

Differentiation of Continuous Functions

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Based on
Introduction to Matrix Algebra, Autar Kaw
<https://ma.mathforcollege.com>

Outline

- 1 Approximations of a first derivative
 - Forward Difference Approximation
 - Backward Difference Approximation
 - Taylor Series
 - Central Divided Difference

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Forward Difference Approximation (1)

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\end{aligned}$$

for a finite $\Delta x > 0$

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Forward Difference Approximation (2)

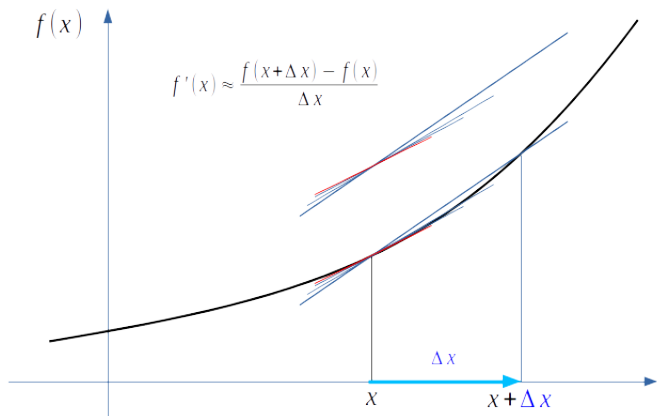


Figure: forward difference approximation

Forward Difference Approximation (3)

a forward difference approximation
as you are taking a point forward from x .

To find the value of $f'(x)$ at $x = x_i$,
we may choose another point Δx forward as $x = x_{i+1}$.

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\begin{aligned} f'(x_i) &\approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x} \\ &= \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \end{aligned}$$

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Backward Difference Approximation (1a)

forward difference approximation

for a finite $\Delta x > 0$,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

backward difference approximation

for a finite $\Delta x < 0$, then $-\Delta x > 0$,

$$\begin{aligned} f'(x) &\approx \frac{f(x - \Delta x) - f(x)}{-\Delta x} \\ &= \frac{f(x) - f(x - \Delta x)}{\Delta x} \end{aligned}$$

Backward Difference Approximation (1b)

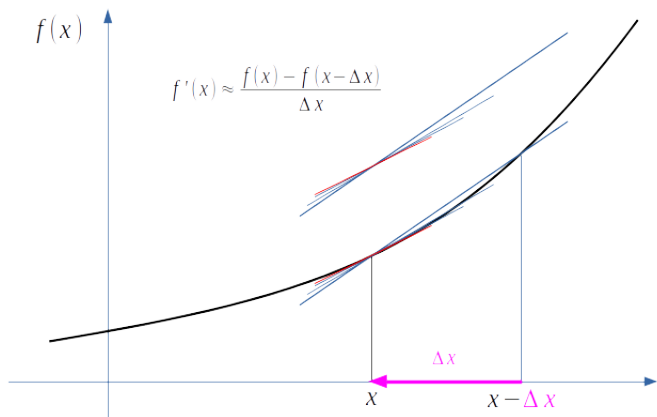


Figure: backward difference approximation (a)

Backward Difference Approximation (2a)

forward difference approximation

for a finite $\Delta x > 0$,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

backward difference approximation

for a finite $\Delta x > 0$, then $-\Delta x < 0$,

$$\begin{aligned} f'(x) &\approx \frac{f(x) - f(x - \Delta x)}{x - (x - \Delta x)} \\ &= \frac{f(x) - f(x - \Delta x)}{\Delta x} \end{aligned}$$

Backward Difference Approximation (2b)

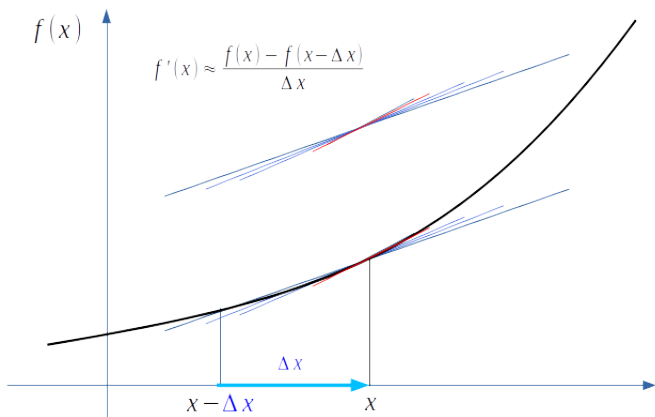


Figure: backward difference approximation (b)

Backward Difference Approximation (3)

a backward difference approximation
as you are taking a point backward from x .

To find the value of $f'(x)$ at $x = x_i$,
we may choose another point Δx backward as $x = x_{i-1}$.

$$f'(x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

$$\begin{aligned} f'(x_i) &\approx \frac{f(x_i) - f(x_{i-1})}{\Delta x} \\ &= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \end{aligned}$$

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- 1 **Approximations of a first derivative**
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Taylor Series (1)

the Taylor series of a function $f(x)$,
that is infinitely differentiable at a point a is the power series

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots$$

Taylor Series (2)

If $f(x)$ is given by a convergent power series in an open disk centred at a , it is said to be analytic in this region.

Thus for x in this region, f is given by a convergent power series

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots$$

Deriving Forward Difference Approximation (1)

A Taylor expansion approximate $f(x)$, using $f(a), f'(a), f''(a), \dots$,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots$$

Let $x_i = a$ and $x_{i+1} = x$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \dots$$

Substituting for convenience $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \dots$$

Deriving Forward Difference Approximation (2)

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \dots$$

$$f(x_{i+1}) - f(x_i) - \frac{f''(x_i)}{2!}(\Delta x)^2 - \dots = f'(x_i)(\Delta x)$$

$$\frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{f''(x_i)}{2!}(\Delta x) - \dots = f'(x_i)$$

$$\frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x) = f'(x_i)$$

Deriving Forward Difference Approximation (3)

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x)$$

the $O(\Delta x)$ term shows that

the error in the approximation is of the order of Δx

- for forward difference approximation

$$x_i = a, \quad x_{i+1} = x, \quad \Delta x = x_{i+1} - x_i$$

- for backward difference approximation

$$x_i = a, \quad x_{i-1} = x, \quad \Delta x = x_i - x_{i-1}$$

both forward and backward divided difference approximation of the first derivative are accurate on the order of $O(\Delta x)$

to get better approximations?

another method to approximate the first derivative is called

the Central Divided Difference Approximation of the first derivative

Deriving Backward Difference Approximation (1)

A Taylor expansion approximate $f(x)$, using $f(a), f'(a), f''(a), \dots$,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots$$

Let $x_i = a$ and $x_{i-1} = x$

$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)}{2!}(x_{i-1} - x_i)^2 + \dots$$

Substituting for convenience $\Delta x = x_i - x_{i-1}$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \dots$$

Deriving Forward Difference Approximation (2)

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \dots$$

$$f'(x_i)(\Delta x) = f(x_i) - f(x_{i-1}) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \dots$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + \frac{f''(x_i)}{2!}(\Delta x) - \dots$$

=

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + O(\Delta x)$$

Deriving Central Divide Approximation (1)

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots$$

Let $x_i = a$ and $x_{i+1} = x$, and substitute $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \dots$$

Let $x_i = a$ and $x_{i-1} = x$, and substitute $\Delta x = x_i - x_{i-1}$

$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)}{2!}(x_{i-1} - x_i)^2 + \dots$$

Deriving Central Divide Approximation

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 + \dots$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 \dots$$

subtracting eq(2) from eq(1)

$$f(x_{i+1}) - f(x_{i-1}) = 2f'(x_i)(\Delta x) + \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 + \dots$$

$$2f'(x_i)(\Delta x) = f(x_{i+1}) - f(x_{i-1}) - \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 - \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2(\Delta x)} - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^2 - \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O((\Delta x)^2)$$

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Central Divided Approximation

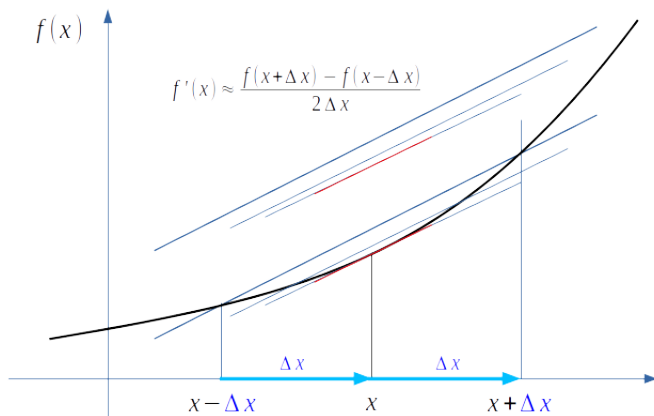


Figure: central difference approximation

Tangent Lines

- as $h \rightarrow 0$, $Q \rightarrow P$
and the **secant line** \rightarrow the **tangent line**
- the slope of the **tangent line**

$$\begin{aligned}m_{\text{tangent}} &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{(a+h) - a} \\ &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}\end{aligned}$$

