

Matched Filter (3C)

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Gaussian Random Process

Thermal Noise zero-mean white Gaussian random process

$n(t)$ random function
the value at time t is characterized by
Gaussian probability density function

$$\Rightarrow z(t) = a + n(t)$$

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma}\right)^2\right]$$

$$\Rightarrow p(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a}{\sigma}\right)^2\right]$$

σ^2 variance of n

$\sigma = 1$ normalized (standardized)
Gaussian function

Central Limit Theorem

sum of statistically independent random variables
approaches Gaussian distribution
regardless of individual distribution functions

White Gaussian Noise (1)

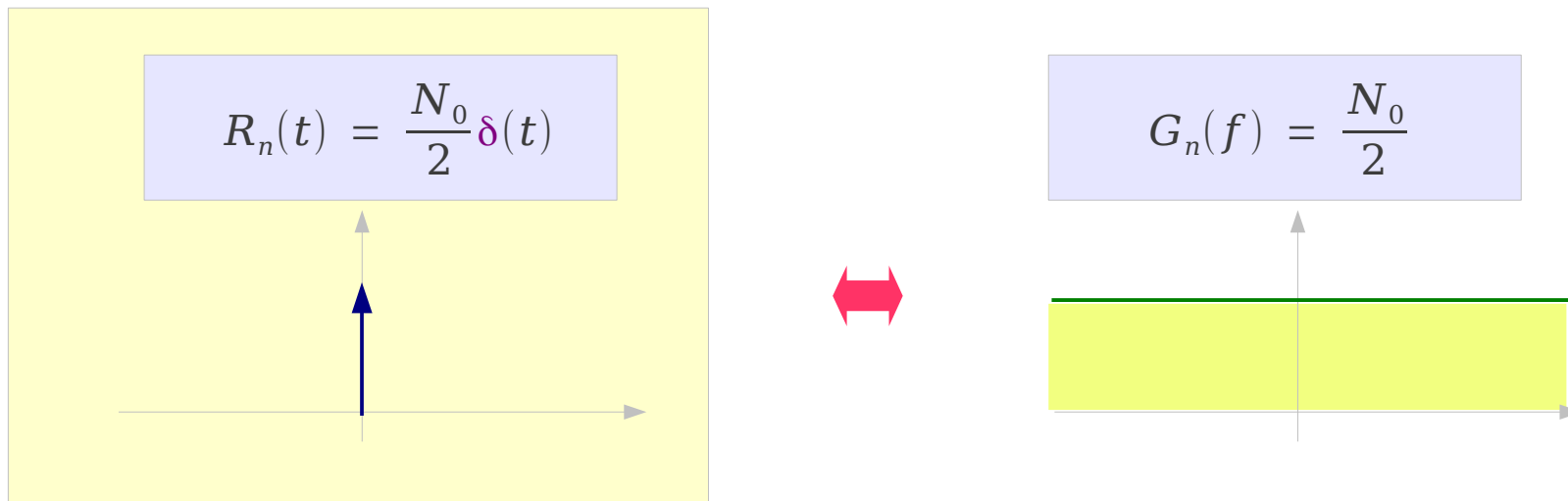
Thermal Noise

power spectral density is the same for all frequencies

$$G_n(f) = \frac{N_0}{2} \text{ watts / hertz}$$

equal amount of noise power per unit bandwidth

uniform spectral density \rightarrow White Noise



$\delta(t)$ totally uncorrelated, noise samples are independent
memoryless channel

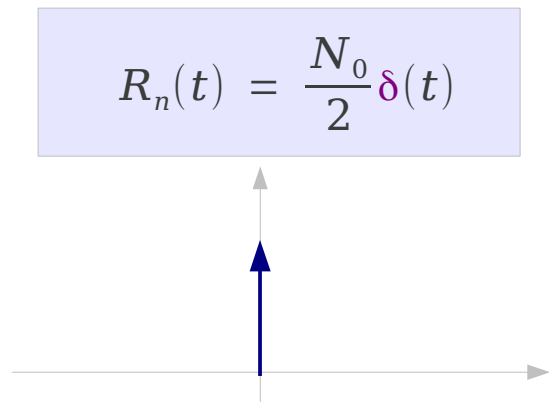
White Gaussian Noise (2)

Thermal Noise

power spectral density is the same for all frequencies

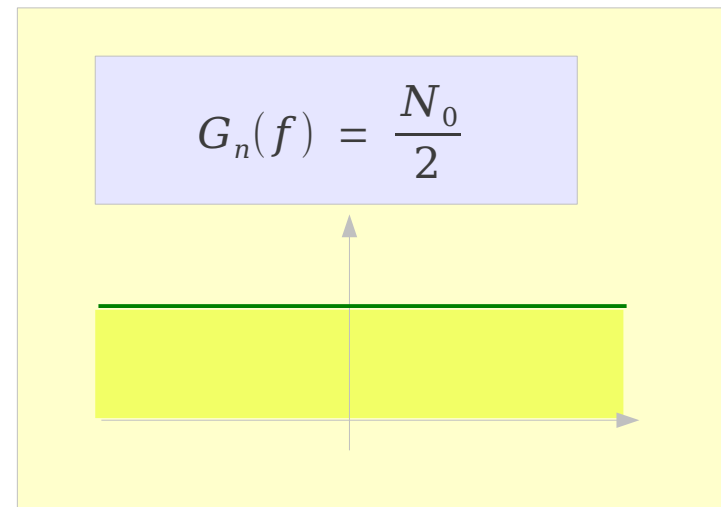
$$G_n(f) = \frac{N_0}{2} \text{ watts / hertz} \quad \text{equal amount of noise power per unit bandwidth}$$

uniform spectral density \rightarrow White Noise



average power

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt = \int_{-\infty}^{+\infty} G_x(f) df$$

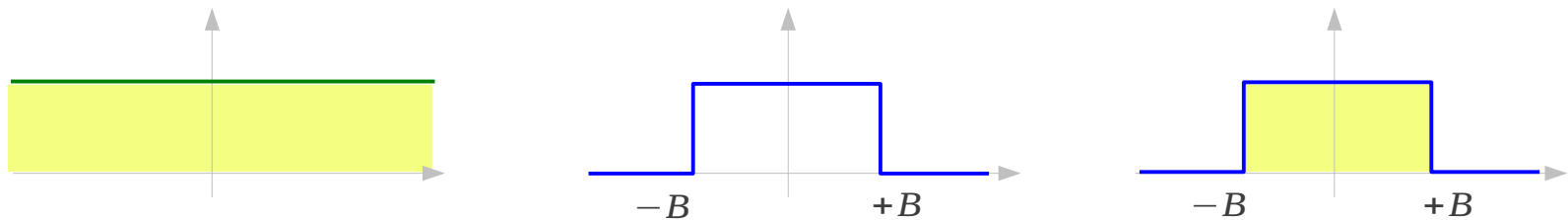
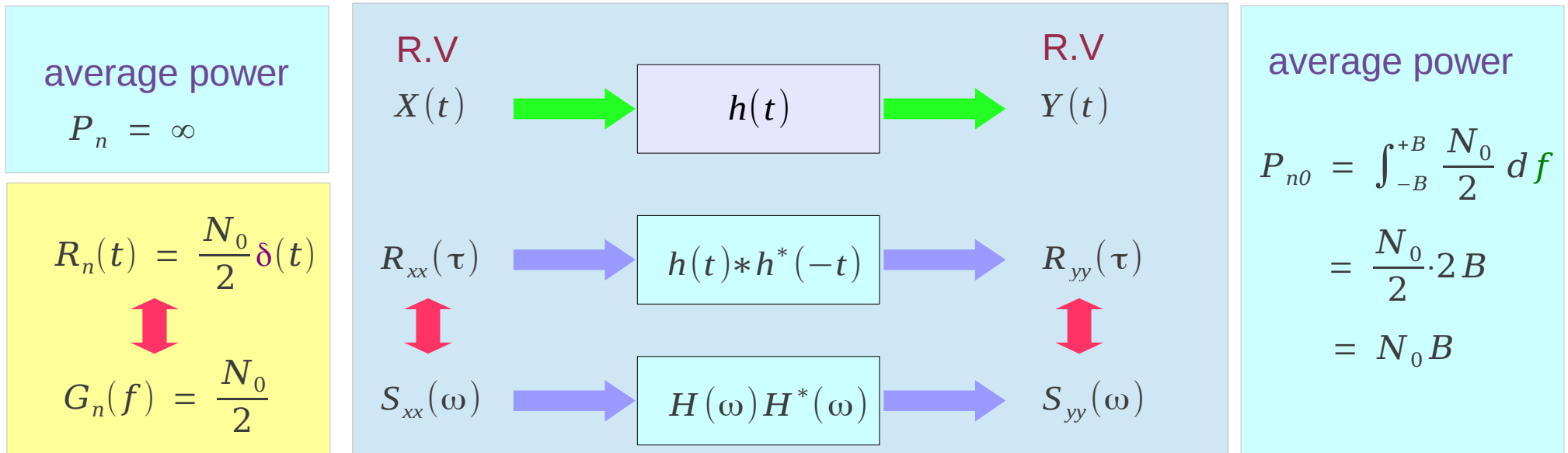


$$P_n = \int_{-\infty}^{+\infty} \frac{N_0}{2} df = \infty$$

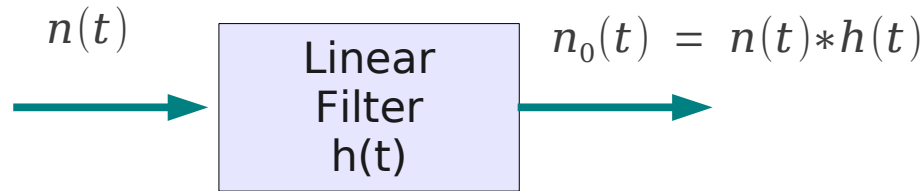
White Gaussian Noise (3)

Additive White Gaussian Noise (AWGN)

additive and no multiplicative mechanism



White Gaussian Noise (4)



$$G_n(f) = \frac{N_0}{2}$$

$$G_{n_0}(f) = G_n(f) |H(f)|^2 = \begin{cases} \frac{N_0}{2} |H(f)|^2 & \text{for } |f| < f_u \\ 0 & \text{otherwise} \end{cases}$$

Average output noise power

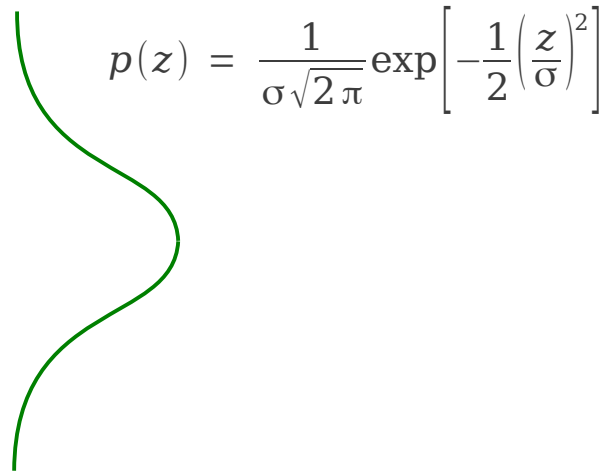
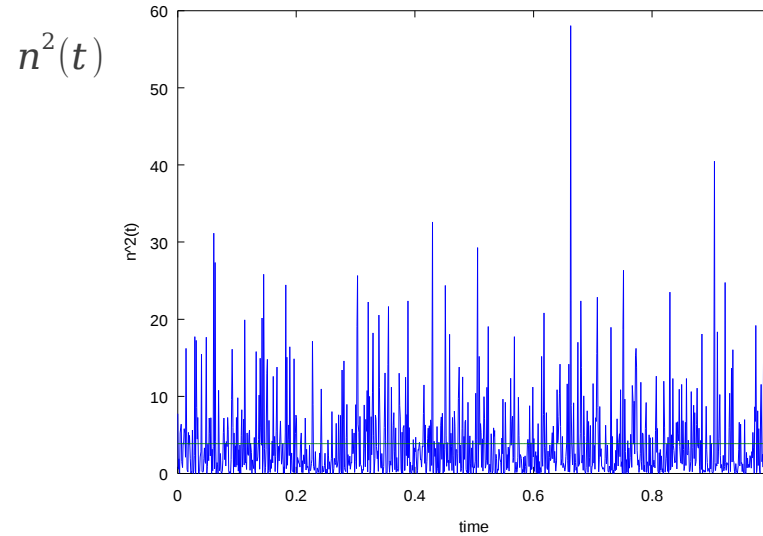
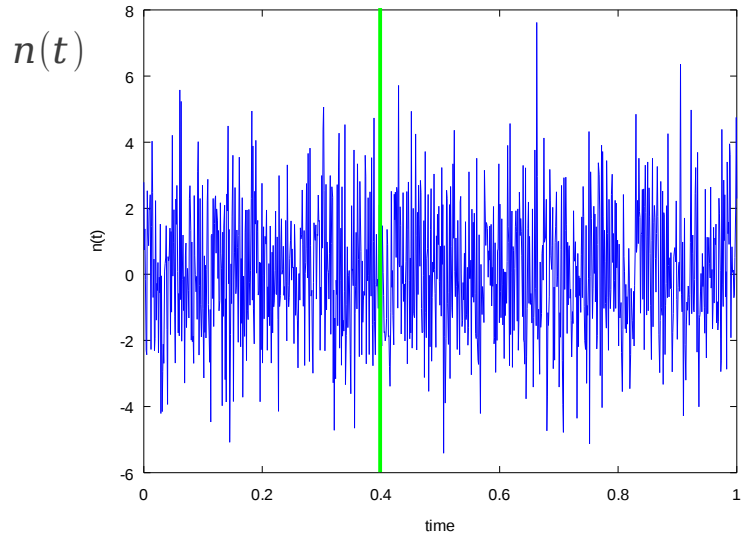
$$\sigma_0^2 = \overline{n_0^2(t)} = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2 df$$

RMS

$$\sigma_0 = \sqrt{\overline{n_0^2(t)}} = \sqrt{\frac{1}{T} \int_{-T/2}^{+T/2} n_0^2(t) dt}$$

Gaussian Random Process

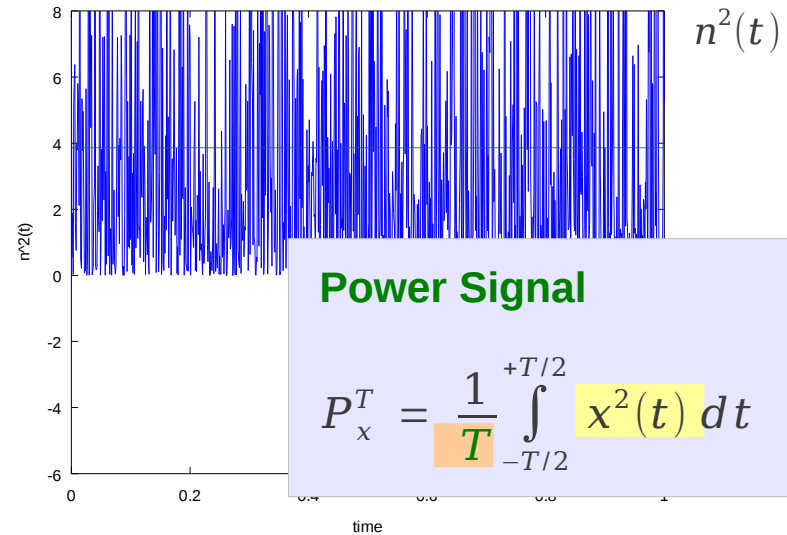
$n(t)$



$$p(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right]$$

$$m = 0$$

$$\sigma^2 \neq 0$$

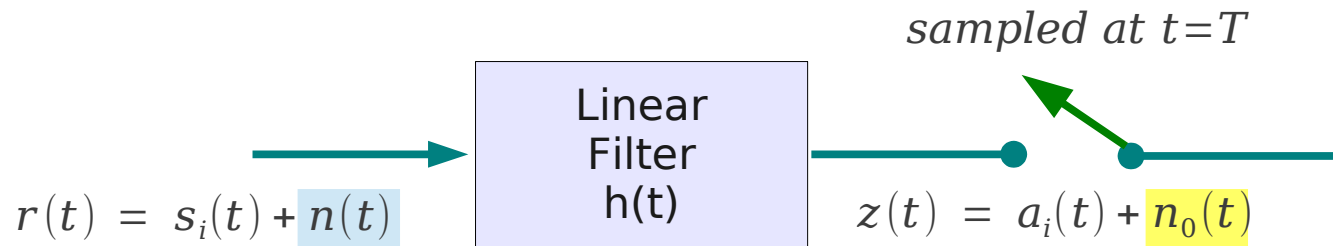


Power Signal

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$

Matched Filter (1)

to find a filter $h(t)$ that gives **max** signal-to-noise ratio



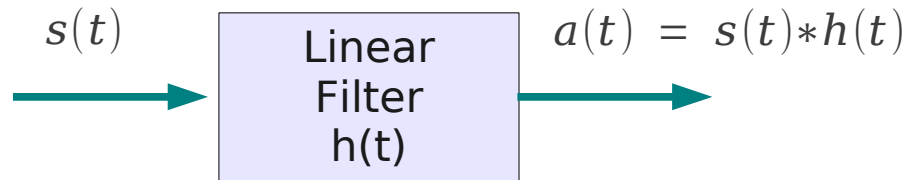
$$\frac{a_i^2(T)}{n_0^2(t)}$$

variance of $n_0(t)$ $\rightarrow \sigma_0^2$ avg noise power

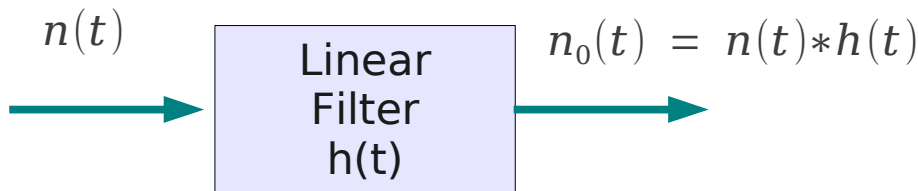
$$\frac{\text{instantaneous signal power}}{\text{average noise power}} \rightarrow \left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2}$$

assume $H_0(f)$ a filter transfer function that maximizes $\left(\frac{S}{N}\right)_T$

Matched Filter (2)



$$S(f) \quad A(f) = S(f)H(f) \quad \longleftrightarrow \quad a(t) = \int_{-\infty}^{+\infty} S(f)H(f)e^{j2\pi ft} df$$



$$G_n(f) = \frac{N_0}{2} \quad G_{n_0}(f) = G_n(f) |H(f)|^2 = \begin{cases} \frac{N_0}{2} |H(f)|^2 & \text{for } |f| < f_u \\ 0 & \text{otherwise} \end{cases}$$

Average output noise power $\sigma_0 = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2 df$

Matched Filter (3)

instantaneous signal power a_i^2 ← $a(t) = \int_{-\infty}^{+\infty} S(f)H(f)e^{j2\pi ft} df$

average output noise power $\sigma_0 = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2 df$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left| \int_{-\infty}^{+\infty} H(f)S(f)e^{j2\pi fT} df \right|^2}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2 df}$$

Does not depend on the particular shape of the waveform

Cauchy Schwarz's Inequality

$$\left| \int_{-\infty}^{+\infty} f_1(x)f_2(x) dx \right|^2 \leq \int_{-\infty}^{+\infty} |f_1(x)|^2 dx \int_{-\infty}^{+\infty} |f_2(x)|^2 dx \quad \text{'=' holds when } f_1(x) = kf_2^*(x)$$

$$\left| \int_{-\infty}^{+\infty} H(f)S(f)e^{j2\pi ft} dx \right|^2 df \leq \int_{-\infty}^{+\infty} |H(f)|^2 df \int_{-\infty}^{+\infty} |S(f)e^{j2\pi fT}|^2 df \quad |e^{j2\pi fT}| = 1$$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left| \int_{-\infty}^{+\infty} H(f)S(f)e^{j2\pi fT} df \right|^2}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2 df} \leq \frac{\int_{-\infty}^{+\infty} |H(f)|^2 df \int_{-\infty}^{+\infty} |S(f)e^{j2\pi fT}|^2 df}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2 df} = \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 df$$

Matched Filter (4)

Two-sided power spectral density of input noise $\rightarrow \frac{N_0}{2}$

Average noise power $\sigma_0 = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2 df$

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{\left| \int_{-\infty}^{+\infty} H(f) S(f) e^{+j2\pi f T} df \right|^2}{N_0/2 \int_{-\infty}^{+\infty} |H(f)|^2 df}$$

Cauchy Schwarz's Inequality

$$\left(\frac{S}{N}\right)_T \leq \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 df$$


$$\max \left(\frac{S}{N}\right)_T = \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 df = \frac{2E}{N_0}$$

input signal energy

power spectral density
of input noise

does not depend on the particular
shape of the waveform

Matched Filter (5)

$$\left| \int_{-\infty}^{+\infty} H(f) S(f) e^{+j2\pi ft} dx \right|^2 df \leq \int_{-\infty}^{+\infty} |H(f)|^2 df \int_{-\infty}^{+\infty} |S(f) e^{+j2\pi f T}|^2 df \quad \left(\frac{S}{N} \right)_T \leq \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 df$$


$$\max \left(\frac{S}{N} \right)_T = \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 df = \frac{2E}{N_0}$$

when complex conjugate relationship exists

$$H(f) = H_0(f) = k S^*(f) e^{-j2\pi f T}$$

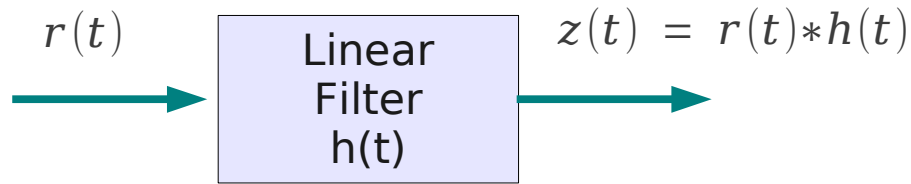


$$h(t) = h_0(t) = \begin{cases} k s(T-t) & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

$H_0(f)$ a filter transfer function that maximizes $\left(\frac{S}{N} \right)_T$

impulse response : delayed version of the mirror image of the signal waveform

Convolution vs. Correlation Realization



$$\begin{aligned}
 z(t) &= \int_0^t r(\tau) h(t-\tau) d\tau \\
 &= \int_0^t r(\tau) s(T-(t-\tau)) d\tau \\
 &= \int_0^t r(\tau) s(T-t+\tau) d\tau \\
 z(T) &= \int_0^T r(\tau) s(\tau) d\tau
 \end{aligned}$$

convolution $z(t) = \int_0^t r(\tau) s(T-t+\tau) d\tau$ $z(T) = \int_0^T r(\tau) s(\tau) d\tau$

a sine-wave amplitude modulated by a linear ramp

↕

correlation $z(t) = \int_0^t r(\tau) s(\tau) d\tau$ $z(T) = \int_0^T r(\tau) s(\tau) d\tau$

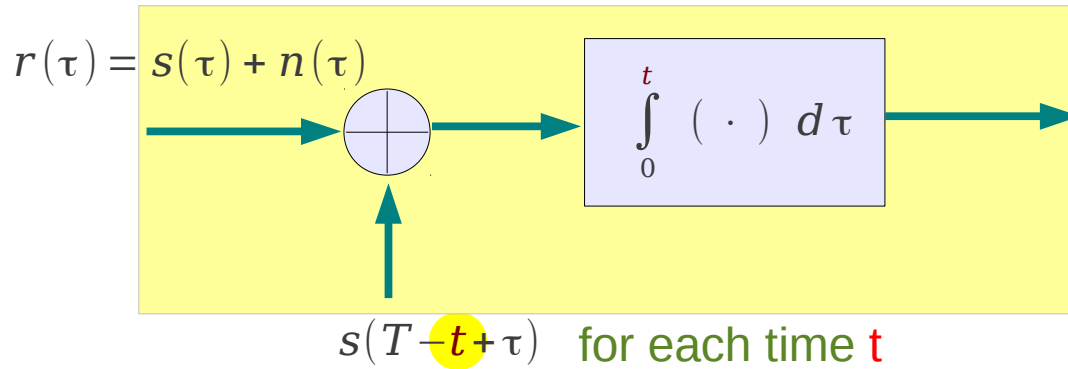
a linear ramp output

↕

shift position

fixed position

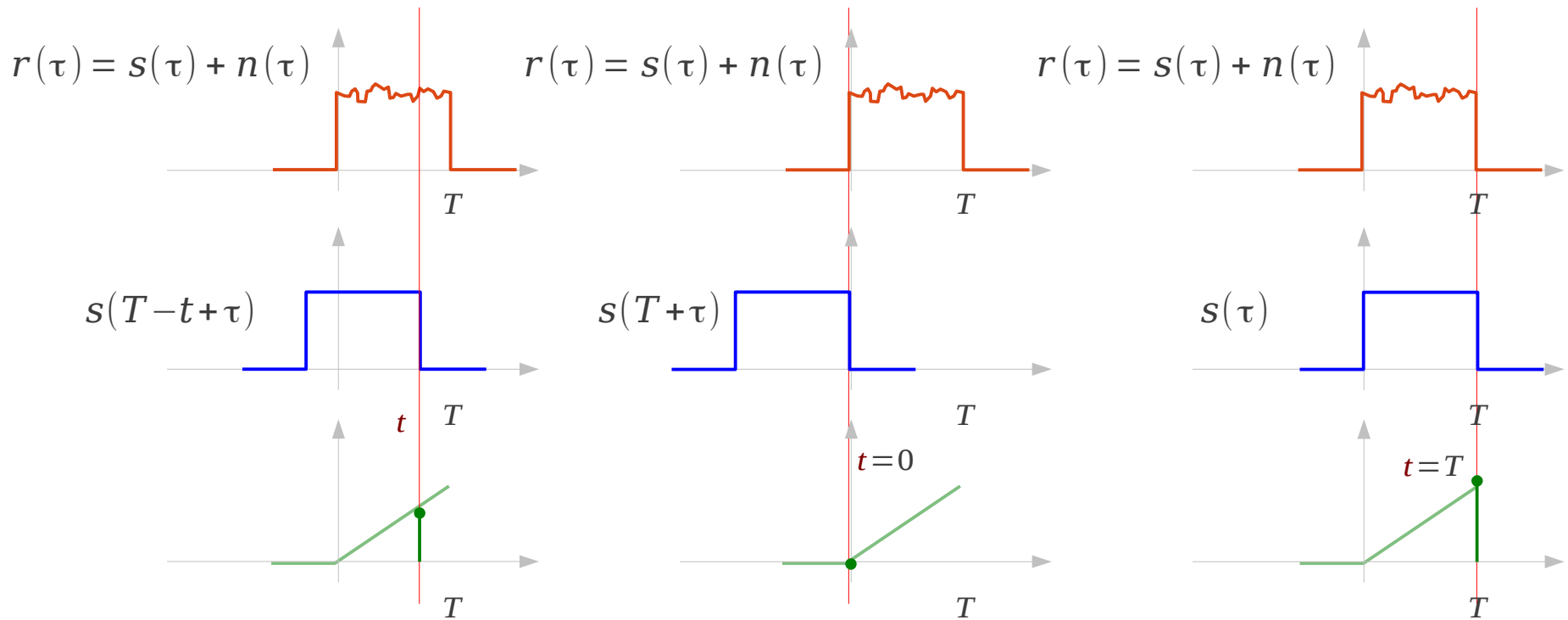
Convolution Realization



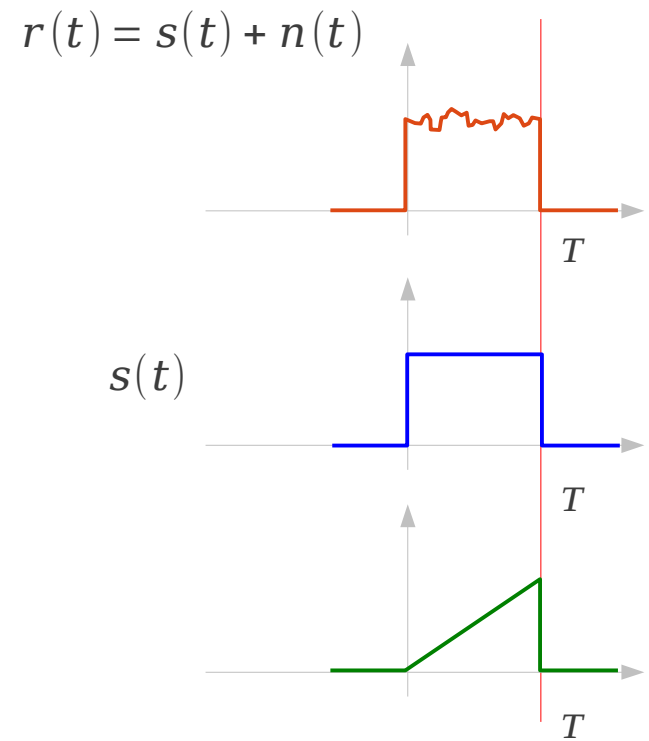
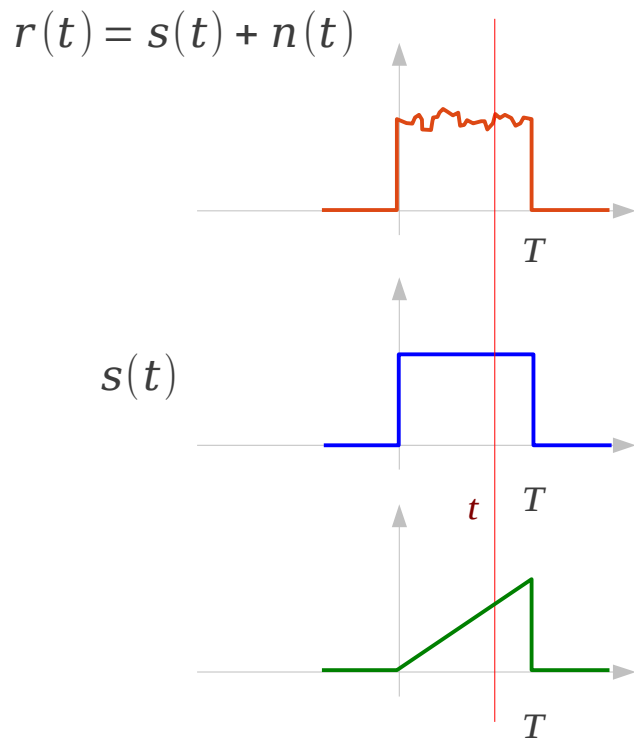
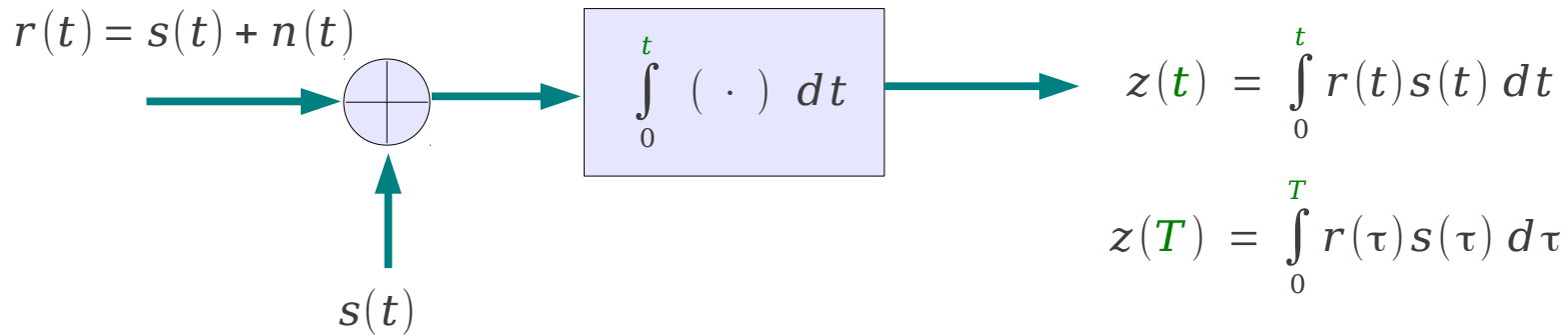
$$z(t) = \int_0^t r(\tau) h(t - \tau) d\tau$$

$$= \int_0^t r(\tau) s(T - t + \tau) d\tau$$

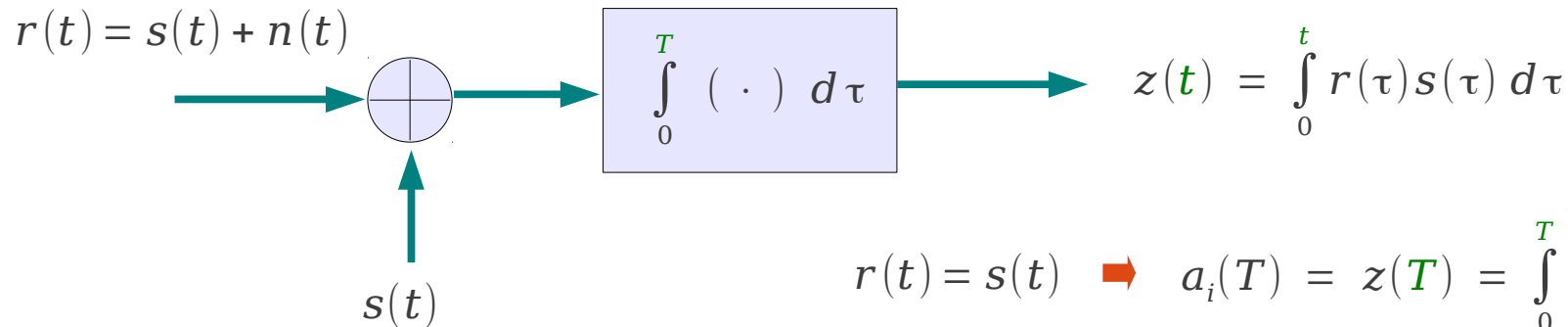
$$z(T) = \int_0^T r(\tau) s(\tau) d\tau$$



Correlation Realization (1)



Correlation Realization (2)



$$\begin{aligned}
 r(t) = s(t) &\rightarrow a_i(T) = z(T) = \int_0^T s^2(\tau) d\tau = E \\
 \sigma_0^2 = \mathbf{E}[n_0^2(t)] &= \mathbf{E}\left[\int_0^T n(t)s(t) dt \int_0^T n(\tau)s(\tau) d\tau\right] \\
 &= \mathbf{E}\left[\iint_0^T \underline{n(t)n(\tau)} s(t)s(\tau) dt d\tau\right] \\
 &= \iint_0^T \mathbf{E}[\underline{n(t)n(\tau)}] s(t)s(\tau) dt d\tau \\
 &= \iint_0^T \frac{N_0}{2} \delta(t - \tau) s(t)s(\tau) dt d\tau \\
 &= \frac{N_0}{2} \int_0^T s^2(t) dt = \frac{N_0}{2} E
 \end{aligned}$$

convolution

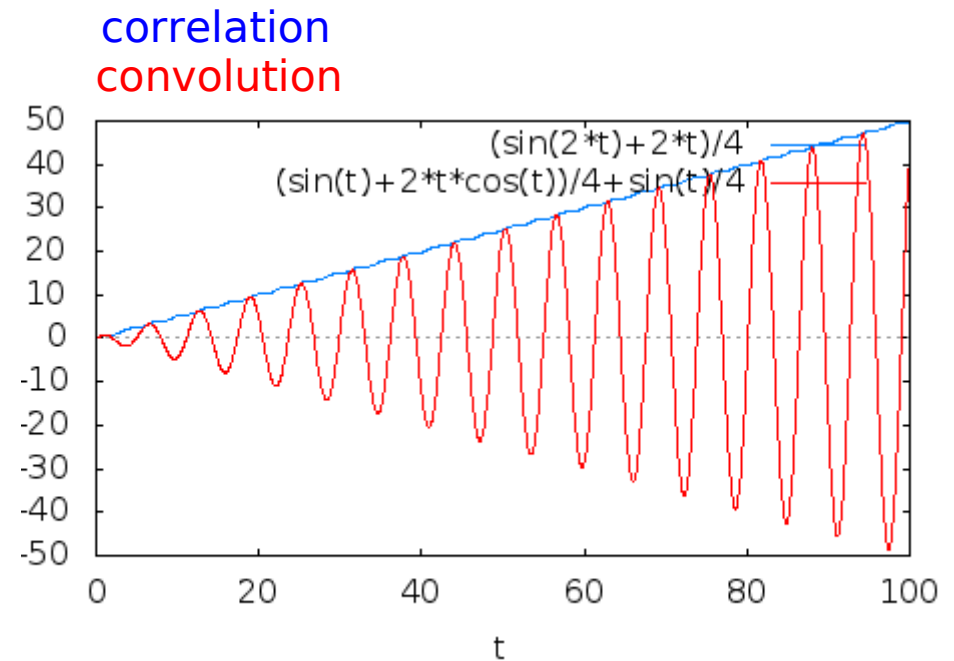
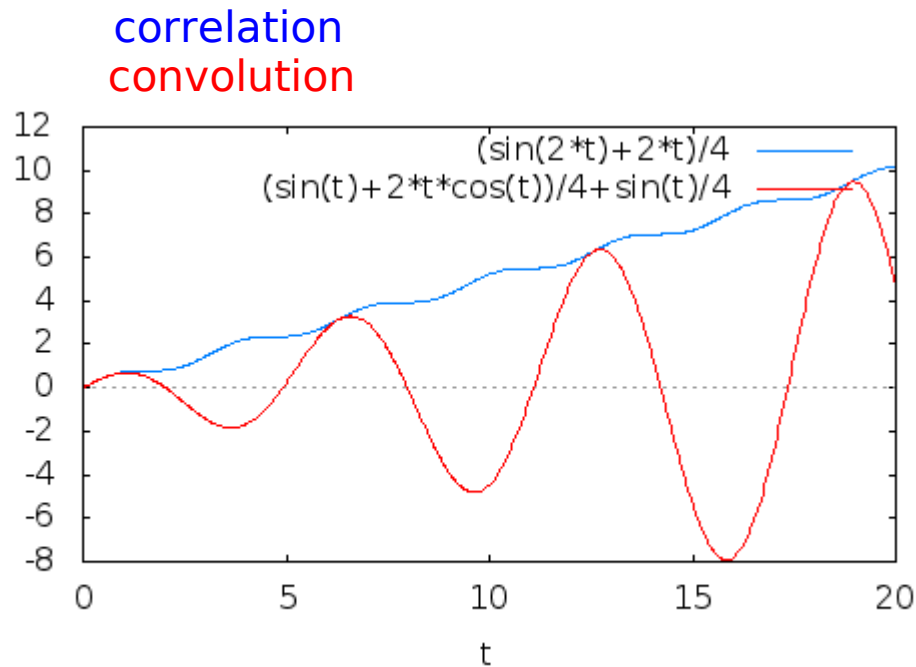
$$\frac{a_i^2(T)}{n_0^2(t)} \left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2}$$

$$\max \left(\frac{S}{N}\right)_T = \frac{2}{N_0} \int_{-\infty}^{+\infty} |S(f)|^2 df = \frac{2E}{N_0}$$

correlation

$$\left(\frac{S}{N}\right)_T = \frac{a_i^2}{\sigma_0^2} = \frac{E^2}{\frac{N_0}{2} E} = \frac{2E}{N_0}$$

Correlation and Convolution Examples (1)



z : integrate(cos(x)*cos(2*%pi - t +x), x, 0, t);

convolution

$(\sin(t)+2*t*\cos(t))/4+\sin(t)/4$

correlation

z : integrate(cos(x)*cos(x), x, 0, t);

$(\sin(2*t)+2*t)/4$

Correlation and Convolution Examples (2)

$$s(t) = \begin{cases} A \cos(\omega_0 t) & 0 \leq t < T \\ 0 & \text{elsewhere} \end{cases}$$

$$z(t) = \int_0^t r(\tau) s(T-t+\tau) d\tau$$

when $r(t) = s(t)$

$$\begin{aligned} z(t) &= \int_0^t s(\tau) s(T-t+\tau) d\tau \\ &= A^2 \int_0^t \cos(\omega_0 \tau) \cos(\omega_0 (T-t+\tau)) d\tau \\ &= \frac{A^2}{2} \int_0^t \cos(\omega_0 (T-t)) + \cos(\omega_0 (T-t+2\tau)) d\tau \\ &= \frac{A^2}{2} \left[\cos(\omega_0 (T-t)) \tau - \frac{1}{2\omega_0} \sin(\omega_0 (T-t+2\tau)) \right]_0^t \\ &= \frac{A^2}{2} \left[\cos(\omega_0 (T-t)) t - \frac{1}{2\omega_0} \{ \sin(\omega_0 (T+t)) - \sin(\omega_0 (T-t)) \} \right] \end{aligned}$$

$$\begin{aligned} z(t) &= \int_0^t r(\tau) h(t-\tau) d\tau \\ &= \int_0^t r(\tau) s(T-(t-\tau)) d\tau \\ &= \int_0^t r(\tau) s(T-t+\tau) d\tau \end{aligned}$$

Correlation and Convolution Examples (2)

$$s(t) = \begin{cases} A \cos(\omega_0 t) & \\ 0 & \text{elsewhere} \end{cases}$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) x(t-\tau) d\tau = \int_{-\infty}^{+\infty} x(t_0-\tau) x(t-\tau) d\tau$$

when $r(t) = s(t)$

$$A=T=1, n=4, \omega_0=8\pi$$

$$\begin{aligned} y(t) &= A^2 \int_0^t \cos[\omega_0(t-\tau)] \cos[\omega_0(-\tau+T)] d\tau \\ &= \frac{A^2}{2} \int_0^t \cos[\omega_0(t-T)] + \cos[\omega_0(-2\tau+t+T)] d\tau \\ &= \frac{A^2}{2} \left\{ t \cos[\omega_0(t-T)] - \frac{1}{2\omega_0} \cos[\omega_0(-2\tau+t+T)] \Big|_0^T \right\} \\ &= \frac{A^2}{2} \left\{ t \cos(\omega_0 t) + \frac{1}{\omega_0} \sin(\omega_0 t) \right\} \quad (0 \leq t \leq T) \\ &= \frac{A^2}{2} \left\{ t(2T-t) \cos(\omega_0 t) - \frac{1}{\omega_0} \sin(\omega_0 t) \right\} \quad (T \leq t \leq 2T) \end{aligned}$$

$$\begin{aligned} z(t) &= \int_0^t r(\tau) h(t-\tau) d\tau \\ &= \int_0^t r(\tau) s(T-(t-\tau)) d\tau \\ &= \int_0^t r(\tau) s(T-t+\tau) d\tau \end{aligned}$$

$$\begin{aligned} y(t) &= A^2 \int_0^t \cos(\omega_0 \tau) d\tau \\ &= \frac{A^2}{2} \int_0^t [1 + \cos(2\omega_0 \tau)] d\tau \\ &= A^2 \frac{t}{2} + \frac{A^2}{4\omega_0} [\sin(2\omega_0 \tau)]_0^t \\ &= A^2 \frac{t}{2} + \frac{A^2}{4\omega_0} [\sin(2\omega_0 t)] \quad (0 \leq t \leq T) \end{aligned}$$

References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] B. Sklar, "Digital Communications: Fundamentals and Applications"
- [4] W. Etten, "Introduction to Random Signals and Noise"