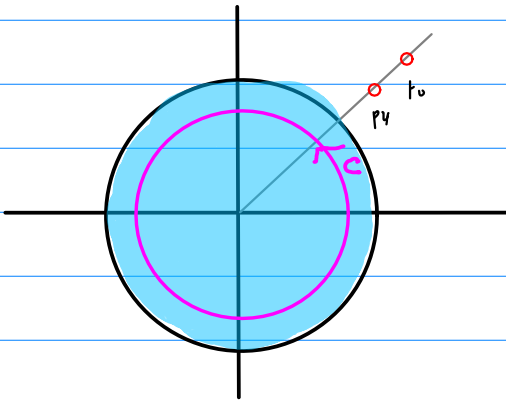
$$\sum_{n=-1}^{-\infty} \left( 1 - \left(\frac{1}{2}\right)^{n+1} \right) z^n$$



$$\sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \cdot z^n$$

L.S at  $z=0$

Z.T.



$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$X(z) = \sum_{k=0}^{\infty} x_k z^{-k}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz$$

$$= \sum_k \text{Res} \left( \frac{f(z)}{z^{n+1}}, z_k \right)$$

$$x_n = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} dz$$

$$= \sum_k \text{Res} (X(z) z^{n-1}, z_k)$$

Poles  $z_k$

$$n \geq 0 \quad z_1, z_2, z_3, \circ$$

$$n < 0 \quad z_1, z_2, z_3$$

Poles  $z_k$

$$n > 0 \quad z_1, z_2, z_3$$

$$n \leq 0 \quad z_1, z_2, z_3, \circ$$

## Z-transform

$$z_m = 0$$

$$\begin{aligned} x[n] &= \frac{1}{2\pi i} \oint_C f(z) z^{n-1} dz \\ &= \sum_k \text{Res}(f(z) z^{n-1}, z_k) \end{aligned}$$

$$n > 0 \quad z_k : \{\text{poles of } f(z)\}$$

$$n = 0 \quad z_k : \{\text{poles of } f(z)\} + \{z = 0\}$$
$$z^{n-1} = z^{-1} = \frac{1}{z}$$

$x[n]$  includes  $u[n] \rightarrow X[z]$  contains  $z$  on its numerator

Also, think about modified partial fraction  $\frac{X[z]}{z}$

## Laurent Expansion

expansion at  $z_m$

$$\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}} dz \\ &= \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) \end{aligned}$$

$$z_m = 0$$

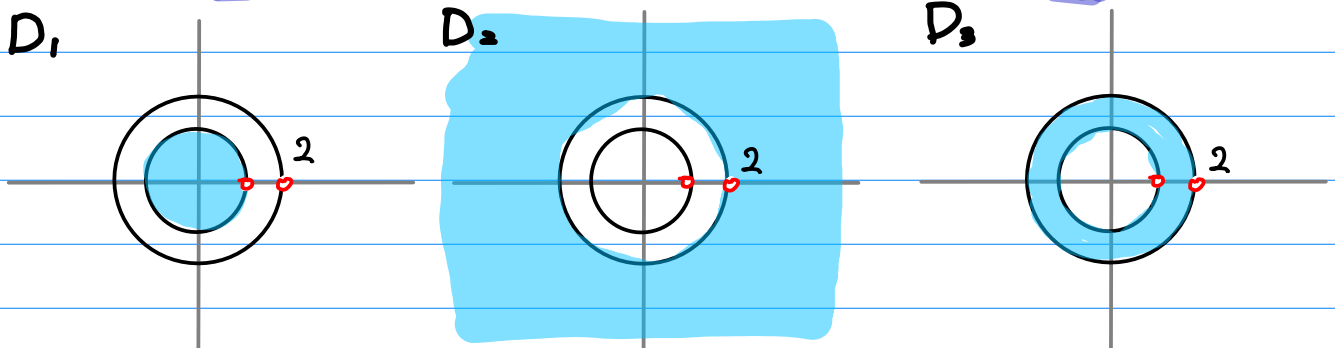
$$\begin{aligned} a_n^{\{0\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz \\ &= \sum_k \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right) \end{aligned}$$

$$\begin{aligned} a_{-n}^{\{0\}} &= \frac{1}{2\pi i} \oint_C f(z) z^{n-1} dz \\ &= \sum_k \text{Res}(f(z) z^{n-1}, z_k) \end{aligned}$$

$$\begin{aligned} a_{-n}^{\{0\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{-n+1}} dz \\ &= \sum_k \text{Res}\left(\frac{f(z)}{z^{-n+1}}, z_k\right) \end{aligned}$$

L.S. first

$$f(z) = \frac{-1}{(z-1)(z-2)}$$



$$a_n? \\ \parallel \\ x_n$$



$$X(z)?$$



$$a_n? \\ \parallel \\ x_n$$



$$X(z)?$$



$$a_n? \\ \parallel \\ x_n$$



$$X(z)?$$

$$X(z) = ?$$

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

Complex Variables and Ap  
Brown & Churchill

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

$$D_1 : |z| < 1$$

$$D_2 : 2 < |z|$$

$$D_3 : 1 < |z| < 2$$

I

$$D_1 \quad |z| < 1$$

$$|z| < 1, \quad \left|\frac{z}{2}\right| < 1$$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-\left(\frac{z}{1}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(\left(\frac{1}{2}\right)^{n+1} - 1\right) z^n \quad |z| < 1$$

II

$$D_2 \quad 2 < |z|$$

$$\left|\frac{1}{z}\right| < 1, \quad \left|\frac{2}{z}\right| < 1$$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} = \sum_{n=1}^{\infty} \left(1 - \left(\frac{1}{2}\right)^{n-1}\right) z^{-n}$$

III

$$D_3 \quad 1 < |z| < 2$$

$$\Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{\frac{1}{z}}{1-\left(\frac{1}{z}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

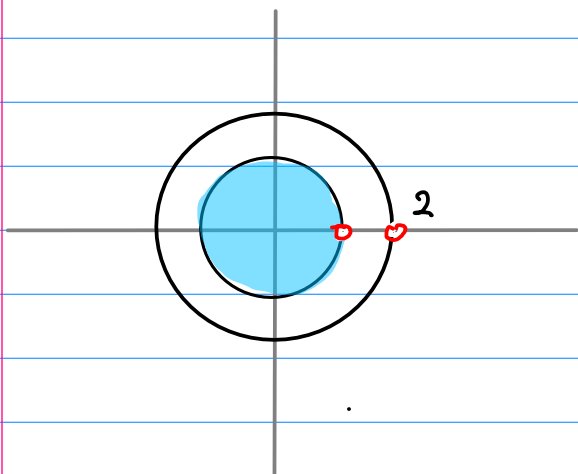
$$(n < 1)$$

$$(n \geq 0)$$

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

Ⓘ  $D_1$   $|z| < 1$

causal

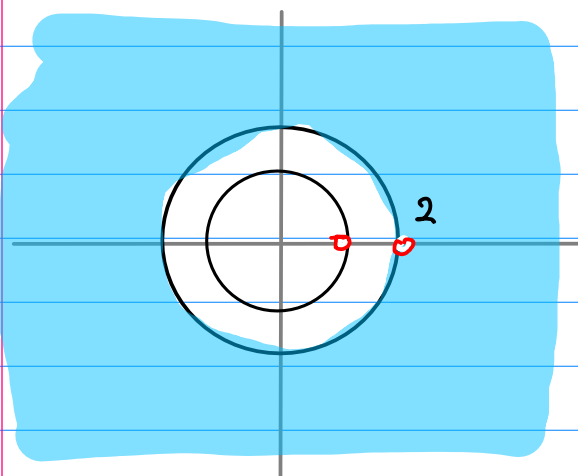
 $a_n = 0$  ( $n \leq 0$ )

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

Ⓜ  $D_3$   $|z| > 2$

anti-causal

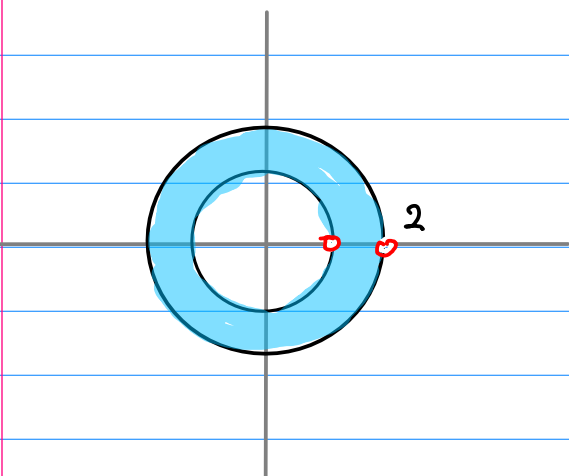
 $a_n = 0$  ( $n \geq 0$ )

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

$$a_n = \begin{cases} 0 & (n \geq 0) \\ 1 - \left(\frac{1}{2}\right)^{n+1} & (n < 0) \end{cases}$$

Ⓝ  $D_2$   $1 < |z| < 2$

two-sided



$$f(z) = \frac{-1}{(z-1)(z-2)}$$

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{cases}$$

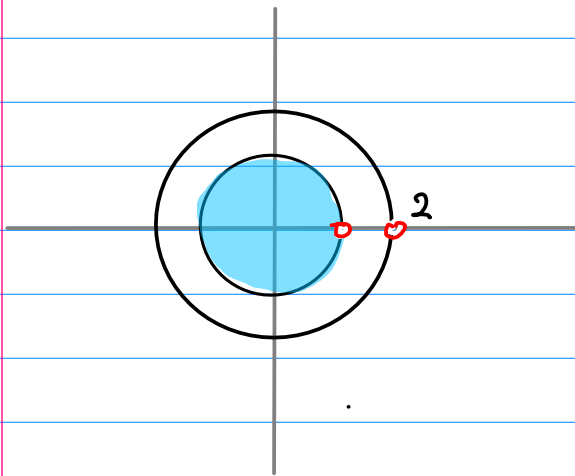
L.S. first

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}$$

①  $D_1$   $|z| < 1$

causal

$a_n = 0$  ( $n \leq 0$ )



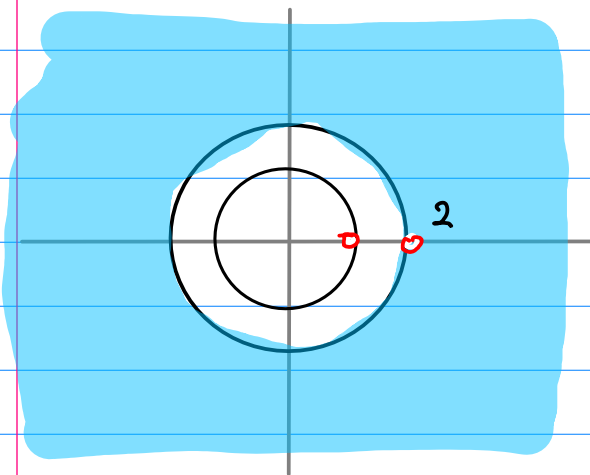
$$\left| \frac{z}{1} \right| < 1 \quad \left| \frac{z}{2} \right| < 1$$

$$\begin{aligned} f(z) &= \frac{-1}{1 - \left(\frac{z}{1}\right)} + \frac{\left(\frac{1}{2}\right)}{1 - \left(\frac{z}{2}\right)} \\ &= -\sum_{n=0}^{\infty} (1)^n z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n \\ &= \sum_{n=0}^{\infty} \left[ \left(\frac{1}{2}\right)^{n+1} - 1 \right] z^n \end{aligned}$$

②  $D_3$   $|z| > 2$

anti-causal

$a_n = 0$  ( $n \geq 0$ )

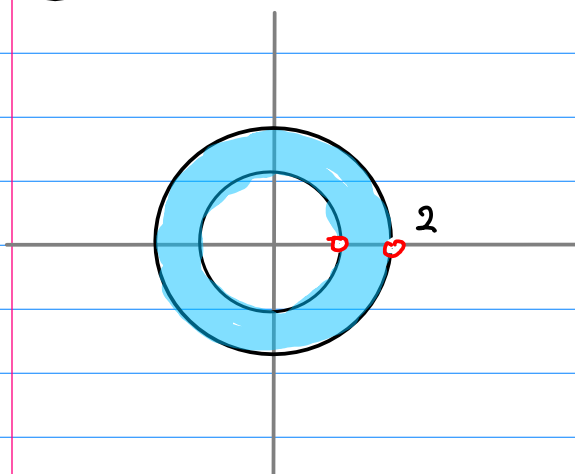


$$\left| \frac{1}{z} \right| < 1 \quad \left| \frac{2}{z} \right| < 1$$

$$\begin{aligned} f(z) &= \frac{\frac{1}{z}}{1 - \left(\frac{1}{z}\right)} - \frac{\frac{2}{z}}{1 - \left(\frac{2}{z}\right)} \\ &= \sum_{n=0}^{\infty} 1^n z^{-n-1} - \sum_{n=0}^{\infty} 2^n z^{-n-1} \\ &= \sum_{n=-1}^{-\infty} \left( 1 - \left(\frac{1}{2}\right)^{n+1} \right) z^n \end{aligned}$$

③  $D_2$   $1 < |z| < 2$

two-sided



$$\left| \frac{1}{z} \right| < 1 \quad \left| \frac{z}{2} \right| < 1$$

$$\begin{aligned} f(z) &= \frac{\frac{1}{z}}{1 - \left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{2}\right)}{1 - \left(\frac{z}{2}\right)} \\ &= \sum_{n=0}^{\infty} 1^n z^{-n-1} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n \\ &= \sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n \end{aligned}$$

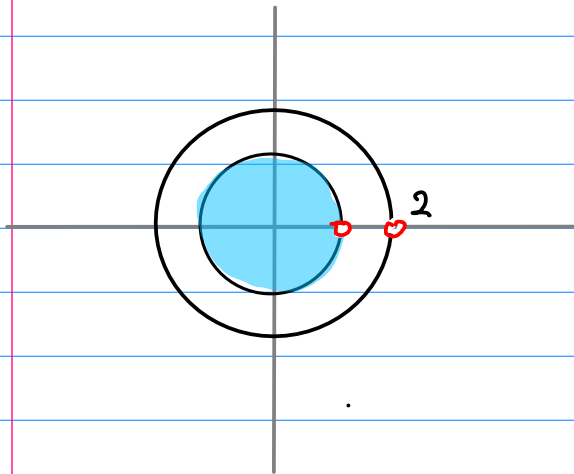


$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

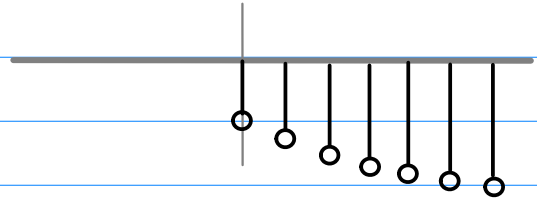
Ⓘ  $D_1$   $|z| < 1$

causal

$a_n = 0$  ( $n \leq 0$ )

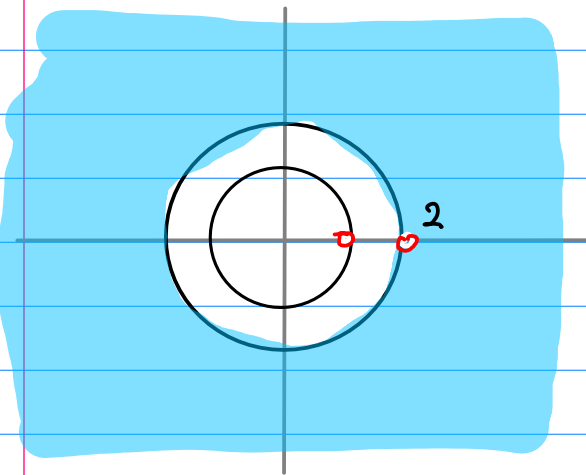


$$f(z) = \sum_{n=0}^{\infty} \left[ \left(\frac{1}{2}\right)^{n+1} - 1 \right] z^n$$

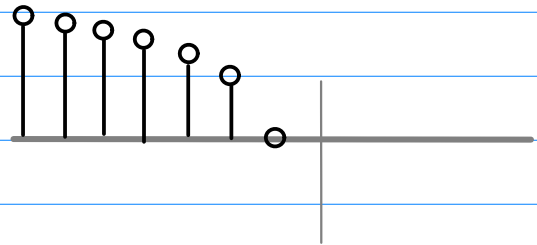


Ⓜ  $D_3$   $|z| > 2$

anti-causal  $a_n = 0$  ( $n \geq 0$ )

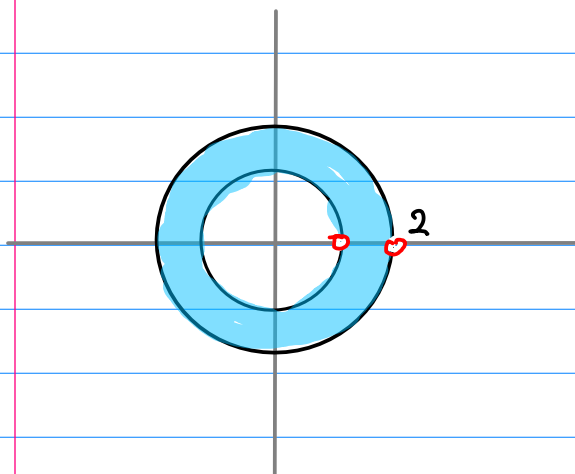


$$f(z) = \sum_{n=-1}^{-\infty} \left[ 1 - \left(\frac{1}{2}\right)^{n+1} \right] z^n$$

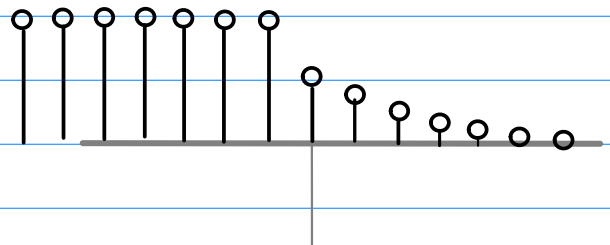


Ⓝ  $D_2$   $1 < |z| < 2$

two-sided



$$f(z) = \sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n$$

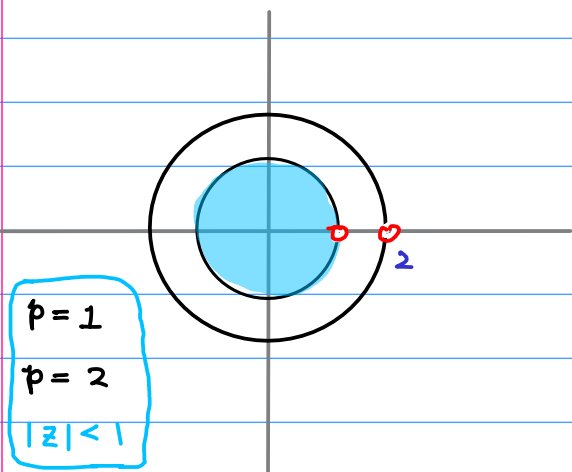


$$f(z) = \frac{-1}{(z-1)(z-2)}$$

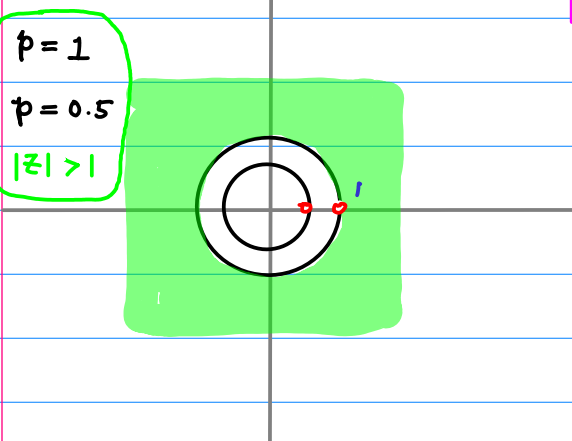
$$X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

L.S. first  
 $\textcircled{I} - 1$

$$\textcircled{I} \quad D_1 \quad |z| < 1 \quad \left[ \left| \frac{z}{2} \right| < 1, \quad \left| \frac{z}{2} \right| < 1 \right]$$



$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\ &= \frac{1}{1-\left(\frac{z}{2}\right)} + \frac{\frac{1}{2}}{1-\left(\frac{z}{2}\right)} \\ &= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \\ &= \sum_{n=0}^{\infty} \left[ 2^{-n-1} - 1 \right] z^n \quad |z| < 1 \\ &= \sum_{n=0}^{\infty} \left[ \left(\frac{1}{2}\right)^{n+1} - 1 \right] z^n \quad |z| < 1 \end{aligned}$$

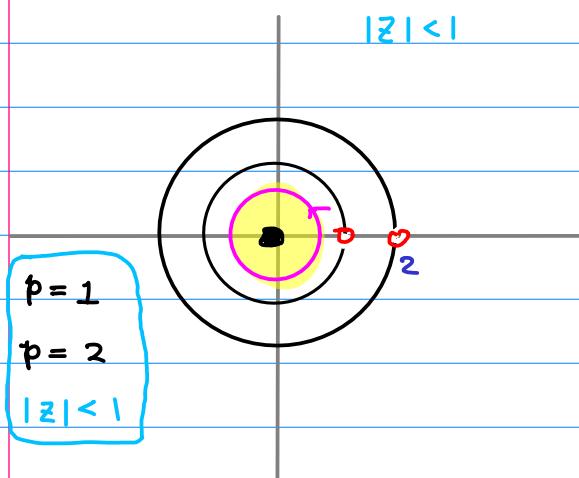


$$\begin{aligned} X(z) = f(z^{-1}) &= \sum_{n=0}^{\infty} \left[ \left(\frac{1}{2}\right)^{n+1} - 1 \right] z^{-n} \quad |z| > 2 \\ &= \sum_{n=0}^{\infty} 2^{-n-1} z^{-n} - \sum_{n=0}^{\infty} 1 \cdot z^{-n} \\ &= \frac{\frac{1}{2}}{1-\left(\frac{1}{2z}\right)} - \frac{1}{1-\left(\frac{1}{z}\right)} \\ &= \frac{0.5z}{z-0.5} - \frac{z}{z-1} \\ &= \frac{0.5z - 0.5 - z + 0.5}{(z-0.5)(z-1)} \cdot z \\ &= \frac{-0.5z^2}{(z-1)(z-0.5)} \end{aligned}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

$$f(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = X(z)$$

$$a_n \leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$$



$$p=1$$

$$p=2$$

$$|z| < 1$$

$$a_n = \sum_{k=1}^M \operatorname{Res} \left( \frac{f(z)}{z^{n+1}}, z_k \right)$$

$$= \operatorname{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right)$$

$$\frac{f(z)}{z^{n+1}} = \left( \frac{1}{z-1} - \frac{1}{z-2} \right) \frac{1}{z^{n+1}}$$

$n \geq 0$  then the pole  $z=0$

$$\frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{d}{dz} \left( (z-1)^{-1} - (z-2)^{-1} \right) = (-1) \left( (z-1)^{-2} - (z-2)^{-2} \right)$$

$$\frac{d^2}{dz^2} \left( (z-1)^{-1} - (z-2)^{-1} \right) = (-1)(-2) \left( (z-1)^{-3} - (z-2)^{-3} \right)$$

$$\frac{d^3}{dz^3} \left( (z-1)^{-1} - (z-2)^{-1} \right) = (-1)(-2)(-3) \left( (z-1)^{-4} - (z-2)^{-4} \right)$$

$$\frac{d^n}{dz^n} \left( (z-1)^{-1} - (z-2)^{-1} \right) = (-1)^n n! \left( (z-1)^{-n-1} - (z-2)^{-n-1} \right)$$

$$\frac{1}{n!} \lim_{z \rightarrow 0} \frac{d^n}{dz^n} z^{n+1} \left( (z-1)^{-1} - (z-2)^{-1} \right) \frac{1}{z^{n+1}} = \frac{1}{n!} \lim_{z \rightarrow 0} \frac{d^n}{dz^n} \left( (z-1)^{-1} - (z-2)^{-1} \right)$$

$$= \frac{1}{n!} \lim_{z \rightarrow 0} (-1)^n n! \left( (z-1)^{-n-1} - (z-2)^{-n-1} \right)$$

$$= (-1)^n \lim_{z \rightarrow 0} \left( (z-1)^{-n-1} - (z-2)^{-n-1} \right)$$

$$= (-1)^n \left( (-1)^{-n-1} - (-2)^{-n-1} \right)$$

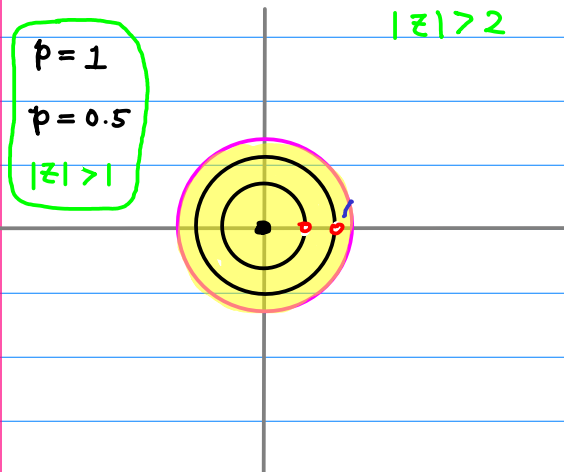
$$= -1 + 2^{-n-1}$$

$$a_n = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \left( \left(\frac{1}{2}\right)^{n+1} - 1 \right) z^n$$

$$x_n \leftarrow X(z) = \frac{-0.5 z^2}{(z-1)(z-0.5)}$$

L.S. first  
① - 3



$$x[n]$$

$$= \frac{1}{2\pi i} \int_C X(z) z^{n+1} dz$$

$$= \sum_{j=1}^k \text{Res} (X(z) z^{n+1}, z_j)$$

$$X(z) = \frac{-0.5 z^2}{(z-1)(z-0.5)}$$

$$X(z) z^{n+1} = \frac{-0.5 z^{n+2}}{(z-1)(z-0.5)}$$

$$\text{Res} (X(z) z^{n+1}, 1) = (z-1) \frac{-0.5 z^{n+2}}{(z-1)(z-0.5)} \Big|_{z=1} = -1$$

$$\text{Res} (X(z) z^{n+1}, \frac{1}{2}) = (z-0.5) \frac{-0.5 z^{n+2}}{(z-1)(z-0.5)} \Big|_{z=0.5} = + \left(\frac{1}{2}\right)^{n+1}$$

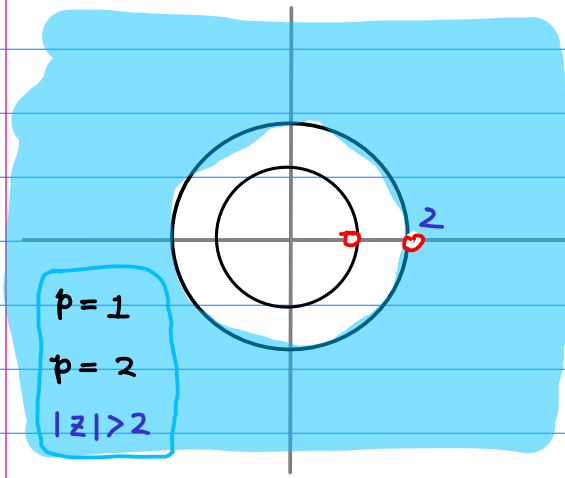
$$x[n] = \left(\frac{1}{2}\right)^{n+1} - 1$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

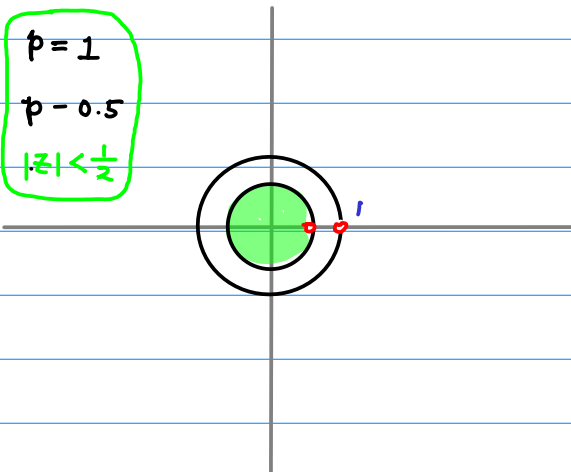
$$X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

L.S. first  
 $\textcircled{\text{II}} - 1$

$$\textcircled{\text{II}} \quad D_2 \quad |z| > 2 \quad \left[ \left| \frac{1}{z} \right| < 1, \quad \left| \frac{2}{z} \right| < 1 \right]$$



$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\ &= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{2}{z}\right)^n \\ &= \sum_{n=0}^{\infty} z^{-n-1} - \sum_{n=0}^{\infty} 2^n z^{-n-1} \\ &= \sum_{n=1}^{\infty} z^{-n} - \sum_{n=1}^{\infty} 2^{n-1} z^{-n} \\ &= \sum_{n=-1}^{\infty} [1 - 2^{n-1}] z^n = \sum_{n=-1}^{\infty} \left[1 - \left(\frac{1}{z}\right)^{n+1}\right] z^n \end{aligned}$$



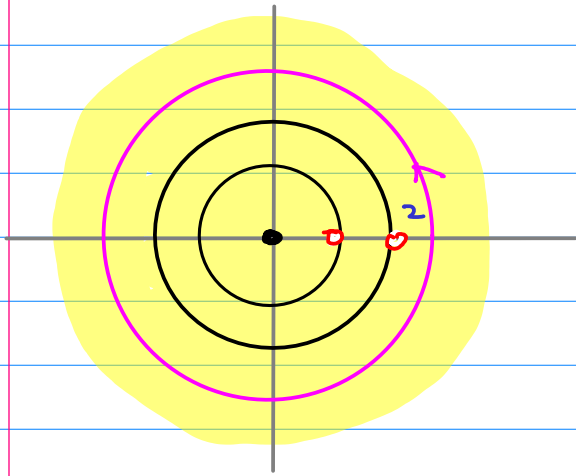
$$\begin{aligned} X(z) = f(z^{-1}) &= \sum_{n=-1}^{\infty} [1 - 2^{n-1}] z^{-n} \\ &= \sum_{n=1}^{\infty} [1 - 2^{n-1}] z^n \\ &= \sum_{n=1}^{\infty} 1 \cdot z^n - \sum_{n=1}^{\infty} 2^{n-1} z^n \\ &= \frac{z}{1-z} - \frac{z}{1-2z} \\ &= -\frac{z}{z-1} + \frac{0.5z}{z-0.5} \\ &= \frac{-z+0.5+0.5z-0.5}{(z-1)(z-0.5)} z \\ &= \frac{-0.5z^2}{(z-1)(z-0.5)} \end{aligned}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

$$f(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = X(z)$$

$$a_n \leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$$

L.S. first  
 $\textcircled{\text{II}} - 2$



$$\begin{aligned} a_n &= \sum_{k=1}^M \text{Res} \left( \frac{f(z)}{z^{n+1}}, z_k \right) \\ &= \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right) \\ &+ \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 1 \right) \\ &+ \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 2 \right) \end{aligned}$$

$$\text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 1 \right) = \lim_{z \rightarrow 1} (z-1) \frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

$$\text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 2 \right) = \lim_{z \rightarrow 2} (z-2) \frac{-1}{(z-1)(z-2)z^{n+1}} = -\frac{1}{2^{n+1}}$$

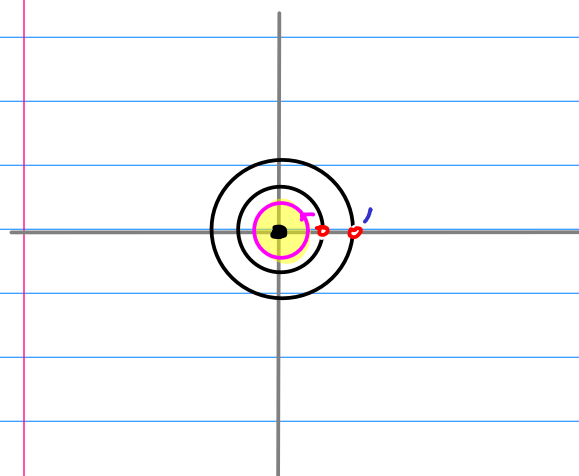
$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\text{Res} \left( \frac{f(z)}{z^{n+1}}, 0 \right)$
1	1	1	1	1	1	$\text{Res} \left( \frac{f(z)}{z^{n+1}}, 1 \right)$
$-2^2$	$-2^1$	$-2^0$	$-2^{-1}$	$-2^{-2}$	$-2^{-3}$	$\text{Res} \left( \frac{f(z)}{z^{n+1}}, 2 \right)$
$1-2^2$	$1-2$	0	0	0	0	
$-(-3)-1$	$-(-2)-1$	$-(-1)-1$				

$$a_n = 1 - 2^{-n-1} \quad n < 0$$

$$\begin{aligned} f(z) &= \sum_{n=-1}^{-\infty} (1 - 2^{-n-1}) z^n = \sum_{n=-1}^{-\infty} \left( 1 - \left(\frac{1}{2}\right)^{n+1} \right) z^n \\ &= \sum_{n=1}^{\infty} (1 - 2^{n-1}) z^{-n} \end{aligned}$$

$$x_n \leftarrow X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

L.S. first  
 (II) -3



$$\begin{aligned} x[n] &= \sum_{j=1}^k \text{Res} (X(z)z^{n-1}, z_j) \\ &= \text{Res} \left( \frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, 0 \right) \\ &= \text{Res} \left( z^{n+1} \left( \frac{1}{z-0.5} - \frac{1}{z-1} \right), 0 \right) \end{aligned}$$

$$\frac{1}{n!} \lim_{z \rightarrow z_0} \frac{d^n}{dz^n} (z-z_0)^{n+1} f(z) \quad (n+1)\text{th order pole at } z=0$$

$n = -1$		$(-2^0 + 1)$
$n = -2$	$z^1 z^{-1} ((z-0.5)^{-1} - (z-1)^{-1}) =$	$0! (-2^1 + 1)$
$n = -3$	$\frac{d}{dz} z^2 z^{-2} ((z-0.5)^{-1} - (z-1)^{-1}) =$	$1! (-2^2 + 1)$
$n = -4$	$\frac{d^2}{dz^2} z^3 z^{-3} ((z-0.5)^{-1} - (z-1)^{-1}) =$	$2! (-2^3 + 1)$
$n = -k$	$\frac{d^{k-2}}{dz^{k-2}} z^{k-1} z^{-k+1} ((z-0.5)^{-1} - (z-1)^{-1}) = (-1)(-2) \dots (-(k-2)) ((z-0.5)^{-k+1} - (z-1)^{-k+1})$	$(k-2)! (-2^{k+1} + 1)$
$x_{-k} =$	$\frac{1}{(k-2)!} \lim_{z \rightarrow 0} \frac{d^{k-2}}{dz^{k-2}} z^{k-1} z^{-k+1} ((z-0.5)^{-1} - (z-1)^{-1}) = (-2^{k+1} + 1)$	$(k \geq 1) \quad (n \leq -1)$

$$x_n = (-2^{-n-1} + 1) \quad (n < 0)$$

$$\begin{aligned} f(z) &= \sum_{n=-1}^{-\infty} (1 - 2^{-n-1}) z^n = \sum_{n=-1}^{-\infty} \left( 1 - \left(\frac{1}{2}\right)^{n+1} \right) z^n \\ &= \sum_{n=1}^{\infty} (1 - 2^{-n-1}) z^{-n} \end{aligned}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

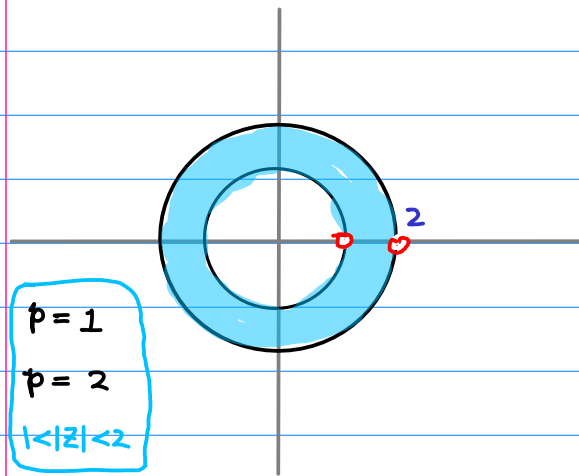
L.S. first

III -1

III

$$D_3 \quad 1 < |z| < 2$$

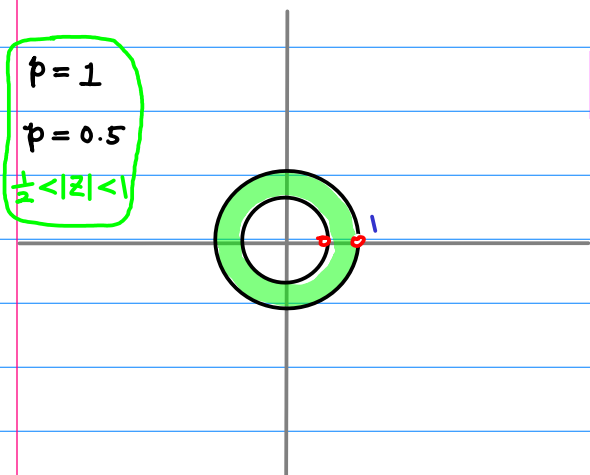
$$\left[ \left| \frac{1}{z} \right| < 1, \quad \left| \frac{z}{2} \right| < 1 \right]$$



$$\begin{aligned}
 f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\
 &= \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{z}{2}\right)} \\
 &= \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{z}{2}\right)^n \\
 &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \\
 &= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}
 \end{aligned}$$

$$= \sum_{n=-\infty}^{-1} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$[ 1 \ (n < 0), \ 2^{-n-1} \ (n \geq 0) ]$$



$$X(z) = f(z^{-1}) = \sum_{n=-\infty}^{-1} z^{-n} + \sum_{n=0}^{\infty} \frac{z^{-n}}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^{-n}}{2^{n+1}}$$

$$= \frac{z}{1-z} - \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{1}{2z}\right)}$$

$$= \frac{-z}{z-1} + \frac{0.5z}{z-0.5}$$

$$= \frac{-z+0.5 + 0.5z-0.5}{(z-1)(z-0.5)} \cdot z$$

$$= \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

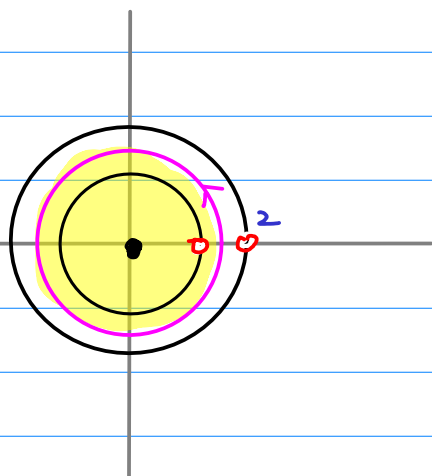
$$f(z^{-1}) = \frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \frac{-z^2}{(1-z)(1-2z)} = \frac{-0.5z^2}{(z-1)(z-0.5)} = X(z)$$



$$a_n \leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$$

L.S. first

III -2



$$a_n = \sum_{k=1}^M \text{Res} \left( \frac{f(z)}{z^{n+1}}, z_k \right)$$

$$= \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right)$$

$$+ \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 1 \right)$$

$$\frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\begin{aligned} \frac{1}{n!} \lim_{z \rightarrow 0} \frac{d^n}{dz^n} ((z-1)^{-1} - (z-2)^{-1}) &= (-1)^n \lim_{z \rightarrow 0} ((z-1)^{-n-1} - (z-2)^{-n-1}) \\ &= (-1)^n ((-1)^{-n-1} - (-2)^{-n-1}) \\ &= -1 + 2^{-n-1} \end{aligned}$$

$$\text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 1 \right) = \lim_{z \rightarrow 1} (z-1) \frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\text{Res} \left( \frac{f(z)}{z^{n+1}}, 0 \right)$
1	1	1	1	1	1	$\text{Res} \left( \frac{f(z)}{z^{n+1}}, 1 \right)$
1	1	1	$2^{-1}$	$2^{-2}$	$2^{-3}$	

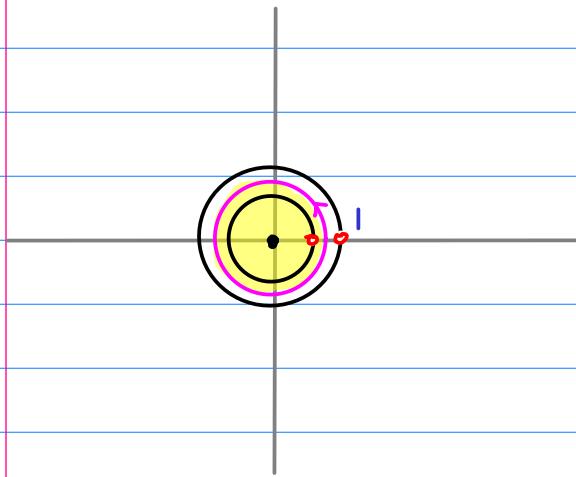
$$\begin{cases} a_n = 2^{-n-1} & n \geq 0 \\ a_n = 1 & n < 0 \end{cases} \quad \begin{cases} 2^{-n-1} z^n \\ z^{-n} \end{cases}$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$x_n \leftarrow X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

L.S. first

III -3



$$\begin{aligned} x[n] &= \sum_{j=1}^2 \text{Res} \left( X(z)z^{n-1}, z_j \right) \\ &= \text{Res} \left( \frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, 0 \right) \\ &\quad + \text{Res} \left( \frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2} \right) \end{aligned}$$

$$\text{Res} \left( \frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2} \right) = \lim_{z \rightarrow 0.5} \frac{-0.5z^{n+1}}{(z-1)} = \left(\frac{1}{2}\right)^{n+1} = 2^{-n-1}$$

$$\text{Res} \left( \frac{-0.5z^{n+1}}{(z-1)(z-0.5)}, 0 \right) = (-2^{-n-1} + 1) \quad (n < 0)$$

$$(n < 0) \quad x_n = (-2^{-n-1} + 1) - 2^{-n-1} = 1$$

$$(n \geq 0) \quad x_n = 2^{-n-1}$$

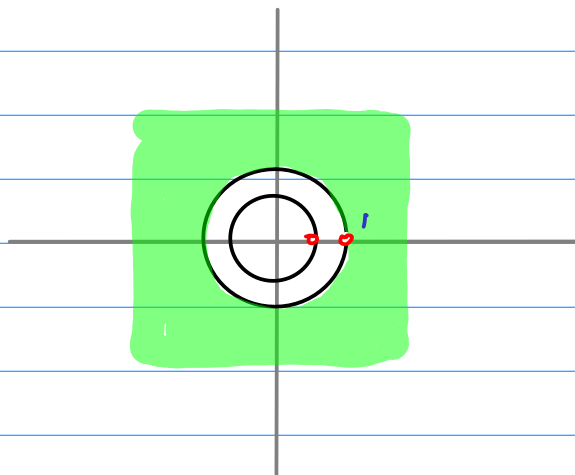
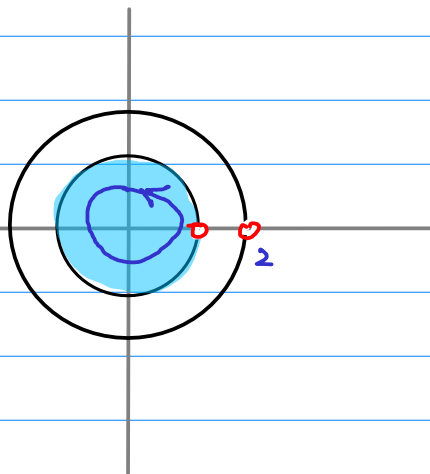
$$(n < 0) \quad a_n = 1$$

$$(n \geq 0) \quad a_n = 2^{-n-1}$$

L.S. first

$a_n, x_n$  using Res  $\leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$  L.S. first

①  $D_1 \quad |z| < 1 \quad \left[ \left| \frac{z}{1} \right| < 1, \left| \frac{z}{2} \right| < 1 \right]$



$p=0$	$\rightarrow$	$\infty$
$p=1$	$\rightarrow$	$p=1$
$p=2$	$\rightarrow$	$p=0.5$
$\infty$	$\rightarrow$	$p=0$

$$a_n = \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right)$$

$$x_n = \text{Res} \left( \frac{-0.5 z^{n+1}}{(z-1)(z-0.5)}, 0 \right)$$

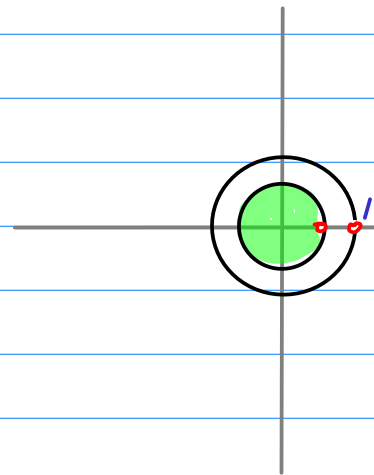
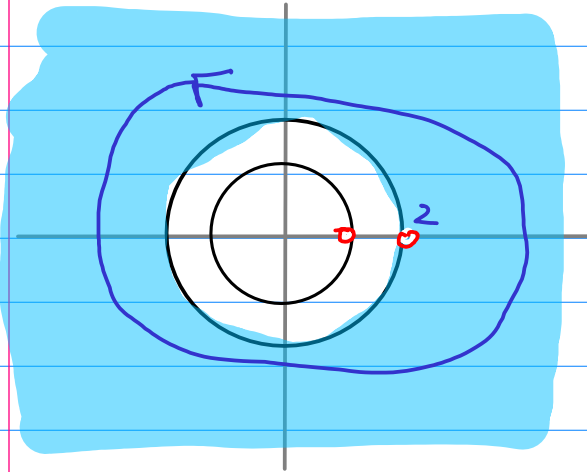
$$= \text{Res} \left( \frac{-0.5 z^{n+1}}{(z-1)(z-0.5)}, 1 \right)$$

$$+ \text{Res} \left( \frac{-0.5 z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2} \right)$$

L.S. first

$$a_n, x_n \text{ using Res} \leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$$

$$\textcircled{\text{II}} D_2 \quad |z| > 2 \quad \left[ \left| \frac{1}{z} \right| < 1, \quad \left| \frac{2}{z} \right| < 1 \right]$$



$p=0$	$\rightarrow$	$\infty$
$p=1$	$\rightarrow$	$p=1$
$p=2$	$\rightarrow$	$p=0.5$
$\infty$	$\rightarrow$	$p=0$

$$a_n = \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right)$$

$$+ \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 1 \right)$$

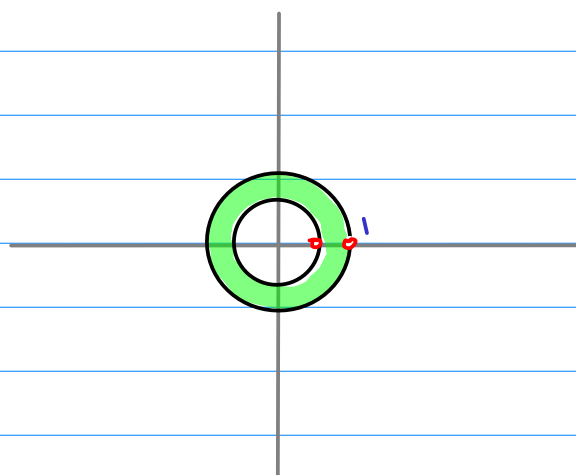
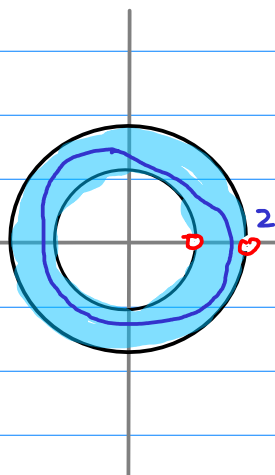
$$+ \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 2 \right)$$

$$x_n = \text{Res} \left( \frac{-0.5 z^{n+1}}{(z-1)(z-0.5)}, 0 \right)$$

$$a_n, x_n \text{ using Res} \leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$$

L.S. first

$$\textcircled{III} D_3 \quad 1 < |z| < 2 \quad \left[ \left| \frac{1}{z} \right| < 1, \quad \left| \frac{z}{2} \right| < 1 \right]$$

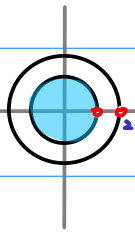


$p=0$	$\rightarrow$	$\infty$
$p=1$	$\rightarrow$	$p=1$
$p=2$	$\rightarrow$	$p=0.5$
$\infty$	$\rightarrow$	$p=0$

$$a_n = \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right) + \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 1 \right)$$

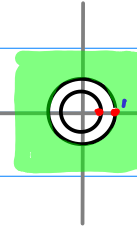
$$x_n = \text{Res} \left( \frac{-0.5 z^{n+1}}{(z-1)(z-0.5)}, 0 \right) + \text{Res} \left( \frac{-0.5 z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2} \right)$$

$$a_n, x_n \text{ using Res} \leftarrow f(z) = \frac{-1}{(z-1)(z-2)}$$

 $(n \geq 0)$ 

$$a_n = -1 + 2^{-n-1}$$

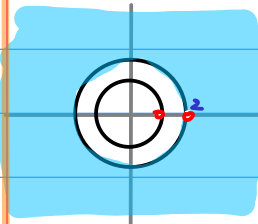
$$a_n = \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right)$$



$$x_n = -1 + 2^{-n-1}$$

$$x_n = \text{Res} \left( \frac{-0.5 z^{n+1}}{(z-1)(z-0.5)}, 1 \right)$$

$$+ \text{Res} \left( \frac{-0.5 z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2} \right)$$

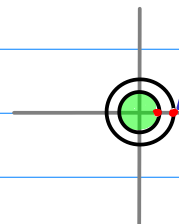
 $(n < 0)$ 

$$1 - 2^{-n-1}$$

$$a_n = \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right)$$

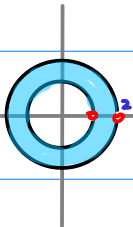
$$+ \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 1 \right)$$

$$+ \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 2 \right)$$



$$1 - 2^{-n-1}$$

$$x_n = \text{Res} \left( \frac{-0.5 z^{n+1}}{(z-1)(z-0.5)}, 0 \right)$$

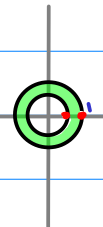
 $(n \geq 0)$  $(n < 0)$ 

$$a_n = 2^{-n-1}$$

$$a_n = 1$$

$$a_n = \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 0 \right)$$

$$+ \text{Res} \left( \frac{-1}{(z-1)(z-2)z^{n+1}}, 1 \right)$$



$$x_n = 2^{-n-1}$$

$$x_n = 1$$

$$x_n = \text{Res} \left( \frac{-0.5 z^{n+1}}{(z-1)(z-0.5)}, 0 \right)$$

$$+ \text{Res} \left( \frac{-0.5 z^{n+1}}{(z-1)(z-0.5)}, \frac{1}{2} \right)$$

L.S. first

