

# Z Transform (H.1)

## Definition

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Based on  
Complex Analysis for Mathematics and Engineering  
J. Mathews

# z - Transform

$$X(z) = \sum_{k=-\infty}^{+\infty} x[k] z^{-k}$$

$$z = |r| e^{j2\pi F} = |r| e^{j\Omega}$$

$$x[n] \longleftrightarrow X(z)$$

One Sided z-transform

$$X(z) = \sum_{k=0}^{+\infty} x[k] z^{-k}$$

# Inverse z-Transform

$$X(z) = \mathcal{Z}\left[\{x_n\}_{n=0}^{\infty}\right] = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x_n = x[n] = \mathcal{Z}^{-1}[X(z)] = \frac{1}{2\pi i} \int_C X(z) z^{n+1} dz$$

# Admissible Form of $z$ -transform

$$X(z) = \sum_{k=0}^{\infty} x[n] z^{-n}$$

admissible  $z$ -transform

if  $X(z)$  is a rational function

$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z^1 + b_2 z^2 + \dots + b_{p-1} z^{p-1} + b_p z^p}{a_0 + a_1 z^1 + a_2 z^2 + \dots + a_{q-1} z^{q-1} + a_q z^q}$$

$P(z)$  : a polynomial of degree  $p$

$Q(z)$  : a polynomial of degree  $q$

D: Simply connected domain

C: Simple closed contour (CCW) in D

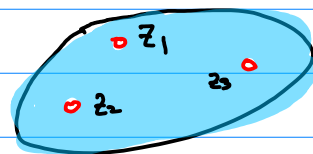
if  $f(z)$  is analytic inside C and on C  
except at the points  $z_1, z_2, \dots, z_k$  in C

then

$$\frac{1}{2\pi i} \int_C f(z) dz = \sum_{j=1}^k \text{Res}(f(z), z_j)$$

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f(z), z_k)$$

finite number  $k$  of  
singular points  $z_k$

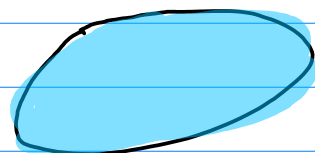


$$\oint_C f(z) dz = 0 \quad \text{if } f(z) \text{ is continuous in } D \text{ and}$$

$f(z) = F'(z) : F(z) \text{ is an antiderivative of } f(z)$   
fundamental theorem of calculus

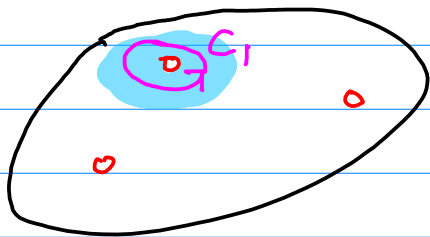
$$\oint_C f(z) dz = 0 \quad \text{if } f(z) \text{ is analytic within and on } C$$

no singularity

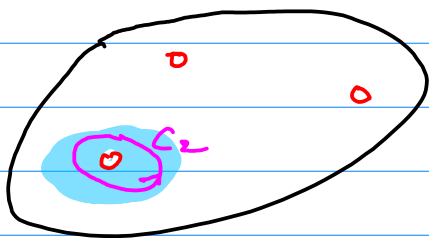


$$f(z) = \sum_{n=n_1}^{\infty} a_n^{(m)} (z - z_m)^n$$

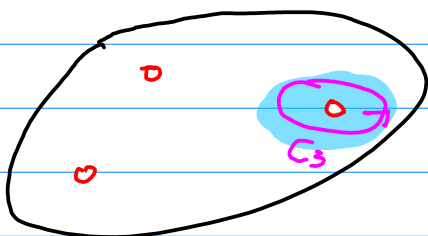
$$\begin{aligned} a_n^{(m)} &= \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} dz' \\ &= \sum_k \text{Res} \left( \frac{f(z)}{(z - z_m)^{n+1}}, z_k \right) \quad z_k \text{ within } C \\ &= \frac{1}{n!} f^{(n)}(z_m) \quad n \geq 0 \end{aligned}$$



$a_n^{(0)}$  expansion at  $z_0$



$a_n^{(1)}$  expansion at  $z_1$



$a_n^{(2)}$  expansion at  $z_2$



$$a_n^{(m)} = \text{Res}(f(z), z_m)$$

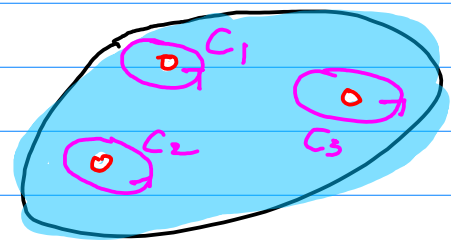
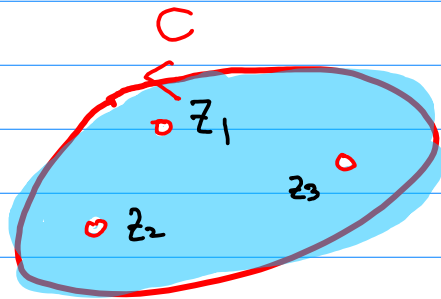
the residue of  $f(z)$  at  $z_m$  using  $C_m$

assumed that

there are several ( $m$ ) singularities (poles) of  $f(z)$  in a region

but that

$C$  is taken to enclose only the pole  $z_m$  :  $C_m$



## Laurent's Theorem

$f$ : analytic within the annular domain  $D$

$$r < |z - z_0| < R$$

then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k, \text{ valid for } r < |z - z_0| < R$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds, \quad k = 0, \pm 1, \pm 2, \dots$$

$C$ : a simple closed curve  
that lies entirely within  $D$   
that encloses  $z_0$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) ds \quad \rightarrow \quad \oint_C f(s) ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) ds = \text{Res}(f(z), z_0)$$

$$= \begin{cases} \lim_{z \rightarrow z_0} (z - z_0) f(z) & \text{(simple)} \\ \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & \text{(order } n) \end{cases}$$

## Cauchy's Residue Theorem

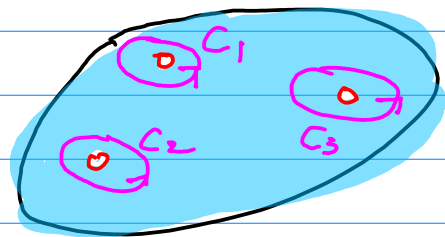
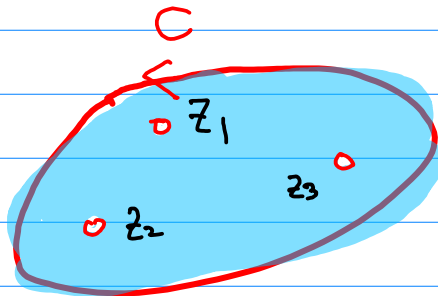
$f(z)$ : **analytic** on and within  $C$   
except a finite number of **singular points**  
 $z_1, z_2, \dots, z_n$  within  $C$

then

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f(z), z_k)$$

$D$ : a simply connected domain

$C$ : a simple closed contour in  $D$



$z_1$

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_1)^k$$

$$a_{-1}^{z_1} = \frac{1}{2\pi i} \oint_{C_1} f(s) ds = \text{Res}(f(z), z_1)$$

$z_2$

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_2)^k$$

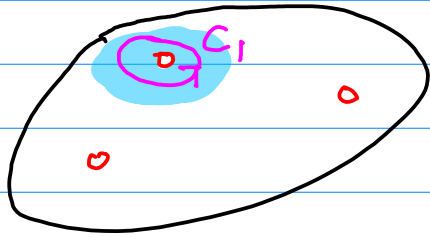
$$a_{-1}^{z_2} = \frac{1}{2\pi i} \oint_{C_2} f(s) ds = \text{Res}(f(z), z_2)$$

$z_3$

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_3)^k$$

$$a_{-1}^{z_3} = \frac{1}{2\pi i} \oint_{C_3} f(s) ds = \text{Res}(f(z), z_3)$$

$z_1$

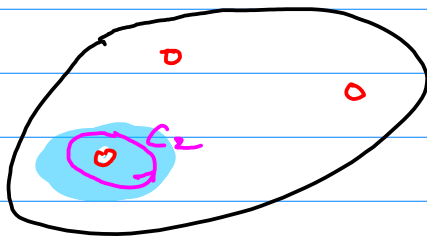


Laurent series expansion at  $z_1$

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_1)^k$$

$$a_{-1} = \frac{1}{2\pi i} \oint_{C_1} f(s) ds = \text{Res}(f(z), z_1)$$

$z_2$

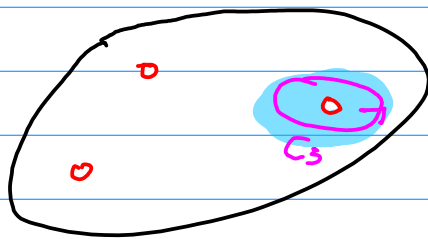


Laurent series expansion at  $z_2$

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_2)^k$$

$$a_{-1} = \frac{1}{2\pi i} \oint_{C_2} f(s) ds = \text{Res}(f(z), z_2)$$

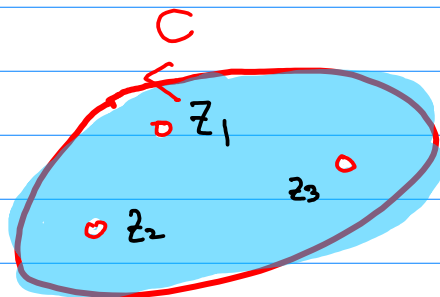
$z_3$



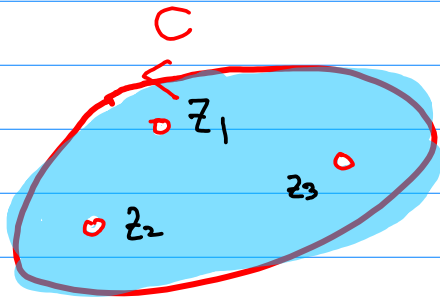
Laurent series expansion at  $z_3$

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_3)^k$$

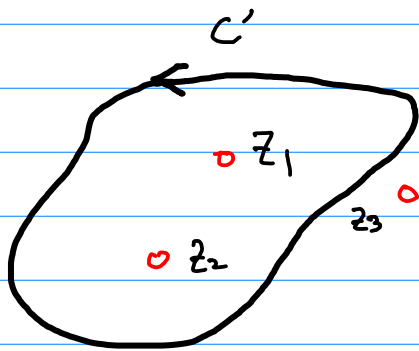
$$a_{-1} = \frac{1}{2\pi i} \oint_{C_3} f(s) ds = \text{Res}(f(z), z_3)$$



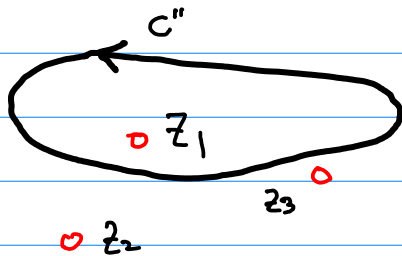
$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f(z), z_k)$$



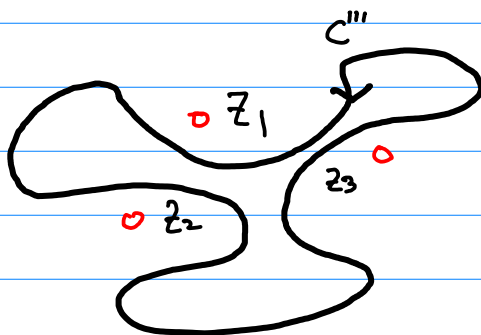
$$\int_C f(z) dz = 2\pi i \operatorname{Res}(f(z), z_1) + 2\pi i \operatorname{Res}(f(z), z_2) + 2\pi i \operatorname{Res}(f(z), z_3)$$



$$\int_{C'} f(z) dz = 2\pi i \operatorname{Res}(f(z), z_1) + 2\pi i \operatorname{Res}(f(z), z_2)$$

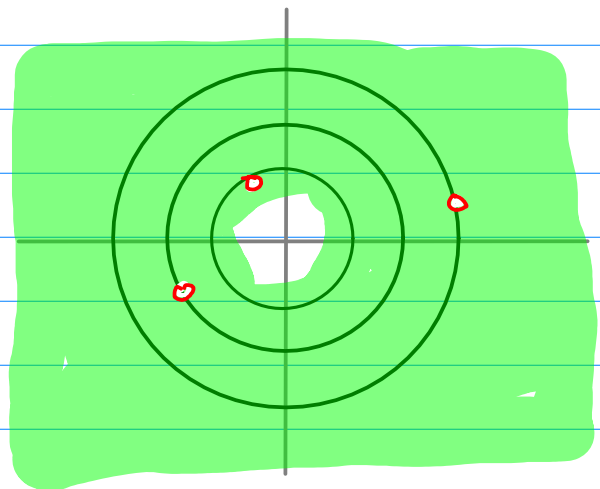
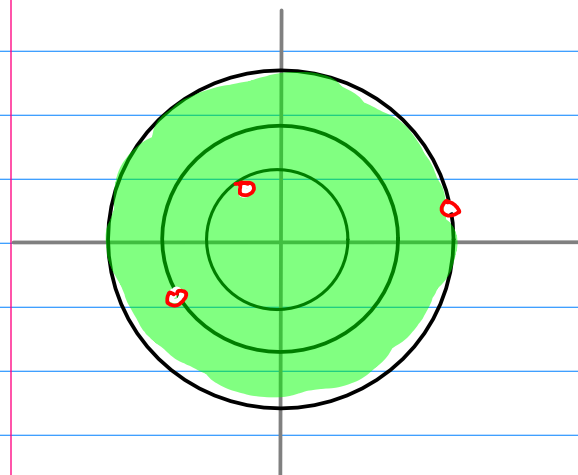
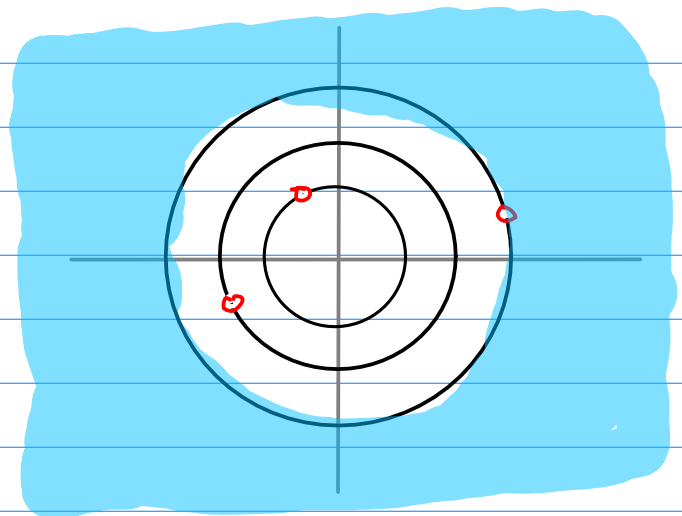
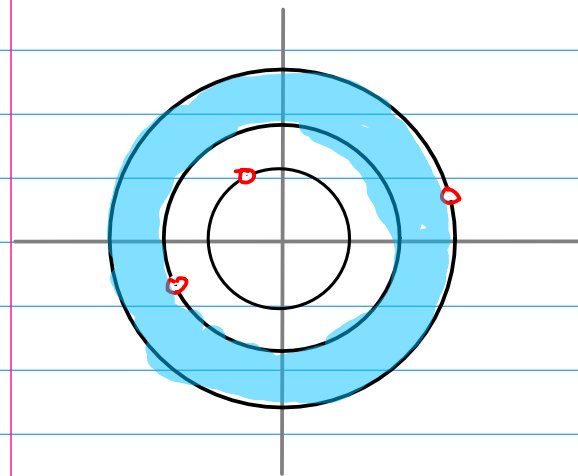
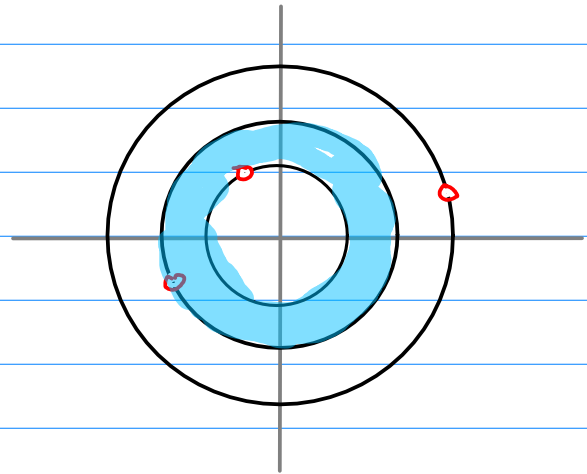
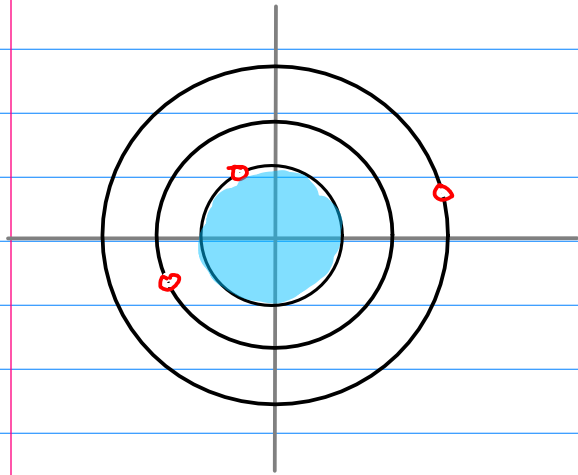


$$\int_{C''} f(z) dz = 2\pi i \operatorname{Res}(f(z), z_1)$$



$$\int_{C'''} f(z) dz = 0$$

# Different $D$ , Different Laurent Series



$z$ -transform

$$f(z) = \sum_{n=n_1}^{\infty} a_n (z - z_m)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} dz'$$

$$= \sum_k \text{Res} \left( \frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$



$C$  is in the same region of analyticity of  $f(z)$   
typically a circle centered on  $z_m$

$z_k$  within  $C$  : singularities of  $\frac{f(z)}{(z - z_m)^{n+1}}$

$n_1 = n_{f,m}$  depends on  $f(z)$ ,  $z_m$

$a_n$  depends on  $f(z)$ ,  $z_m$ , region of analyticity

Whether  $f(z)$  is singular at  $z = z_m$  or not

or at other points between  $z$  and  $z_m$

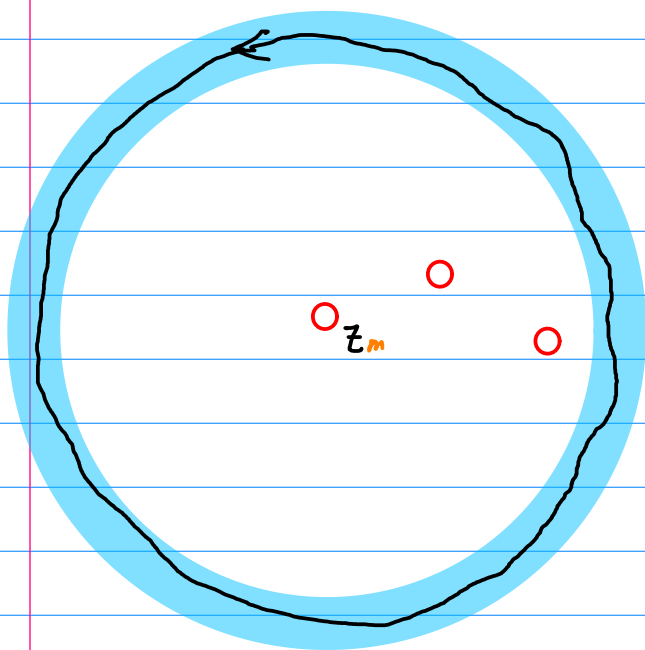
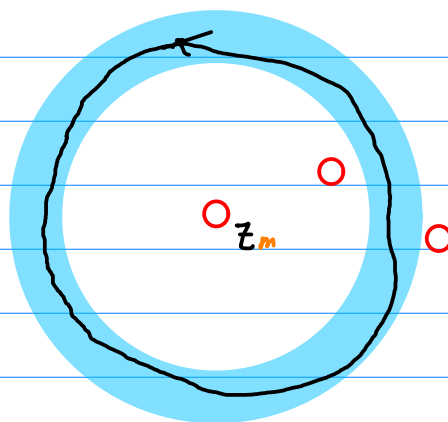
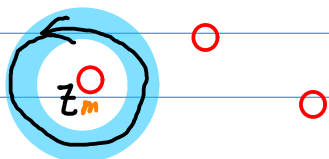
We can expand  $f(z)$  about any point  $z_m$

over powers of  $(z - z_m)$ .

$$f(z) = \sum_{n=-\infty}^{\infty} a_n^{(m)} (z - z_m)^n$$

$$a_n^{(m)} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} dz'$$

$$= \sum_k \operatorname{Res} \left( \frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$





$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} dz'$$

$$= \sum_k \operatorname{Res} \left( \frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$

Analytic at  $z_m$

$$n_1 \geq 0$$

$$\text{general } n_1, \quad z_m = 0$$

Taylor Series

MacLaurin Series

Singular at  $z_m$

$$\text{general } n_1$$

$$\text{general } n_1, \quad z_m = 0$$

Laurent Series

$z$ -Transform

$$f(z) = \sum_{n=-\infty}^{\infty} a_n^{(m)} (z - z_m)^n$$

$$a_n^{(m)} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z' - z_m)^{n+1}} dz'$$

$$= \sum_k \operatorname{Res} \left( \frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$

$$z_m = 0$$

$$a_{-n}^{(0)} = h(n)$$

$$n \rightarrow -n$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(-n) z^n$$

$$h(n) = \frac{1}{2\pi i} \oint_c \frac{H(z')}{z'^{n+1}} dz'$$

$$= \sum_k \operatorname{Res} \left( \frac{H(z)}{z^{n+1}}, z_k \right)$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$h(n) = \frac{1}{2\pi i} \oint_c H(z') z'^{n-1} dz'$$

$$= \sum_k \operatorname{Res} (H(z) z^{n-1}, z_k)$$

$C$  is in the same region of analyticity of  $f(z)$   
typically a circle centered on  $z_m$

$z_k$  within  $C$  : singularities of  $\frac{f(z)}{(z-z_k)^{n+1}}$

$C$  is in the same region of analyticity of  $H(z)$   
typically a circle centered on  $z_m$

generally a circle centered on the origin  
may enclose any or all singularities of  $H(z)$   
often the unit circle

$z_k$  within  $C$  : singularities of  $H(z)z^{n-1}$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \quad z \in \text{R.O.C.}$$

$$h(n) = \frac{1}{2\pi i} \oint_C H(z') z'^{n-1} dz' \quad C \text{ in R.O.C.}$$

$$= \sum_k \text{Res}(H(z) z^{n-1}, z_k)$$

- ① a power series representation of a function  $f(z)$  of a complex variable  $z$
- ② a transform  $H(z)$  of a sequence of 1