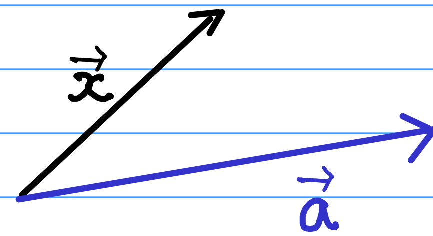
Projection (H.2)

20151218

Projection onto a vector

$\vec{a}$: a non-zero vector in $\mathbb{R}^n$

$\vec{x}$: a vector in $\mathbb{R}^n$



authogonal projection of $\vec{x}$ unto $\text{span}\{\vec{a}\}$

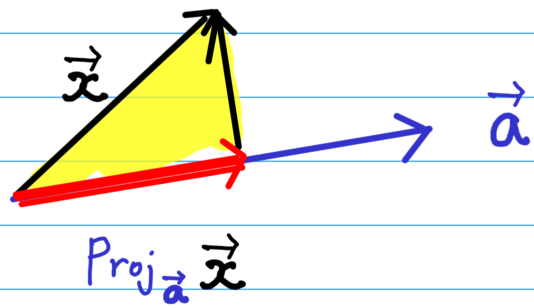
$$\text{Proj}_{\vec{a}}\,\vec{x} = \frac{\vec{x}\cdot\vec{a}}{\|\vec{a}\|}\vec{a}$$

the vector component of $\vec{x}$ along $\vec{a}$

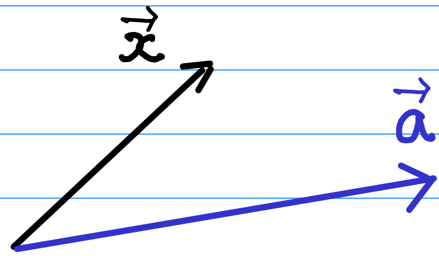
$\Rightarrow$ $\boxed{\text{Proj}_{\vec{a}}\,\vec{x}}$

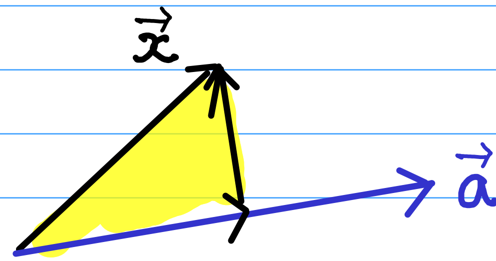
the vector component of $\vec{x}$ orthogonal to $\vec{a}$

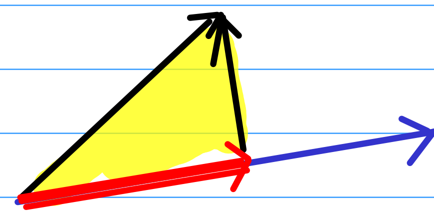
$\Rightarrow$ $\boxed{\vec{x} - \text{Proj}_{\vec{a}}\,\vec{x}}$



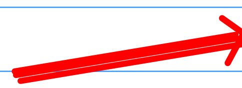
$\vec{x}, \vec{a} \in R^2$



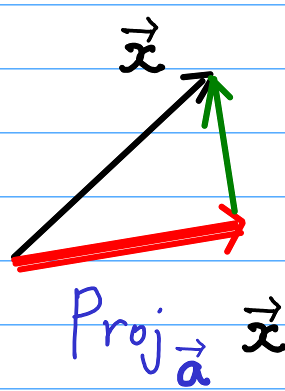




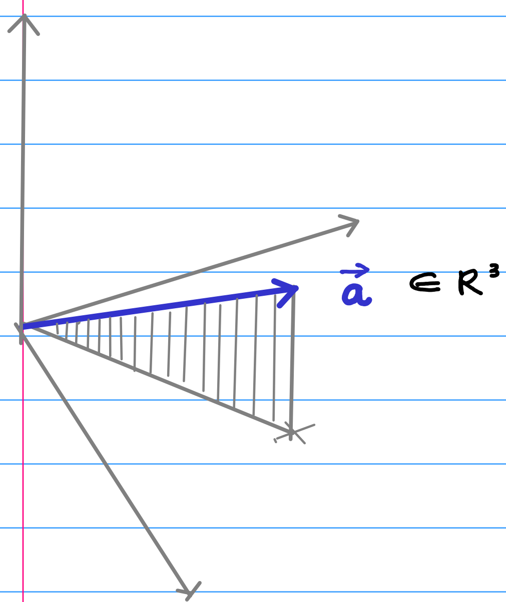
$$\boxed{\operatorname{Proj}_{\vec{a}} \vec{x} = \frac{\vec{x} \cdot \vec{a}}{\|\vec{a}\|} \vec{a}}$$

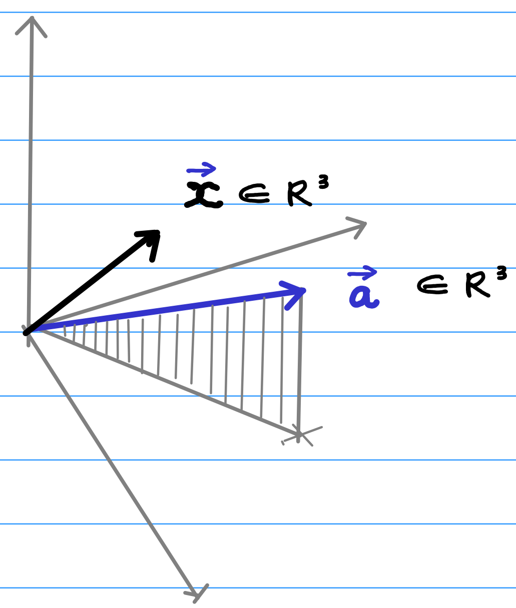


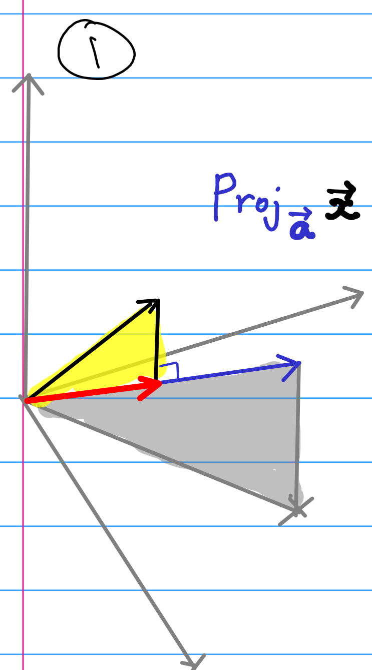
$$\boxed{\vec{x} - \operatorname{Proj}_{\vec{a}} \vec{x}}$$

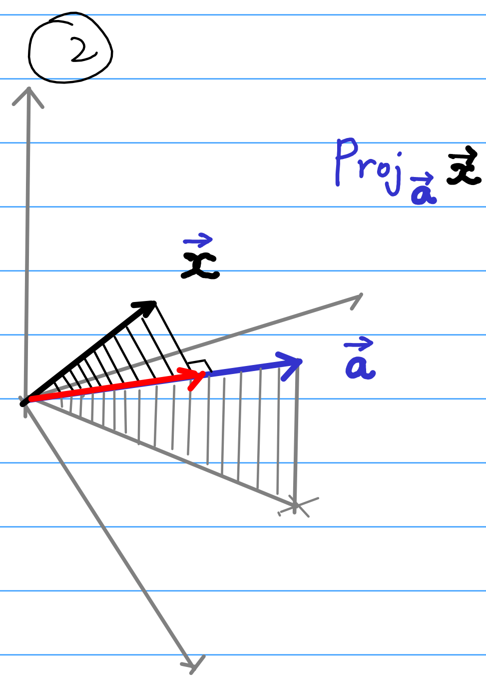


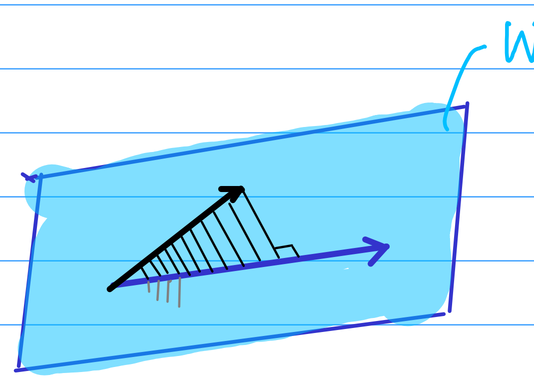
$\vec{x},\ \vec{a} \in R^3$







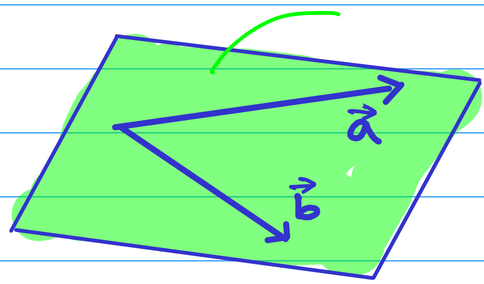


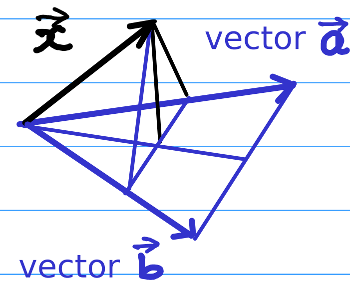


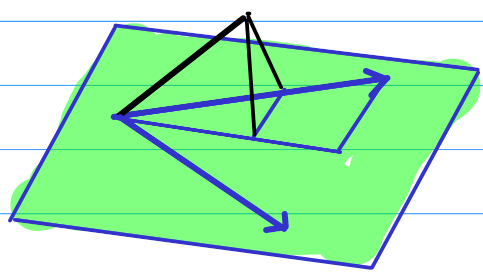
$\mathrm{Proj}_{\vec{a}}\,\vec{x}$, $\quad \mathrm{Proj}_{\vec{b}}\,\vec{x}$, $\quad \mathrm{Proj}_{W}\,\vec{x}$

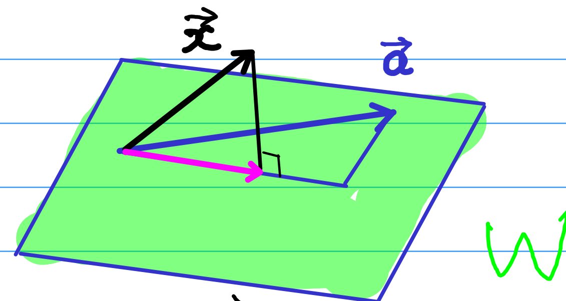
vector $\qquad$ vector $\qquad$ subspace

subspace W

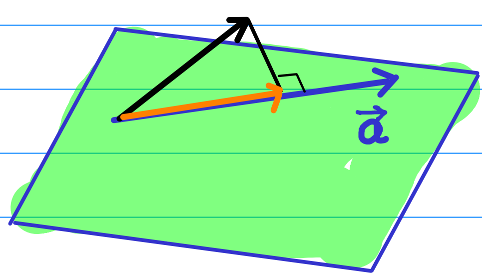




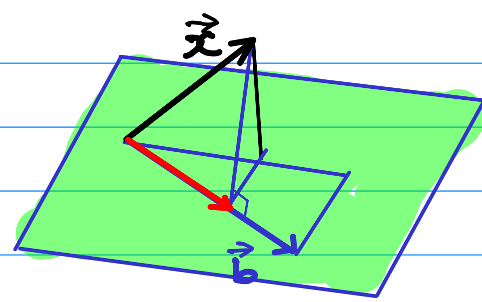




$\mathrm{Proj}_{W}\,\vec{x}$



$\mathrm{Proj}_{\vec{a}}\,\vec{x}$



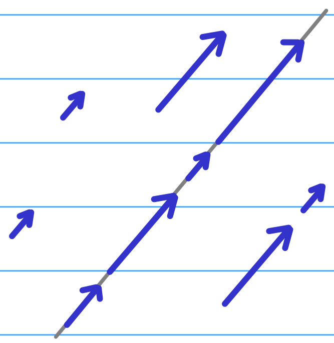
$\mathrm{Proj}_{\vec{b}}\,\vec{x}$

Orthogonal Complement

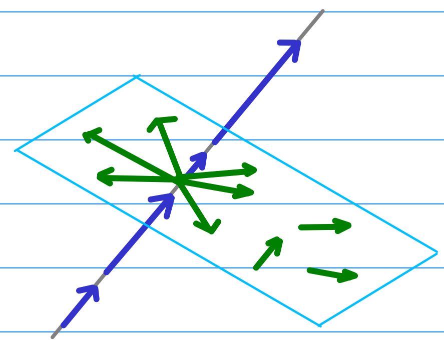
S : a non-empty set in R^n

$S^\perp$: the orthogonal complement of S

$$= \left\{ \begin{array}{l} \text{all vectors in } R^n \\ \text{that are orthogonal} \\ \text{to every vector in } S \end{array} \right\}$$



S



$S^\perp$

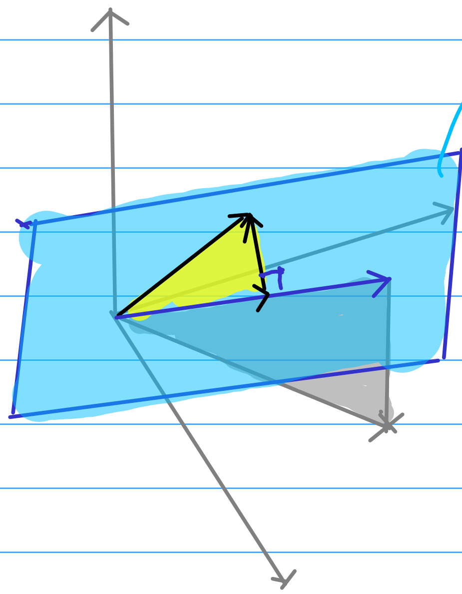
$\dim(S) = 1$

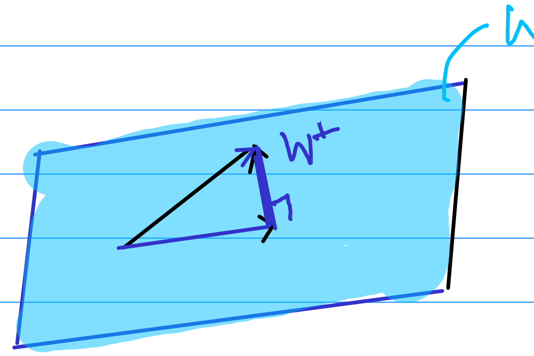
$\dim(S^\perp) = 2$

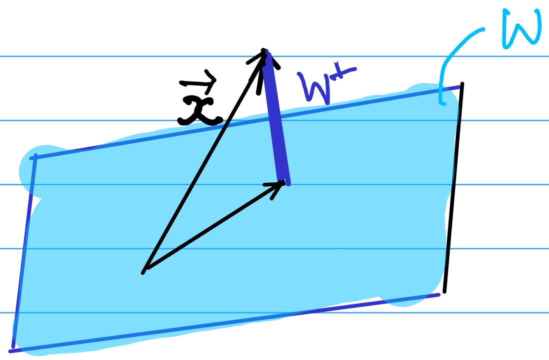
$\dim(S) + \dim(S^\perp) = n$ $\quad R^n$

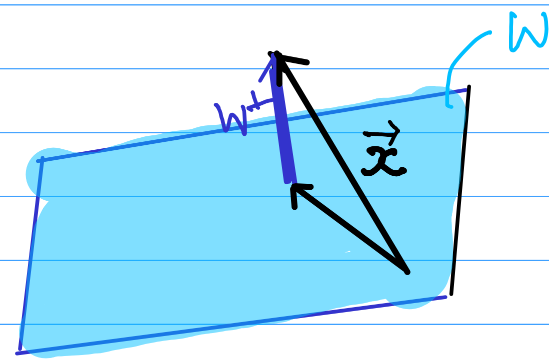
Projection onto a general subspace

Subspace W
Null Space $W^{\perp}$









any Vector $\vec{x}$ can be decomposed into

$$\vec{x} = \vec{x}_1 + \vec{x}_2$$

$$\vec{x}_1 \in W \qquad \vec{x}_2 \in W^{\perp}$$

$$= \mathrm{Proj}_{W}\vec{x} + \mathrm{Proj}_{W^{\perp}}\vec{x}$$

Subspace W
Null Space $W^{\perp}$

Projection Theorem for subspace

W : a subspace of R^n

any vector $\vec{x}$ in R^n can be decomposed into

$$\vec{x} = \vec{x}_1 + \vec{x}_2 \qquad \vec{x}_1 \in W \qquad \vec{x}_2 \in W^{\perp}$$

W, $W^{\perp}$: authogonal complements

$\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_n\}$: a basis for any W

Let $M = [\,\vec{w}_1 \mid \vec{w}_2 \mid \cdots \mid \vec{w}_n\,]$ ····· full rank

$col(M)$, $null(M^T)$: authogonal complements any vector $\vec{x}$ in R^n can be decomposed into

$$\vec{x} = \vec{x}_1 + \vec{x}_2 \qquad \vec{x}_1 \in W \qquad \vec{x}_2 \in W^{\perp}$$

$\Rightarrow$ $\boxed{\vec{x}_1 \in col(M),}$ $\boxed{\vec{x}_2 \in null(M^T)}$

$\equiv$ $\equiv$

$\boxed{\vec{x}_1 = M\vec{v} \text{ for some } \vec{v}}$ $\boxed{M^T\vec{x}_2 = \vec{0}}$

$$M^T(\vec{x} - \vec{x}_1) = \vec{0}$$

$$M^T(\vec{x} - M\vec{v}) = \vec{0}$$

$$\vec{v} = (M^T M)^{-1} M^T \vec{x}$$

$$= (M^T M)^{-1} M^T(\vec{x}_1 + \vec{x}_2)$$

there exists a unique $\vec{v}$ for any $\vec{x}$

$\Rightarrow$ $\vec{x}_1 = M\vec{v}$ $\qquad M^T\vec{x}_2 = \vec{0}$

$\boxed{\vec{x}_1 \in col(M),}$ $\boxed{\vec{x}_2 \in null(M^T)}$

$$\vec{x} = \vec{x}_1 + \vec{x}_2 \qquad \vec{x}_1 \in W \qquad \vec{x}_2 \in W^{\perp}$$

$\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_n\}$: a basis for any W

$M = [\vec{w}_1 | \vec{w}_2 | \cdots | \vec{w}_n]$ ····· full rank

$\vec{v} = (M^T M)^{-1} M^T \vec{x}$ for any $\vec{x}$ in R^n

$$\begin{cases} \vec{x}_1 & \text{such that } \boxed{\vec{x}_1 = M\vec{v}} \\ \vec{x}_2 & \text{such that } \boxed{M^T \vec{x}_2 = \vec{0}} \end{cases}$$

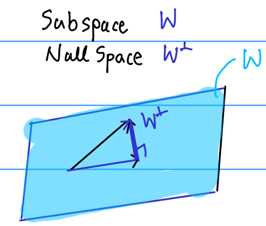
$$\vec{x}_1 = M(M^T M)^{-1} M^T \vec{x}$$

$$\vec{x}_2 = \vec{x} - \vec{x}_1$$

$$\vec{x} = \underset{W}{\underset{\in}{\vec{x}_1}} + \underset{W^T}{\underset{\in}{\vec{x}_2}}$$

W : a subspace of R^n

$\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_n\}$: a basis for W



$$\begin{array}{cc} & k \\ n & \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix} \end{array} = M$$

$$\begin{array}{cc} & k \\ n & \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix} \end{array} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} = M\vec{v}$$

$$= v_1 \begin{bmatrix} \vec{w}_1 \end{bmatrix} + v_2 \begin{bmatrix} \vec{w}_2 \end{bmatrix} + \cdots + v_k \begin{bmatrix} \vec{w}_k \end{bmatrix} \Rightarrow \boxed{\text{linear combination of } \{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_n\}}$$

Col space of M

$$\{M\vec{v}\} = \text{col}(M) = W$$

$\boxed{\text{orthogonal compliments}}$

$$W = \text{col}(M)$$
$$W^\perp = \text{null}(M^T)$$

$$\begin{array}{cc} & k \\ n & \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix} \end{array} = M \qquad \begin{array}{cc} & n \\ k & \begin{bmatrix} \vec{w}_1{}' \\ \vec{w}_2{}' \\ \vdots \\ \vec{w}_k{}' \end{bmatrix} \end{array} = M^T$$

$$\begin{array}{cc} & k \\ n & \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix} \end{array} \begin{bmatrix} v_1 \\ v_2 \\ v_k \end{bmatrix} = M\vec{v} = \begin{bmatrix} \vec{p} \end{bmatrix}$$

a weight vector

$\vec{p} \in$ col(M)

linear combination of $\{ \vec{w}_1, \vec{w}_2, \cdots, \vec{w}_n \}$

$$\begin{array}{cc} & n \\ k & \begin{bmatrix} \vec{w}_1{}' \\ \vec{w}_2{}' \\ \vdots \\ \vec{w}_k{}' \end{bmatrix} \end{array} \begin{bmatrix} \vec{q} \end{bmatrix} = M^T \vec{q} = \vec{0}$$

$\vec{q} \in$ null(M^T)

$$\begin{bmatrix} \vec{p}\,' \end{bmatrix} \begin{bmatrix} \vec{q} \end{bmatrix} = 0$$

$$\boxed{\text{col}(M) \perp \text{null}(M^T)}$$

W : a subspace of R^n

$\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_n\}$: a basis for W

$$[\, \vec{w}_1 \mid \vec{w}_2 \mid \cdots \mid \vec{w}_n \,] = M$$

$\vec{p} = M\vec{v}$ $\quad \{\vec{p}\} \cdots\cdots\!> \mathrm{col}(M) \Rightarrow W$

$\vec{0} = M^T\vec{q}$ $\quad \{\vec{q}\} \cdots\cdots\!> \mathrm{null}(M^T) \Rightarrow W^\perp$

for any $\vec{x}$ in R^n

$$\vec{x} = \underset{\substack{\cap \\ W}}{\vec{p}} + \underset{\substack{\cap \\ W^\perp}}{\vec{q}}$$

$$= \underset{\substack{\cap \\ W}}{\boxed{\mathrm{Proj}_W \vec{x}}} + \underset{\substack{\cap \\ W^\perp}}{\boxed{\mathrm{Proj}_{W^\perp} \vec{x}}}$$

after finding M

$$= \underset{\substack{\cap \\ W}}{\boxed{\mathrm{Proj}_{\mathrm{col}(M)} \vec{x}}} + \underset{\substack{\cap \\ W^\perp}}{\boxed{\mathrm{Proj}_{\mathrm{null}(M^T)} \vec{x}}}$$

Finding x_1 & x_2

$$\vec{x} = \underset{\substack{\cap \\ W}}{\vec{x}_1} + \underset{\substack{\cap \\ W^\perp}}{\vec{x}_2}$$

$$\vec{x}_1 = M\vec{v} \qquad M^T\vec{x}_2 = \vec{0}$$

$$M^T(\vec{x} - \vec{x}_1) = \vec{0}$$

$$M^T(\vec{x} - M\vec{v}) = \vec{0}$$

$$M^T\vec{x} = M^TM\vec{v}$$

$$\vec{v} = (M^TM)^{-1}M^T\vec{x}$$

$$\vec{x}_1 = \mathrm{Proj}_W\vec{x}$$

$$= M\vec{v}$$

$$= M(M^TM)^{-1}M^T\vec{x}$$

$$\vec{x}_2 = \mathrm{Proj}_{W^\perp}\vec{x} \quad \Rightarrow \quad \vec{x} - \vec{x}_1 = \vec{x} - \mathrm{Proj}_W\vec{x}$$

$$= \left(I - M(M^TM)^{-1}M^T\right)\vec{x}$$

$$\vec{x}_1 = \text{Proj}_W \vec{x} = M\vec{v} = M(M^TM)^{-1}M^T\vec{x}$$

$$\vec{x}_2 = \text{Proj}_{W^\perp}\vec{x} = \left(I - M(M^TM)^{-1}M^T\right)\vec{x}$$

$$\begin{array}{cc} & k \\ n & \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix} \end{array} = M \qquad \begin{array}{cc} & n \\ k & \begin{bmatrix} \vec{w}_1' \\ \vec{w}_2' \\ \vdots \\ \vec{w}_k' \end{bmatrix} \end{array} = M^T$$

$$\begin{array}{cc} & n \\ k & \begin{bmatrix} \vec{w}_1' \\ \vec{w}_2' \\ \vdots \\ \vec{w}_k' \end{bmatrix} \end{array} \begin{array}{c} k \\ \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix} \end{array} = \begin{array}{cc} & k \\ k & \boxed{M^TM} \end{array}$$

$$\begin{array}{cc} & k \\ n & \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix} \end{array} \begin{array}{cc} & k \\ k & \boxed{(M^TM)^{-1}} \end{array} \begin{array}{cc} & n \\ k & \begin{bmatrix} \vec{w}_1' \\ \vec{w}_2' \\ \vdots \\ \vec{w}_k' \end{bmatrix} \end{array} = \begin{array}{cc} & n \\ n & \boxed{M(M^TM)^{-1}M^T} \end{array}$$

$\boxed{Ax = 0}$

$$\begin{bmatrix} r_1 \\ r_2 \\ \\ \\ r_m \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} r_1 \cdot x &= 0 \\ r_2 \cdot x &= 0 \\ &\vdots \\ r_m \cdot x &= 0 \end{aligned}}$$

$$(k_1 r_1 + k_2 r_2 + \cdots + k_3 r_m) \cdot x = 0$$

$$\boxed{\text{row}(A) \perp \text{null}(A)}$$

$\boxed{Ax = b}$

$$\begin{bmatrix} c_1 & c_2 & & & & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\boxed{x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = b}$$

$$\boxed{\begin{array}{c} Ax = b \\ \text{consistent} \end{array}} \iff \boxed{b \in \text{col}(A)}$$

Homogeneous System

$$Ax = 0$$

$$\boxed{\text{row}(A) \perp \text{null}(A)}$$

Non-homogeneous System

$$Ax = b$$

$$\boxed{Ax = b \text{ consistent}} \iff \boxed{b \in \text{col}(A)}$$

Orthogonal Compliments

$$\boxed{\begin{array}{l} Ax = 0 \\ \boxed{\text{row}(A) \perp \text{null}(A)} \end{array}}$$

$$\boxed{\begin{array}{l} A^T y = 0 \\ \text{row}(A^T) \perp \text{null}(A^T) \end{array}}$$

$$\boxed{\text{col}(A) \perp \text{null}(A^T)}$$

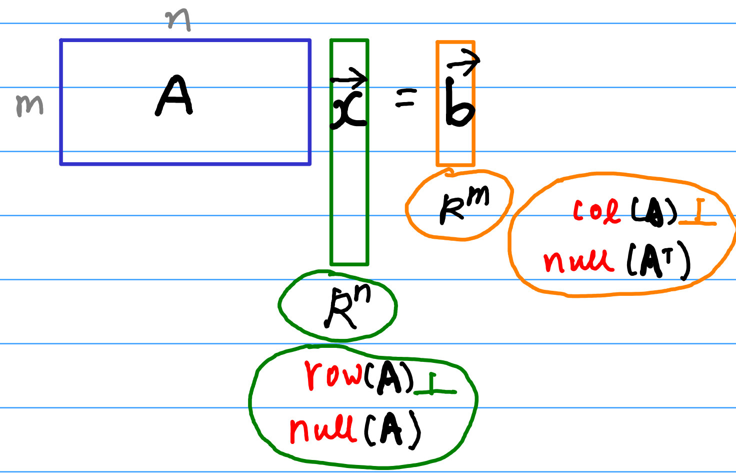
Orthogonal Compliments

$$\boxed{\text{row}(A) \perp \text{null}(A)}$$

Orthogonal Compliments

$$\boxed{\text{col}(A) \perp \text{null}(A^T)}$$

$$\boxed{A\vec{x} = \vec{b}}$$



* any vector $\vec{x}$ in R^n can be decomposed

$$\vec{x} = \vec{x}_{row(A)} + \vec{x}_{null(A)}$$

* any vector $\vec{b}$ in R^m can be decomposed

$$\vec{b} = \vec{b}_{col(A)} + \vec{b}_{null(A^T)}$$

Because

Orthogonal Compliments

$$\boxed{row(A) \perp null(A)}$$

Orthogonal Compliments

$$\boxed{col(A) \perp null(A^T)}$$

$$\boxed{row(A) \perp null(A)} \qquad \boxed{col(A) \perp null(A^T)}$$

$$\boxed{A\vec{x} = \vec{b}}$$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} m$$

$$\vec{x} \in R^n \qquad \vec{b} \in R^m$$

$\vec{x} \in R^n$ any vector

$$\vec{x} = \boxed{Proj_W \vec{x}} + \boxed{Proj_{W^\perp} \vec{x}}$$

$$= \vec{x}_{row(A)} + \vec{x}_{null(A)}$$

$\vec{b} \in R^m$ any vector

$$\vec{b} = \boxed{Proj_W \vec{b}} + \boxed{Proj_{W^\perp} \vec{b}}$$

$$= \vec{b}_{col(A)} + \vec{b}_{null(A^T)}$$

Conceptual Drawing of 4 Fundamental Spaces

Orthogonal Compliments

$$\boxed{\text{row}(A) \perp \text{null}(A)}$$

Orthogonal Compliments

$$\boxed{\text{col}(A) \perp \text{null}(A^T)}$$

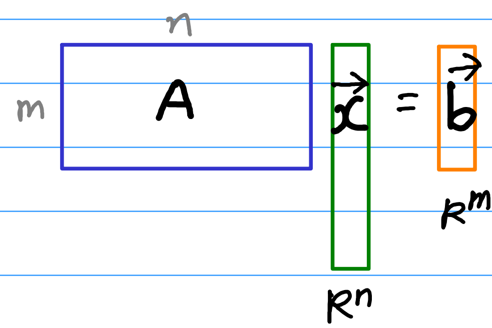
$$\vec{x} \in \mathbb{R}^n$$

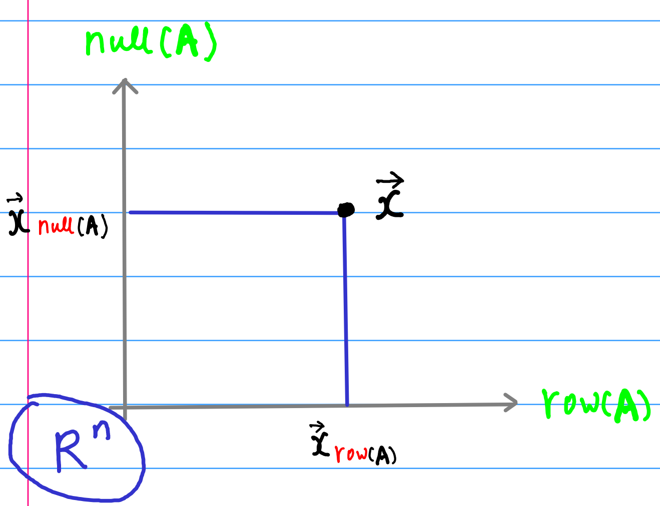
$$\vec{x} = \vec{x}_{\text{row}(A)} + \vec{x}_{\text{null}(A)}$$

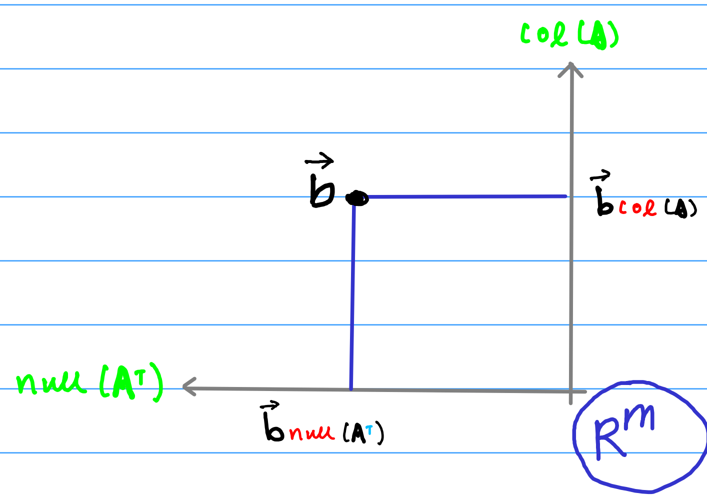
$$\vec{b} \in \mathbb{R}^m$$

$$\vec{b} = \vec{b}_{\text{col}(A)} + \vec{b}_{\text{null}(A^T)}$$

$$\boxed{A\vec{x} = \vec{b}}$$





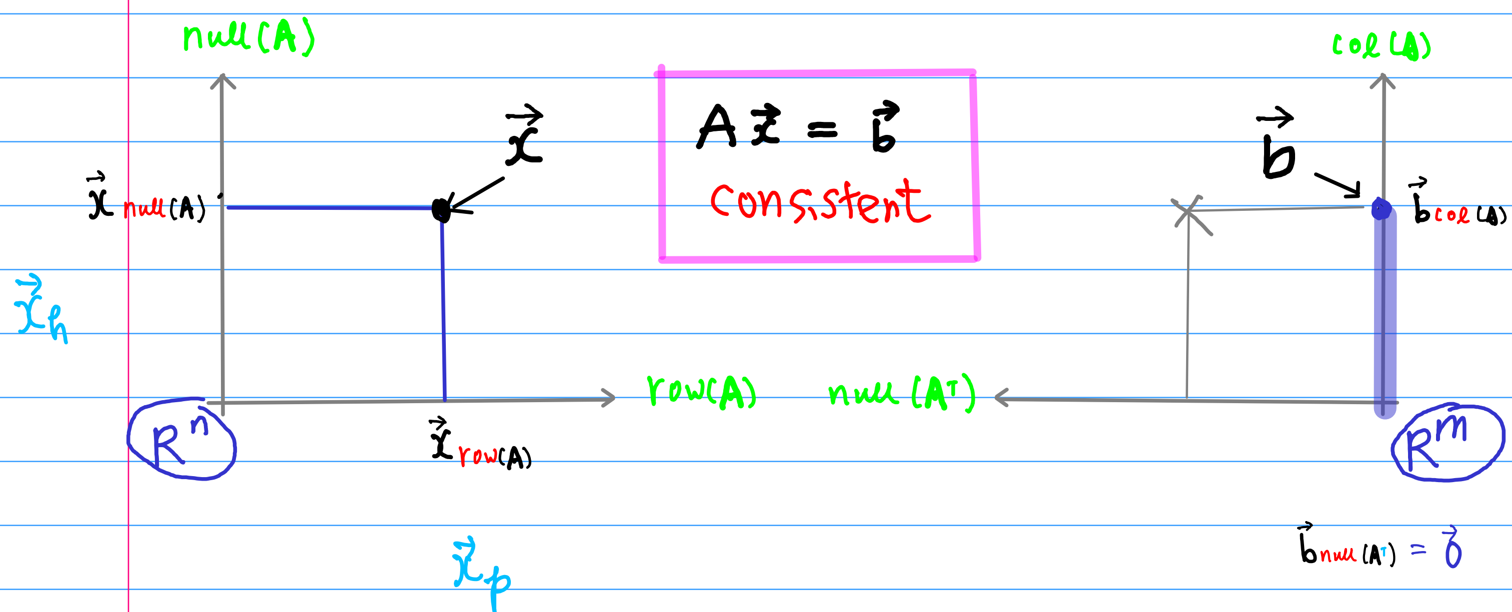


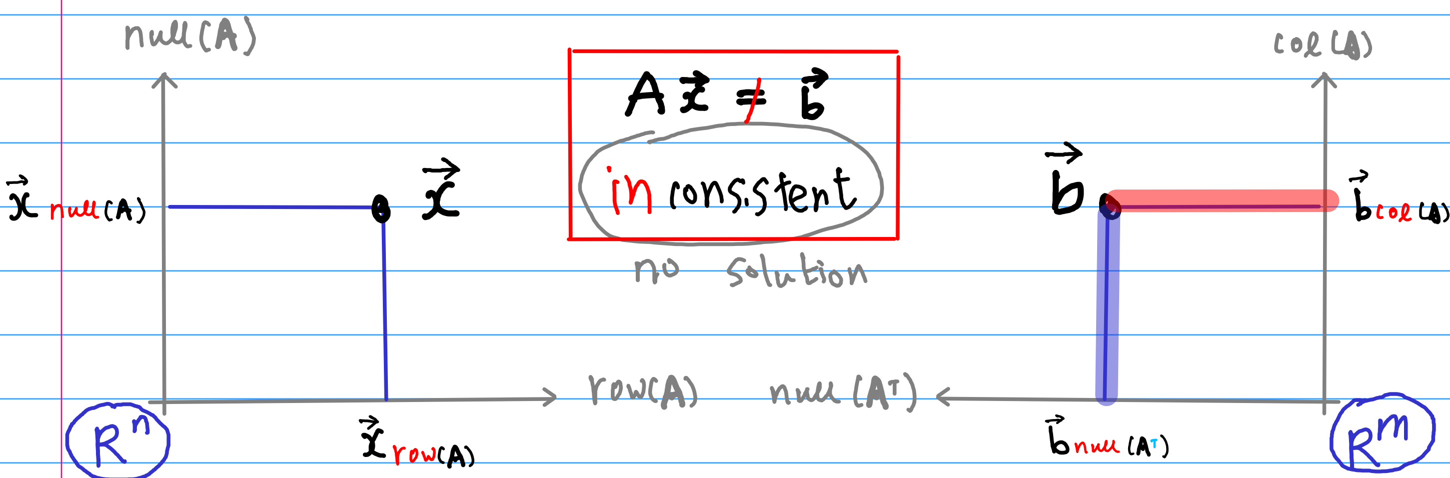
$\boxed{A\vec{x} = \vec{b}}$

Condition for a consistent system

$\vec{b}$ in column space

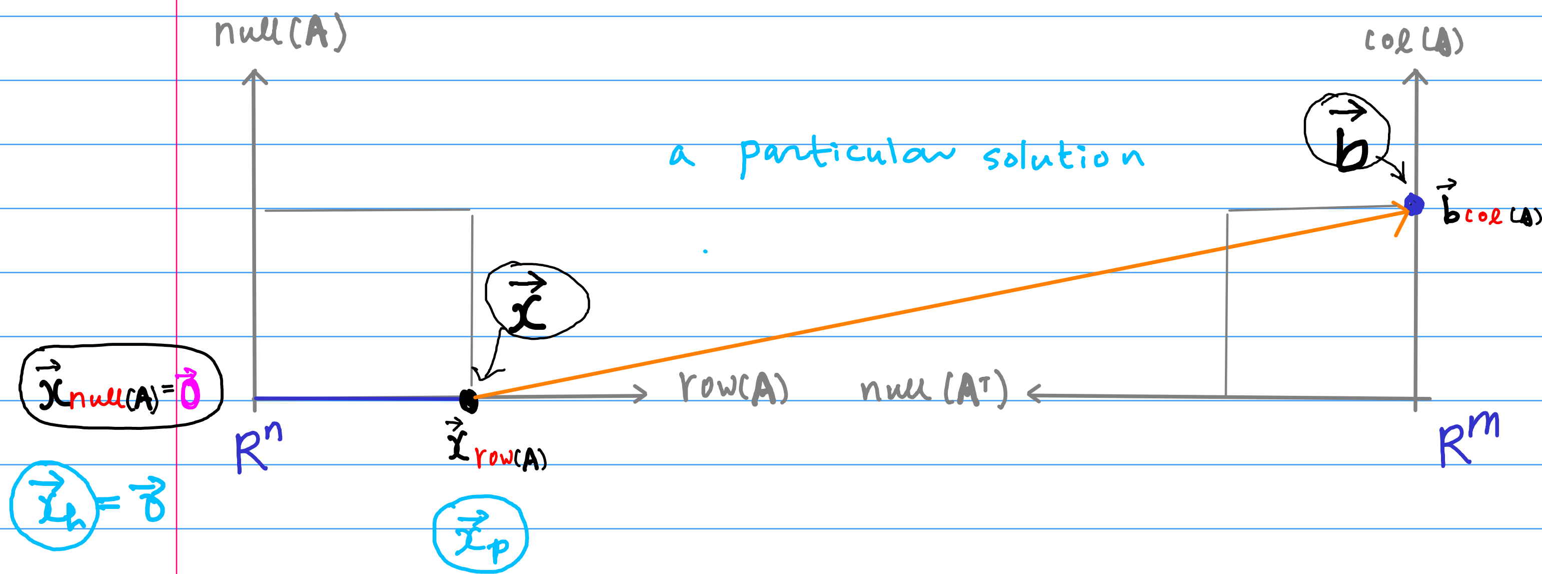
$\Rightarrow \boxed{\vec{b}_{null(A^T)} = \vec{0}}$





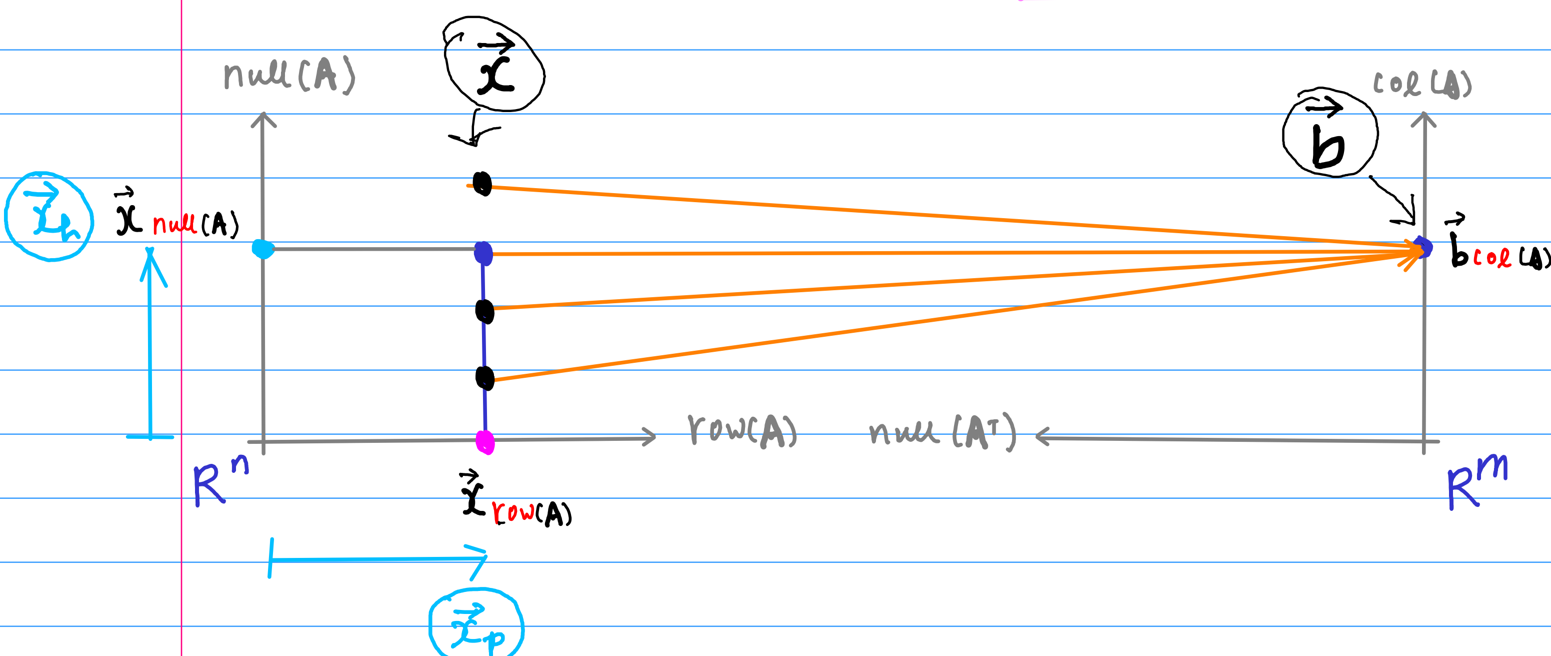
a unique solution

consistent $A\vec{x} = \vec{b}$



many solutions

consistent $A\vec{x} = \vec{b}$



$\text{rank}(A) =$ full column rank

the only <u>one</u> solution of $A\vec{x} = \vec{b}$

a unique solution is in $\text{row}(A)$

$\text{rank}(A) <$ full column rank

infinitely <u>many</u> solutions of $A\vec{x} = \vec{b}$

a unique solution is in $\text{row}(A)$

$$\vec{x} \in R^n$$

$$\vec{x} = \vec{x}_{row(A)} + \vec{x}_{null(A)}$$

$$\vec{b} \in R^m$$

$$\vec{b} = \vec{b}_{col(A)} + \vec{b}_{null(A^T)}$$

$$\boxed{A\vec{x} = \vec{b}}$$

$$\underset{m \times n}{A}\ \underset{R^n}{\vec{x}} = \underset{R^m}{\vec{b}}$$

any $\vec{x} \in R^n$

so the solution of $A\vec{x} = \vec{b}$ can be decomposed

$\vec{x} = \vec{x}_{row(A)} + \vec{x}_{null(A)}$: the assumed solution of $A\vec{x} = \vec{b}$ system

$$\begin{aligned} A\vec{x} &= A\vec{x}_{row(A)} + A\vec{x}_{null(A)} \\ &= A\vec{x}_{row(A)} + \vec{0} \\ &= A\vec{x}_{row(A)} = \vec{b} \end{aligned}$$

$$\boxed{A\vec{x} = \vec{b}} \Rightarrow \boxed{A\vec{x}_{row(A)} = \vec{b}}$$

$\vec{x}_{row(A)}$ is also a solution

$\boxed{A\vec{x} = \vec{b}} \Rightarrow \boxed{A\vec{x}_{row(A)} = \vec{b}}$

$\vec{x}_{row(A)}$ is a solution

the solution is in row(A)

rank(A) = full column rank

the only one solution of $A\vec{x} = \vec{b}$

$\vec{x}_{row(A)}$: the unique solution in row(A)

rank(A) < full column rank

infinitely many solutions of $A\vec{x} = \vec{b}$

$\vec{x}_{row(A)}$: a possible solution in row(A)

the unique solution

$\text{rank}(A) =$ full column rank

the only one solution of $A\vec{x} = \vec{b}$

$\vec{x}_{row(A)}$: the unique solution in row(A)

$\text{rank}(A) <$ full column rank

infinitely many solutions of $A\vec{x} = \vec{b}$

$\vec{x}_{row(A)}$: a possible solution in row(A)

$\Downarrow$

unique

To show the uniqueness, assume

$\vec{x}_r$, $\vec{x}_s$: two solutions in row(A)

$$\begin{array}{l} A\vec{x}_r = \vec{b} \\ A\vec{x}_s = \vec{b} \\ \hline A(\vec{x}_r - \vec{x}_s) = \vec{0} \end{array} \quad \Rightarrow \quad (\vec{x}_r - \vec{x}_s) \in \text{null}(A)$$

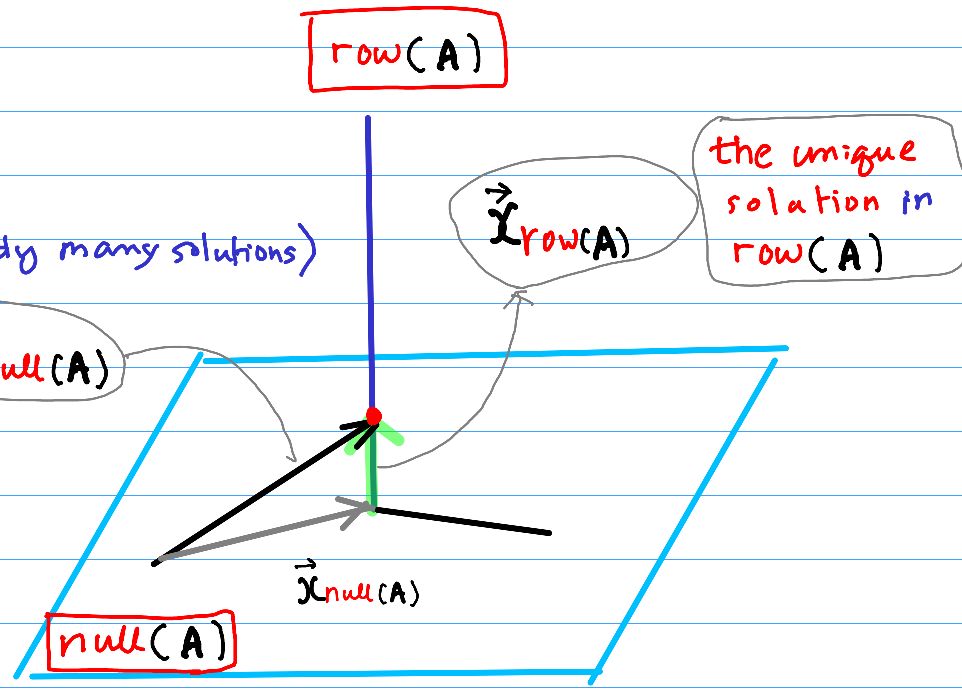
but from the assumption $\Rightarrow$ $(\vec{x}_r - \vec{x}_s) \in \text{row}(A)$

$\text{null}(A) \cap \text{row}(A) = \{\vec{0}\}$ $\quad (\vec{x}_r - \vec{x}_s) = \vec{0}$

$\hookrightarrow$ $\boxed{\vec{x}_r = \vec{x}_s}$

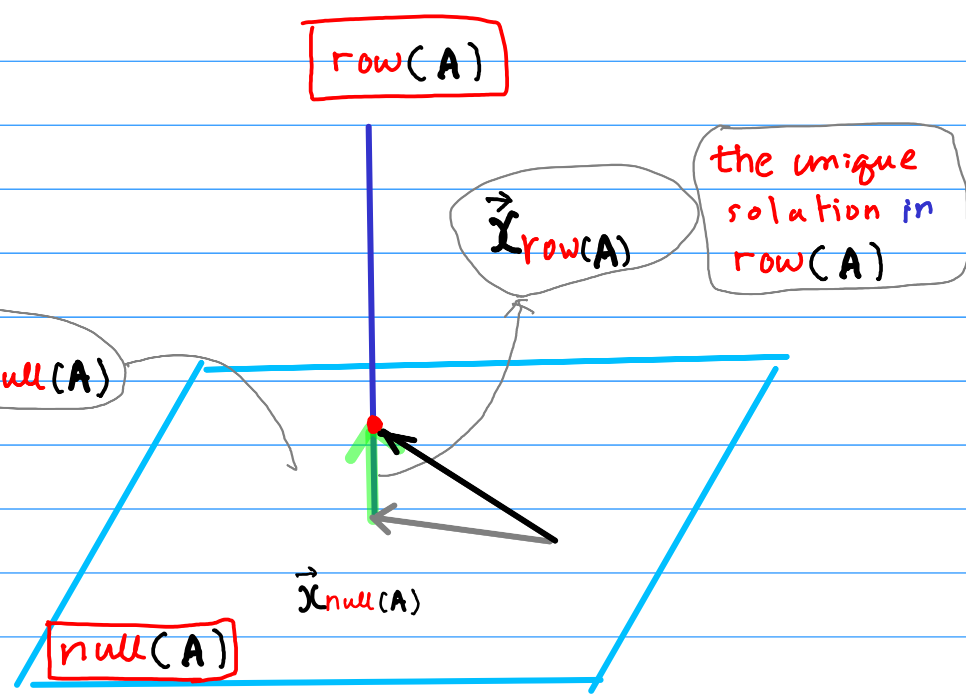
$\vec{x}_{row(A)}$: the unique solution in row(A) a solution. (possibly many solutions)

$$\vec{x} = \vec{x}_{row(A)} + \vec{x}_{null(A)}$$



another solution.

$$\vec{x} = \vec{x}_{row(A)} + \vec{x}_{null(A)}$$



$$\boxed{\|\vec{x}\| \geq \|\vec{x}_{row(A)}\|}$$

Solution Space of **Ax=b** (1)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$

$0 \cancel{=} 1$

$1 \cdot x_1 \quad + 3 \cdot x_3 = -1$

$1 \cdot x_2 - 4 \cdot x_3 = 2$

$1 \cdot x_1 - 5 \cdot x_2 + 1 \cdot x_3 = 4$

Solve for a leading variable	$x_1 = -1 - 3 \cdot x_3$ $x_2 = 2 + 4 \cdot x_3$	$x_1 = 4 + 5 \cdot x_2 - 1 \cdot x_3$
Treat a free variable as a parameter	$x_3 = t$	$x_2 = s \quad x_3 = t$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

null (A)

$$\begin{bmatrix} 1 & -5 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 5s - t - 5s + t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

null(A)

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} t = \begin{bmatrix} -3 + 3 \\ 4 - 4 \\ 0 \end{bmatrix} t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution Space of **Ax=b** (2)

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$x_3 = t$ ⇐ free variable

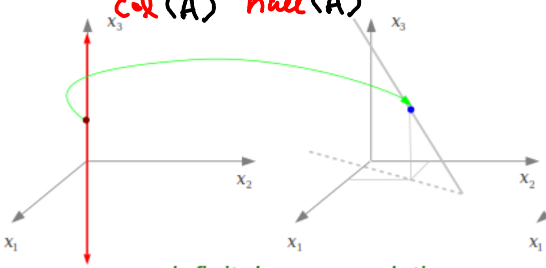
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\in \text{col}(A)} + \underbrace{t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}}_{= \text{null}(A)}$$

$$\begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

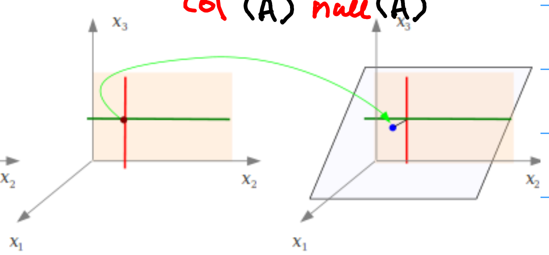
$x_2 = s$ ⇐ free variable

$x_3 = t$ ⇐ free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}}_{\in \text{col}(A)} + \underbrace{s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{= \text{null}(A)}$$



infinitely many solutions



infinitely many solutions

Solution Space of **Ax=b** (3)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} + \boxed{t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}} + \boxed{s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

General Solution of $Ax = b$ → Particular Solution of $Ax = b$ | General Solution of $Ax = 0$ | Particular Solution of $Ax = b$ | General Solution of $Ax = 0$

$t=0$ $\qquad\qquad\qquad\qquad$ $s=0$, $t=0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} + \boxed{t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}} + \boxed{s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

Particular Solution of $Ax = b$ ⋒ Col(A) | General Solution of $Ax = 0$ ⋒ null(A) Line | Particular Solution of $Ax = b$ ⋒ Col(A) | General Solution of $Ax = 0$ ⋒ null(A) Plane

A

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

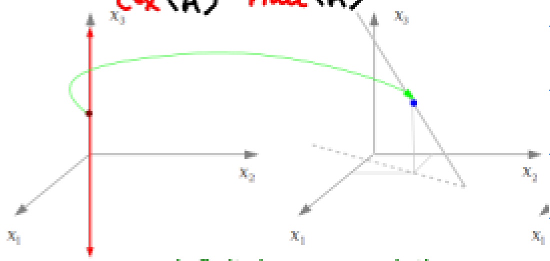
$$1x_1 \quad + 3 \cdot x_3 = -1$$
$$1x_2 - 4 \cdot x_3 = 2$$

$$x_1 = -1 - 3 \cdot x_3$$
$$x_2 = 2 + 4 \cdot x_3$$
$$x_3 = t$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$ ⬅ free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\in \ \text{col}(A)} + t \underbrace{\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}}_{\in \ \text{null}(A)}$$



infinitely many solutions

$$\text{Span}\left\{ \begin{matrix} (1 \quad 0 \quad 3), \\ (0 \quad 1 \quad -4) \end{matrix} \right\} = \text{row}(A)$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$ already in RREF

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \text{col}(A)$$

$$x_1 + \quad 3x_3 = 0$$
$$x_2 - 4x_3 = 0$$

$x_3 \Rightarrow t$

$x_1 = -3t, \quad x_2 = 4t, \quad x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3t \\ 4t \\ t \end{bmatrix}$$

$$\text{Span}\left\{ \begin{bmatrix} -3t \\ 4t \\ t \end{bmatrix} \right\} = \text{null}(A)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -4 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0$

$$\text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \text{null}(A^T)$$

$$\text{span}\left\{\begin{matrix}(1 & 0 & 3), \\ (0 & 1 & -4)\end{matrix}\right\} = \text{row}(A) \qquad \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} = \text{col}(A)$$

$$\text{span}\left\{\begin{bmatrix}-3t\\4t\\t\end{bmatrix}\right\} = \text{null}(A) \qquad \text{span}\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\} = \text{null}(A^T)$$

$$\boxed{\text{row}(A) \perp \text{null}(A)} \qquad \boxed{\text{col}(A) \perp \text{null}(A^T)}$$

$$\boxed{\begin{aligned}&\vec{x} \in R^n \\ &\vec{x} = \vec{x}_{\text{row}(A)} + \vec{x}_{\text{null}(A)}\end{aligned}} \qquad \boxed{\begin{aligned}&\vec{b} \in R^m \\ &\vec{b} = \vec{b}_{\text{col}(A)} + \vec{b}_{\text{null}(A^T)}\end{aligned}}$$

$$\boxed{A\vec{x} = \vec{b}} \qquad \underset{m}{} \overset{n}{\boxed{\;A\;}} \; \underset{R^n}{\vec{x}} = \underset{R^m}{\vec{b}}$$

$$(1 \;\; 0 \;\; 3)\begin{bmatrix}-3t\\4t\\t\end{bmatrix} = 0 \qquad (1 \;\; 0 \;\; 0)\begin{bmatrix}0\\0\\0\end{bmatrix} = 0$$

$$(0 \;\; 1 \;\; -4)\begin{bmatrix}-3t\\4t\\t\end{bmatrix} = 0 \qquad (0 \;\; 1 \;\; 0)\begin{bmatrix}0\\0\\0\end{bmatrix} = 0$$

$$\text{span}\left\{\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\1\\-4\end{bmatrix}\right\} + \text{span}\left\{\begin{bmatrix}-3t\\4t\\t\end{bmatrix}\right\}$$

$\boxed{\text{row}(A) \perp \text{null}(A)}$ $\boxed{\text{col}(A) \perp \text{null}(A^T)}$

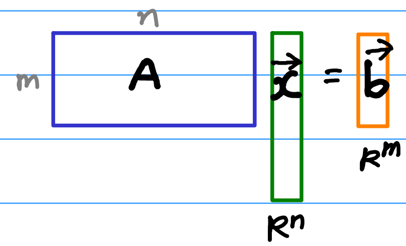
$\vec{x} \in R^n$

$\vec{x} = \vec{x}_{\text{row}(A)} + \vec{x}_{\text{null}(A)}$

$\vec{b} \in R^m$

$\vec{b} = \vec{b}_{\text{col}(A)} + \vec{b}_{\text{null}(A^T)}$

$\boxed{A\vec{x} = \vec{b}}$



$$\text{span}\left\{ \begin{pmatrix} 1 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -4 \end{pmatrix} \right\} = \text{row}(A)$$

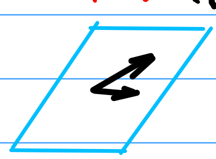
$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \text{col}(A)$$

$$\text{span}\left\{ \begin{bmatrix} -3t \\ 4t \\ t \end{bmatrix} \right\} = \text{null}(A)$$

$$\text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \text{null}(A^T)$$

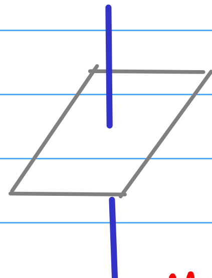
$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \right\}$$

row(A)



$$+$$

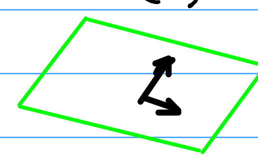
$$\text{span}\left\{ \begin{bmatrix} -3t \\ 4t \\ t \end{bmatrix} \right\}$$



null(A)

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

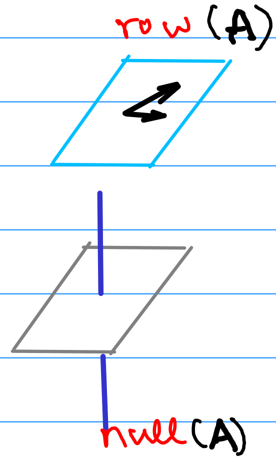
col(A)



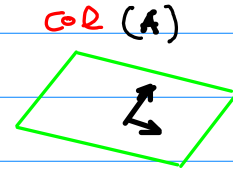
$$+$$

$$\text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

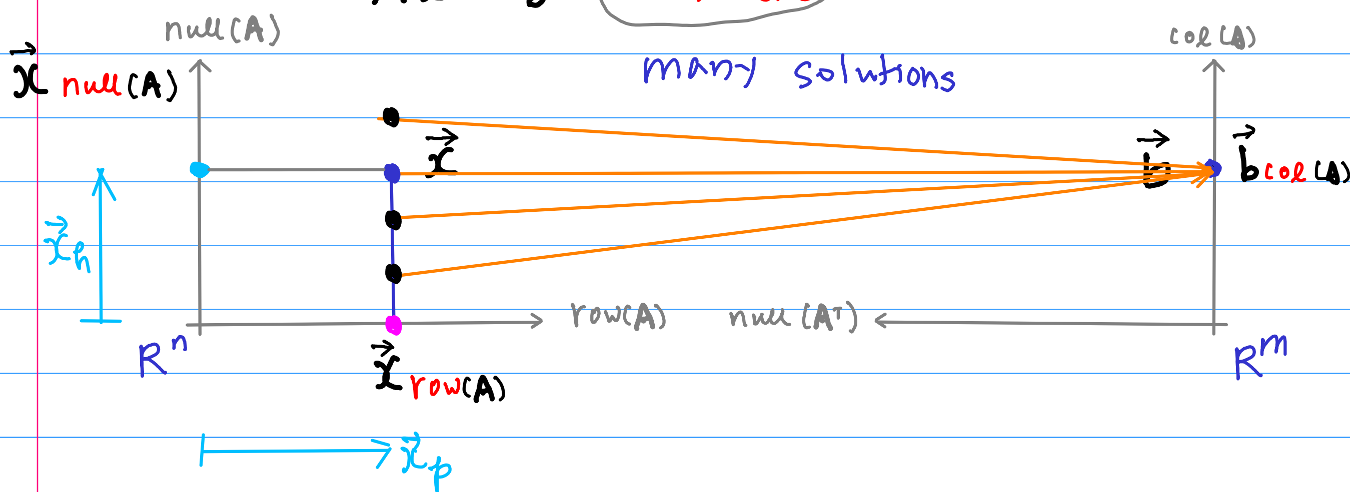
$$\text{span}\left\{\begin{bmatrix}1\\0\\3\end{bmatrix},\begin{bmatrix}0\\1\\-4\end{bmatrix}\right\} + \text{span}\left\{\begin{bmatrix}-3t\\4t\\t\end{bmatrix}\right\}$$

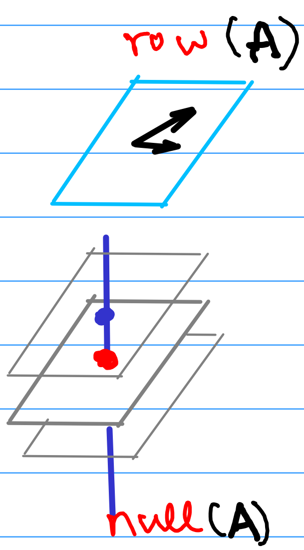


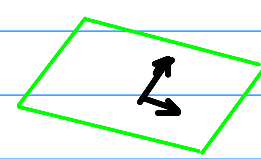
$$\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\} + \text{span}\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$



$A\vec{x} = \vec{b}$ Consistent







Linear System & Inner Product (3)

Non-Homogeneous Linear System $\quad \boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{b} \qquad \boldsymbol{A} : m\times n$

Homogeneous Linear System $\quad \boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{0}$

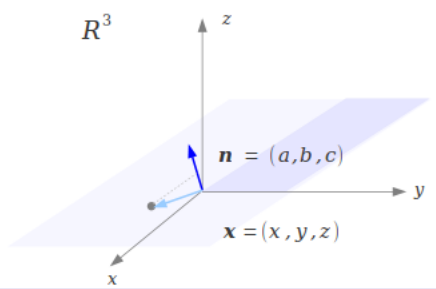
a particular solution

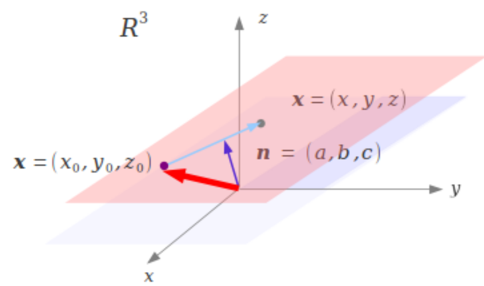
$\boldsymbol{A}\cdot\boldsymbol{x} = \boldsymbol{b}$

solution set consists of all vectors in R^n that are **orthogonal** to every row vector of $\boldsymbol{A}$

+

a particular solution $\boldsymbol{x_0}$ $\qquad \boldsymbol{A}\cdot\boldsymbol{x_0} = \boldsymbol{b}$





Linear System & Inner Product (4)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$3 \begin{cases} 2 \begin{cases} \boldsymbol{r_1}\cdot\boldsymbol{x} = 0 \\ \boldsymbol{r_2}\cdot\boldsymbol{x} = 0 \end{cases} \\ 1 \; \{ \text{a line through the origin} \quad R^1 \end{cases} \qquad 3 \begin{cases} 1 \; \{ \boldsymbol{r_1}\cdot\boldsymbol{x} = 0 \\ 2 \; \{ \text{a plane through the origin} \quad R^2 \end{cases}$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

Solution Space of **Ax=0**

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

the same case

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases} \qquad \begin{cases} x_1 = 4 + 5s - 1t \\ x_2 = s \\ x_3 = t \end{cases}$$

General Solution of $Ax = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

dim(row space of A)
dim(col space of A)

rank(A) = 2 — **rank**(A) = 1

dim(null space of A)

nullity(A) = 1 — **nullity**(A) = 2

$$A = \begin{array}{cc} & n \\ m & \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \end{array}$$

$$A^T = \begin{array}{cc} & m \\ n & \begin{bmatrix} c_1' \\ c_2' \\ \vdots \\ c_n' \end{bmatrix} \end{array}$$

$$A^T A = \begin{array}{cc} & m \\ n & \begin{bmatrix} c_1' \\ c_2' \\ \vdots \\ c_n' \end{bmatrix} \end{array} \begin{array}{cc} & n \\ m & \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \end{array}$$

$$= \begin{array}{cc} & n \\ n & \begin{bmatrix} c_1 \cdot c_1 & c_1 \cdot c_2 & \cdots & c_1 \cdot c_n \\ c_2 \cdot c_1 & c_2 \cdot c_2 & \cdots & c_2 \cdot c_n \\ \vdots & \vdots & & \vdots \\ c_n \cdot c_1 & c_n \cdot c_2 & \cdots & c_n \cdot c_n \end{bmatrix} \end{array}$$

$$A = m\overset{n}{\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}} = m\overset{n}{\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}}$$

$$A^T = n\overset{m}{\begin{bmatrix} c_1' \\ c_2' \\ \vdots \\ c_n' \end{bmatrix}} = n\overset{m}{\begin{bmatrix} r_1' & r_2' & \cdots & r_m' \end{bmatrix}}$$

$$AA^T = m\overset{n}{\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}} \; n\overset{m}{\begin{bmatrix} r_1' & r_2' & \cdots & r_m' \end{bmatrix}}$$

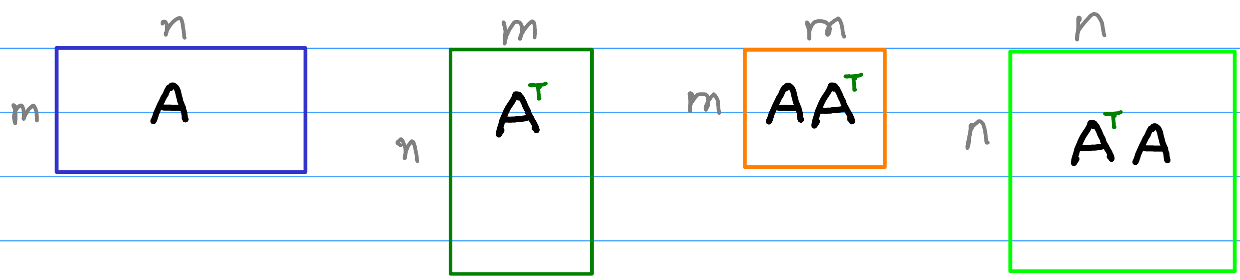
$$= m\overset{m}{\begin{bmatrix} r_1 \cdot r_1 & r_1 \cdot r_2 & \cdots & r_1 \cdot r_m \\ r_2 \cdot r_1 & r_2 \cdot r_2 & \cdots & r_2 \cdot r_m \\ \vdots & \vdots & & \vdots \\ r_m \cdot r_1 & r_m \cdot r_2 & \cdots & r_m \cdot r_m \end{bmatrix}}$$

$$\text{null}(A) = \text{null}(A^TA)$$

$$\text{row}(A) = \text{row}(A^TA)$$

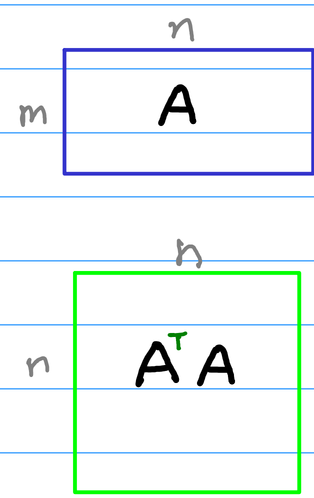
$$\text{col}(A^T) = \text{col}(AA^T)$$

$$\text{rank}(A) = \text{rank}(A^TA)$$



$$Ax = 0$$

$$A^TAx = A^T0 = 0$$



$$\boxed{A}\,x = 0$$

$$\boxed{A^TA}\,x = 0$$

$$\text{rank} \leq m = \min(m, n)$$

$$\text{null}(A) = \text{null}(A^TA)$$

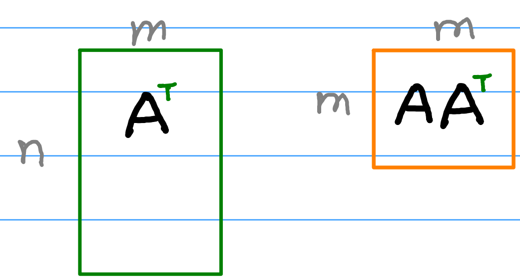
$$\text{row}(A) = \text{row}(A^TA)$$

$$\text{col}(A^T) = \text{col}(AA^T)$$

$$\text{rank}(A) = \text{rank}(A^TA)$$

$\boxed{A}\, x = 0$ $\quad\quad$ $\boxed{A^T}\, y = 0$

$\boxed{A^TA}\, x = 0$ $\quad\quad$ $\boxed{AA^T}\, y = 0$



$\text{rank} \le m = \min(m, n)$

$$\mathrm{null}(A) = \mathrm{null}(A^T A)$$
$$\mathrm{row}(A) = \mathrm{row}(A^T A)$$
$$\mathrm{col}(A^T) = \mathrm{col}(A A^T)$$
$$\mathrm{rank}(A) = \mathrm{rank}(A^T A)$$

$B = A^T$
$A = B^T$

$$\mathrm{null}(B^T) = \mathrm{null}(B B^T)$$
$$\mathrm{row}(B^T) = \mathrm{row}(B B^T)$$
$$\mathrm{col}(B) = \mathrm{col}(B^T B)$$
$$\mathrm{rank}(B^T) = \mathrm{rank}(B B^T)$$

actually, B can any matrix