

All Pass Filter (2A)

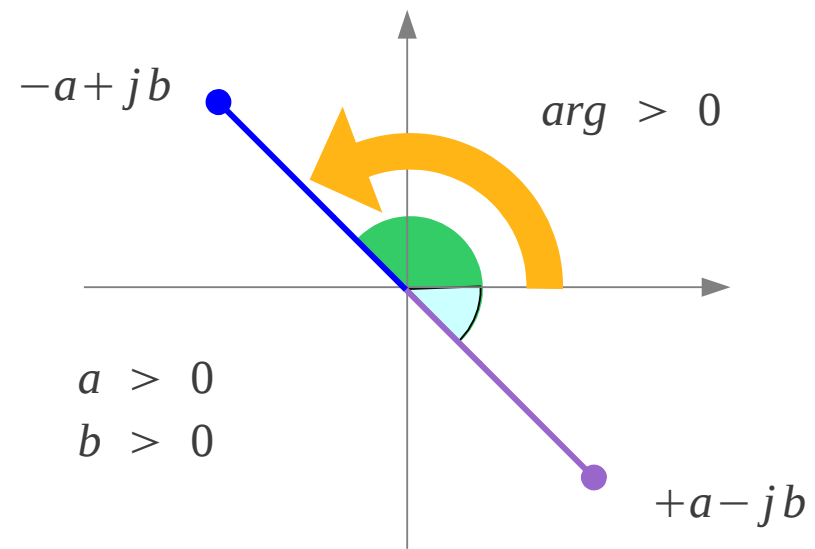
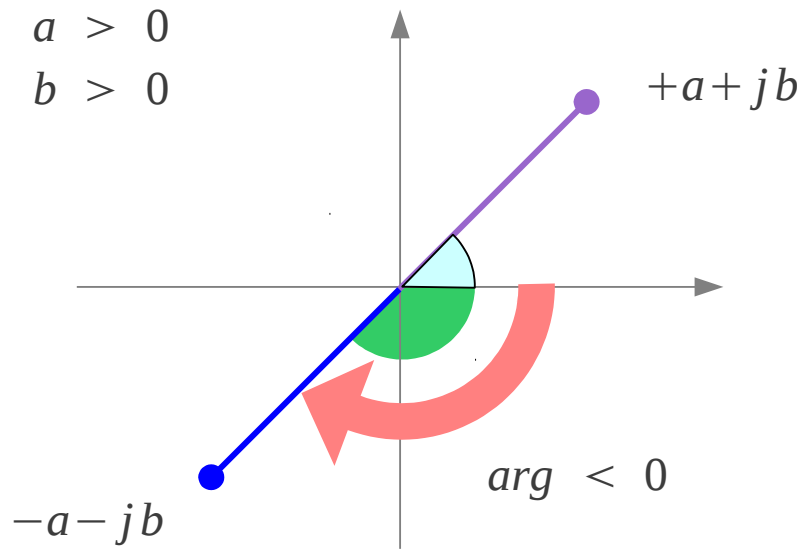
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Phase and $\tan^{-1}(b/a)$ (1)



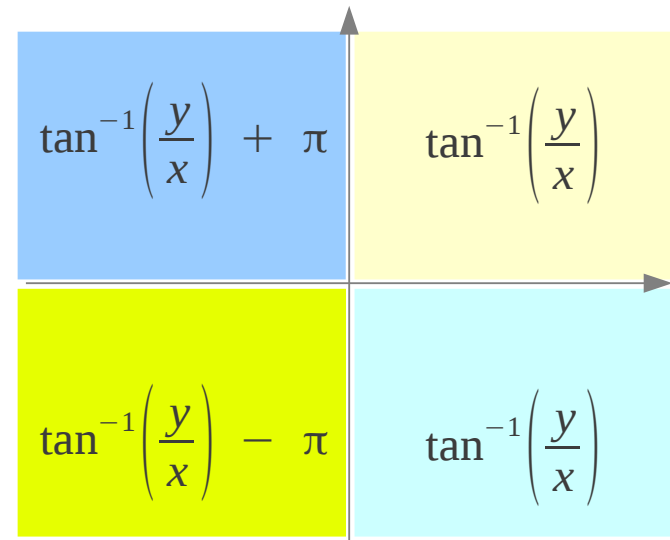
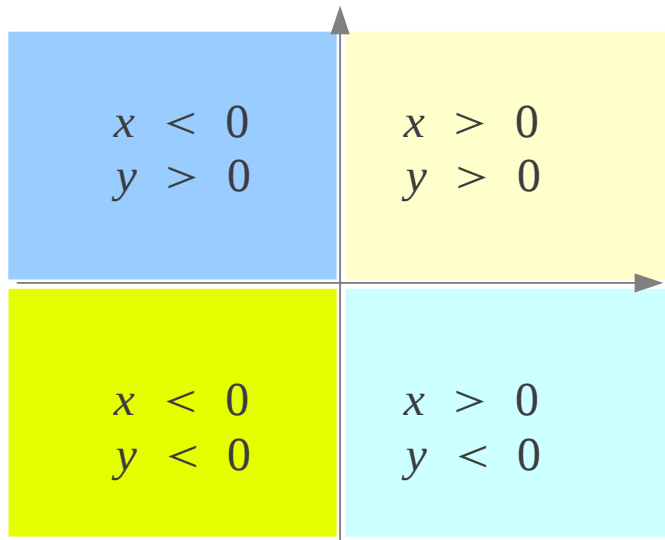
$$\tan^{-1}\left(\frac{-b}{-a}\right) = \tan^{-1}\left(\frac{+b}{+a}\right)$$

$$\tan^{-1}\left(\frac{+b}{-a}\right) = \tan^{-1}\left(\frac{-b}{+a}\right)$$

$$arg\{-a-jb\} = arg\{+a+jb\} - \pi$$

$$arg\{-a+jb\} = arg\{+a-jb\} + \pi$$

Phase and $\tan^{-1}(b/a)$ (2)



Octave function

`atan2(y, x)`

`arg{x + jy}`

| | | | |
|---|---|------------------|---------------------------------|
| } | $\tan^{-1}\left(\frac{y}{x}\right)$ | $(x > 0)$ | <code>atan(y / x)</code> |
| | $\tan^{-1}\left(\frac{y}{x}\right) + \pi$ | $(x < 0, y > 0)$ | <code>atan(y / x) + 3.14</code> |
| | $\tan^{-1}\left(\frac{y}{x}\right) - \pi$ | $(x < 0, y < 0)$ | <code>atan(y / x) - 3.14</code> |

All Pass Filter Example (1)

$$H_{all}(s) = \frac{1 - 2s}{1 + 2s}$$

Flat Magnitude

$$\begin{aligned} \left| \frac{1 - j2\omega}{1 + j2\omega} \right| &= \frac{|1 - j2\omega|}{|1 + j2\omega|} \\ &= \frac{\sqrt{1 + 4\omega^2}}{\sqrt{1 + 4\omega^2}} = 1 \end{aligned}$$

$$|H_{all}(j\omega)| = \frac{\sqrt{1 + 4\omega^2}}{\sqrt{1 + 4\omega^2}} = 1$$

$$\begin{aligned} \frac{1 - j2\omega}{1 + j2\omega} &= \frac{1 - j2\omega}{1 + j2\omega} \cdot \frac{1 - j2\omega}{1 - j2\omega} \\ &= \frac{(1 - 4\omega^2) - j4\omega}{1 + 4\omega^2} \end{aligned}$$

$$\arg\{H_{all}(j\omega)\} = \tan^{-1}\left(\frac{-4\omega}{1 - 4\omega^2}\right)$$

A Pure Phase Shifter

$$\begin{aligned} \arg\left\{\frac{1 - j2\omega}{1 + j2\omega}\right\} &= \arg\{1 - j2\omega\} - \arg\{1 + j2\omega\} \\ &= \tan^{-1}\left(\frac{-2\omega}{1}\right) - \tan^{-1}\left(\frac{2\omega}{1}\right) = -2 \tan^{-1}(2\omega) \end{aligned}$$

$$\arg\{H_{all}(j\omega)\} = -2 \tan^{-1}(2\omega)$$

All Pass Filter Example (2)

$$H_{all}(s) = -\frac{s - 0.5}{s + 0.5} \quad \leftarrow \quad H_{all}(s) = \frac{1 - 2s}{1 + 2s}$$

Flat Magnitude

$$\begin{aligned} \left| \frac{j\omega - 0.5}{j\omega + 0.5} \right| &= \frac{|j\omega - 0.5|}{|j\omega + 0.5|} \\ &= \frac{\sqrt{\omega^2 + 0.25}}{\sqrt{\omega^2 + 0.25}} = 1 \end{aligned}$$

$$|H_{all}(j\omega)| = \frac{\sqrt{1 + 4\omega^2}}{\sqrt{1 + 4\omega^2}} = 1$$

$$\begin{aligned} \frac{-j\omega + 0.5}{+j\omega + 0.5} &= \frac{-j\omega + 0.5}{+j\omega + 0.5} \cdot \frac{-j\omega + 0.5}{-j\omega + 0.5} \\ &= \frac{(-\omega^2 + 0.25) - j\omega}{\omega^2 + 0.25} \end{aligned}$$

$$\arg\{H_{all}(j\omega)\} = \tan^{-1}\left(\frac{-\omega}{0.25 - 4\omega^2}\right)$$

A Pure Phase Shifter

$$\begin{aligned} \arg\left\{\frac{-j\omega + 0.5}{+j\omega + 0.5}\right\} &= \arg\{-j\omega + 0.5\} - \arg\{+j\omega + 0.5\} \\ &= \tan^{-1}\left(\frac{-\omega}{+0.5}\right) - \tan^{-1}\left(\frac{+\omega}{+0.5}\right) = -2 \tan^{-1}(2\omega) \end{aligned}$$

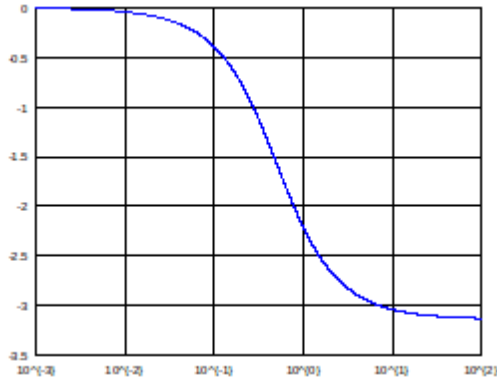
$$\arg\{H_{all}(j\omega)\} = -2 \tan^{-1}(2\omega)$$

$$\begin{aligned} -\arg\left\{\frac{j\omega - 0.5}{j\omega + 0.5}\right\} &= -\arg\{j\omega - 0.5\} + \arg\{j\omega + 0.5\} \\ &= -\tan^{-1}\left(\frac{\omega}{-0.5}\right) + \tan^{-1}\left(\frac{\omega}{0.5}\right) = -(-\tan^{-1}(2\omega) + \pi) + \tan^{-1}(2\omega) \end{aligned}$$

$$\arg\{H_{all}(j\omega)\} = -\pi + 2 \tan^{-1}(2\omega)$$

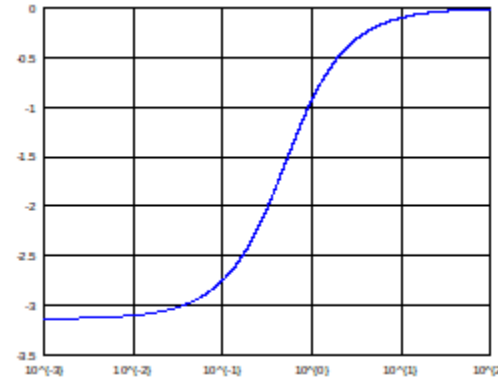
All Pass Filter Example (3)

$$\arg\{H_{all}(j\omega)\} = -2 \tan^{-1}(2\omega)$$



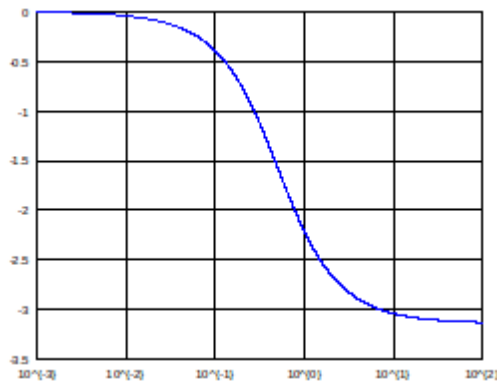
semilogx(...)

$$\arg\{H_{all}(j\omega)\} = -\pi + 2 \tan^{-1}(2\omega)$$



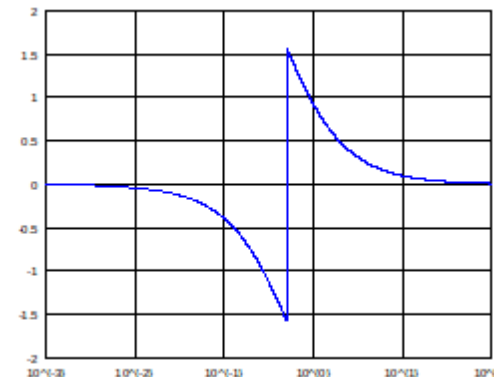
semilogx(...)

$$\arg\{H_{all}(j\omega)\} = -\text{atan2}(4\omega, 1 - 4\omega^2)$$



semilogx(...)

$$\arg\{H_{all}(j\omega)\} = -\text{atan}\left(\frac{4\omega}{1 - 4\omega^2}\right)$$



semilogx(...)

All Pass Filter Example (2)

$$H_{all}(s) = \frac{s - 0.5}{s + 0.5} \quad \Leftrightarrow \quad H_{all}(s) = \frac{1 - 2s}{1 + 2s}$$

Flat Magnitude

$$\begin{aligned} \left| \frac{j\omega - 0.5}{j\omega + 0.5} \right| &= \frac{|j\omega - 0.5|}{|j\omega + 0.5|} \\ &= \frac{\sqrt{\omega^2 + 0.25}}{\sqrt{\omega^2 + 0.25}} = 1 \end{aligned}$$

$$|H_{all}(j\omega)| = \frac{\sqrt{1 + 4\omega^2}}{\sqrt{1 + 4\omega^2}} = 1$$

$$\begin{aligned} \frac{+j\omega - 0.5}{+j\omega + 0.5} &= \frac{+j\omega - 0.5}{+j\omega + 0.5} \cdot \frac{-j\omega + 0.5}{-j\omega + 0.5} \\ &= \frac{(\omega^2 - 0.25) + j\omega}{\omega^2 + 0.25} \end{aligned}$$

$$\arg\{H_{all}(j\omega)\} = \tan^{-1}\left(\frac{\omega}{\omega^2 - 0.25}\right)$$

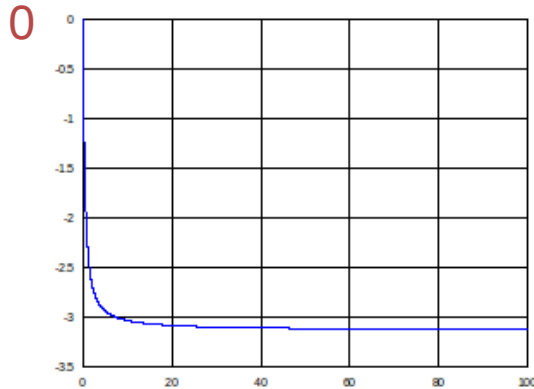
A Pure Phase Shifter

$$\begin{aligned} \arg\left\{\frac{+j\omega - 0.5}{+j\omega + 0.5}\right\} &= \arg\{+j\omega - 0.5\} - \arg\{+j\omega + 0.5\} \\ &= \tan^{-1}\left(\frac{+\omega}{-0.5}\right) - \tan^{-1}\left(\frac{+\omega}{+0.5}\right) = \pi - 2\tan^{-1}(2\omega) \end{aligned}$$

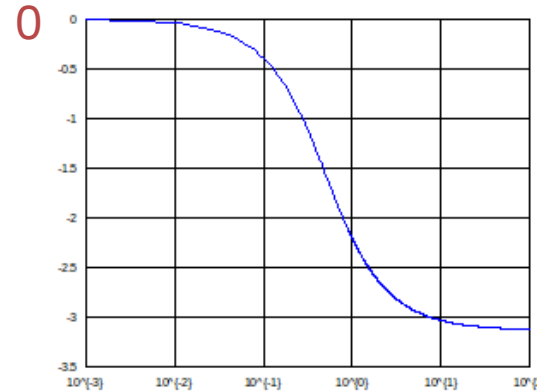
$$\arg\{H_{all}(j\omega)\} = \pi - 2\tan^{-1}(2\omega)$$

Example - All Pass Filter (1)

$$\arg\{H_{all}(j\omega)\} = -2 \tan^{-1}(2\omega)$$

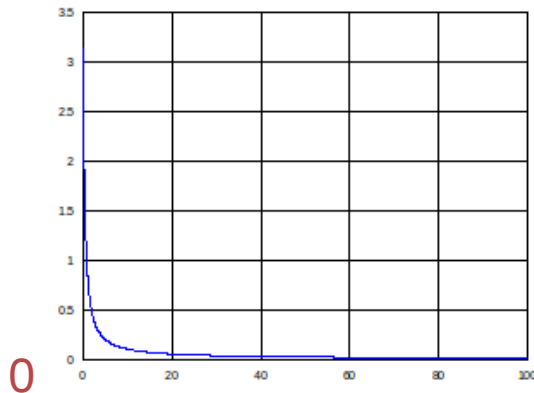


plot(...)

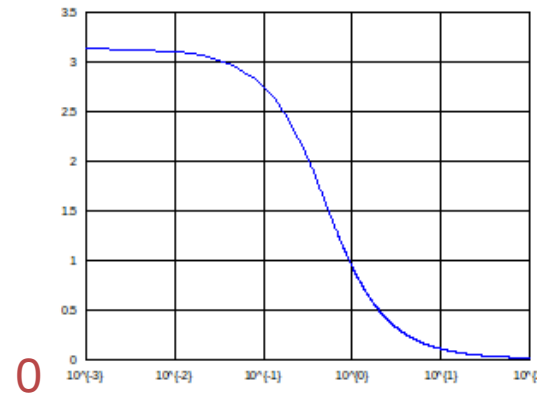


semilogx(...)

$$\arg\{H_{all}(j\omega)\} = \pi - 2 \tan^{-1}(2\omega)$$



plot(...)



semilogx(...)

Analog All Pass Filter (1)

$$H_{all}(s) = \frac{s - a}{s + a}$$

$$H_{all}(j\omega) = \frac{j\omega - a}{j\omega + a}$$

Impulse Response

$$H_{all}(s) = \frac{s - a}{s + a} = \frac{s + a - 2a}{s + a}$$

Flat Magnitude

$$|H_{all}(j\omega)| = \frac{\sqrt{\omega^2 + a^2}}{\sqrt{\omega^2 + a^2}} = 1$$

$$H(s) = 1 - \frac{2a}{(s + a)}$$

Phase Shifter

$$\arg\{H_{all}(j\omega)\} = \pi - 2 \tan^{-1}\left(\frac{\omega}{a}\right)$$



Inverse Laplace Transform

$$h(t) = \delta(t) - e^{-at}$$

Positive Group Delay

$$\begin{aligned} & -\frac{d}{d\omega}(\arg\{H_{all}(j\omega)\}) \\ &= -\frac{d}{d\omega} \left(\pi - 2 \tan^{-1}\left(\frac{\omega}{a}\right) \right) \\ &= \frac{4}{(1 + \omega^2/a^2)} > 0 \end{aligned}$$

Analog All Pass Filter (2)

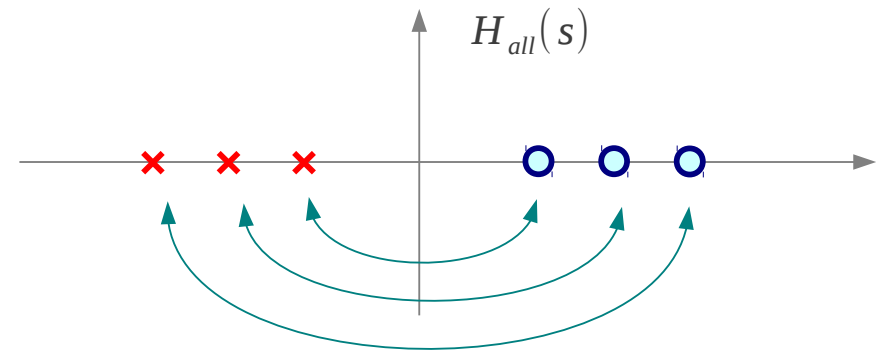
$$G_{all}(s) = \pm \frac{(s - \bar{s}_1)(s - \bar{s}_2) \cdots (s - \bar{s}_n)}{(s - s_1)(s - s_2) \cdots (s - s_n)}$$

Flat Magnitude

A Pure Phase Shifter

zero \bar{s}_i
pole s_i not complex conjugate

$s_i = +a + jb$
 $\bar{s}_i = -a + jb$ only differ in the signs of their real parts



Digital All Pass Filter (1)

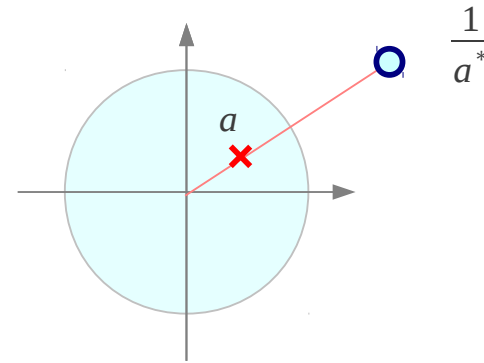
$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} \longrightarrow z^{-1} = a^* \longrightarrow z = \frac{1}{a^*}$$
$$\longrightarrow az^{-1} = 1 \longrightarrow z = a$$

Flat Magnitude
A Pure Phase Shifter

$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega} \frac{1 - a^* e^{+j\omega}}{1 - ae^{-j\omega}}$$

$$(1 - a^* e^{+j\omega})^* = (1 - ae^{-j\omega})$$

$$|H(e^{j\omega})| = |e^{-j\omega}| \left| \frac{1 - a^* e^{+j\omega}}{1 - ae^{-j\omega}} \right| = 1$$

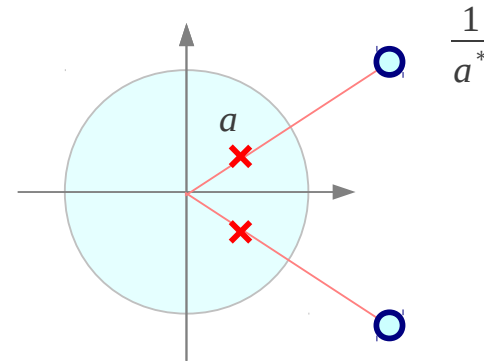


Digital All Pass Filter (2)

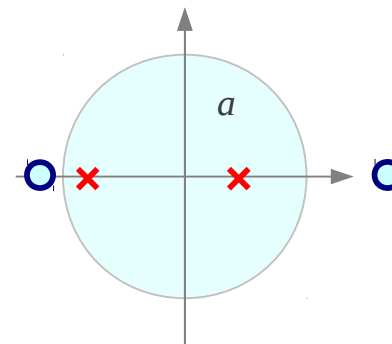
$$H_{all}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

Flat Magnitude
A Pure Phase Shifter

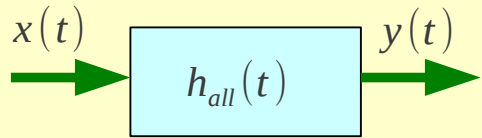
Cascade form of all pass system
for real-valued impulse response
system



Conjugate symmetric $H(e^{j\omega})$



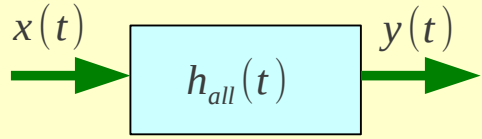
All Pass Filter – Parseval's Theorem



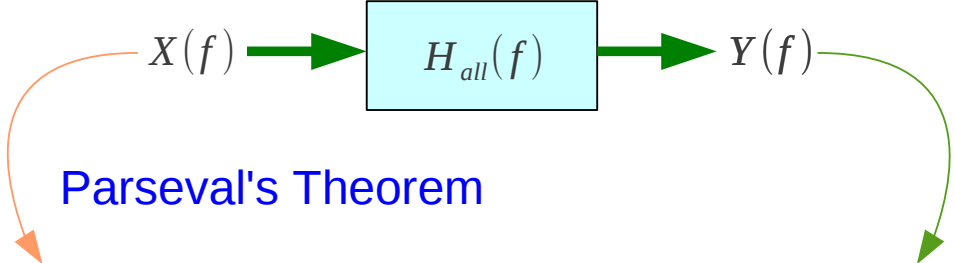
Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |y(t)|^2 dt$$

Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 dt \geq \int_{-\infty}^{t_0} |y(t)|^2 dt$$


The energy build-up in the input is more **rapid** than in the output



Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

$$\int_{-\infty}^{+\infty} |y(t)|^2 dt = \int_{-\infty}^{+\infty} |Y(f)|^2 df$$

$$Y(f) = H_{all}(f) X(f)$$

Allpass Filter

$$|H_{all}(f)| = 1$$

$$\int_{-\infty}^{+\infty} |Y(f)|^2 df = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

$$\int_{-\infty}^{+\infty} |y(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

All Pass Filter – Energy Compaction

Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |y(t)|^2 dt$$

Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 dt \geq \int_{-\infty}^{t_0} |y(t)|^2 dt$$

The energy build-up in the input is more **rapid** than in the output

Truncated input

$$x_1(t) = \begin{cases} x(t) & (t \leq t_0) \\ 0 & (t > t_0) \end{cases}$$

Corresponding output

$$y_1(t) = ?$$

For $t \leq t_0$

$$x_1(t) = x(t)$$



$$y_1(t) = y(t)$$

$$y_1(t) = \int_{-\infty}^{t_0} h(t-\tau)x_1(\tau)d\tau = \int_{-\infty}^t h(t-\tau)x(\tau)d\tau = y(t)$$

For $t > t_0$

$$x_1(t) = 0$$



$$y_1(t) \neq 0$$

$$\int_{-\infty}^{+\infty} |x_1(t)|^2 dt = \int_{-\infty}^{+\infty} |y_1(t)|^2 dt$$

$$\int_{-\infty}^{t_0} |x_1(t)|^2 dt = \int_{-\infty}^{t_0} |y_1(t)|^2 dt + \int_{t_0}^{+\infty} |y_1(t)|^2 dt$$

For $t \leq t_0$

$$\int_{-\infty}^{t_0} |x(t)|^2 dt \geq \int_{-\infty}^{t_0} |y(t)|^2 dt$$

Properties of a Minimum Phase System

Properties of a Minimum Phase System

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] <http://www.libinst.com/tpfd.htm>
- [4] K.Shin, J.K. Hammond, “Fundamentals of Signal Processing for Sound and Vibration Engineers”
- [5] www.radiolab.com.au/DesignFile/DN004.pdf