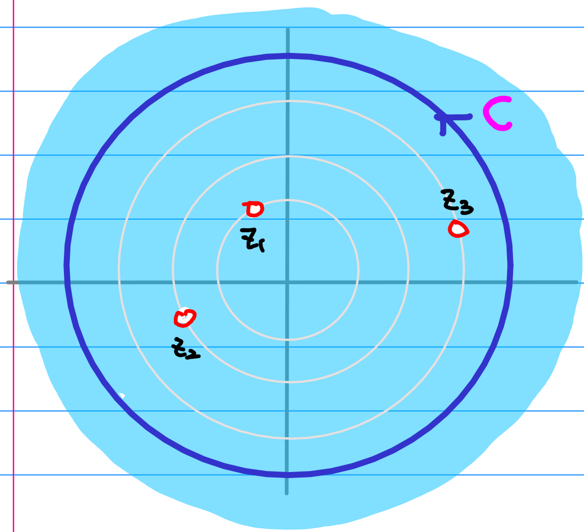
Laurent Series and Geometric Series

20170706

Series Expansion at $z = 0$



$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} z^n}$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} \, dz$$

$$= \sum_k \operatorname{Res}\left(\frac{f(z)}{z^{n+1}}, z_k \right)$$

Poles z_k

$n \geq 0$	$z_1, z_2, z_3, 0$
$n < 0$	z_1, z_2, z_3

* General Series Expansion at $z=0$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n z^n}$$

$$\boxed{\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right) \end{aligned}}$$

* Z-transform

$$\boxed{X(z) = \sum_{k=0}^{\infty} x_k z^{-k}}$$

$$\boxed{\begin{aligned} x_n &= \frac{1}{2\pi i}\oint_C X(z) z^{n-1}\,dz \\ &= \sum_k \mathrm{Res}\left(X(z) z^{n-1}, z_k\right) \end{aligned}}$$

z - Transform	$X(z)$
Laurent Series	$f(z)$

z-Transform $X(z) \longleftrightarrow x_n$

Laurent Series $f(z) \longleftrightarrow a_n$

$$\boxed{X(z) = f(z^{-1}) \longleftrightarrow x_n = a_n}$$

z-Transform $X(z) \longleftrightarrow x_n$

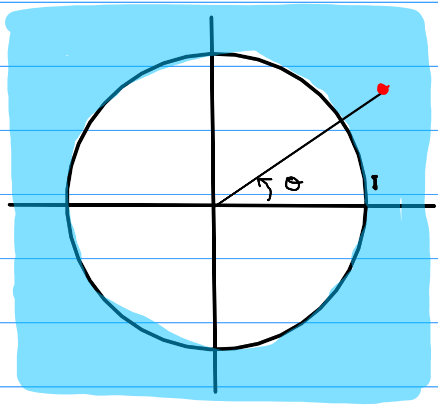
Laurent Series $f(z) \longleftrightarrow a_n$

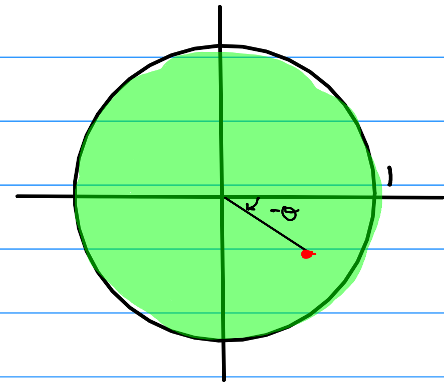
$$\boxed{X(z) = f(z) \longleftrightarrow x_n = a_{-n}}$$

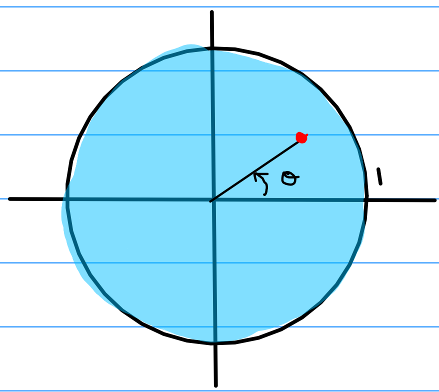
Mapping $w = \frac{1}{z}$

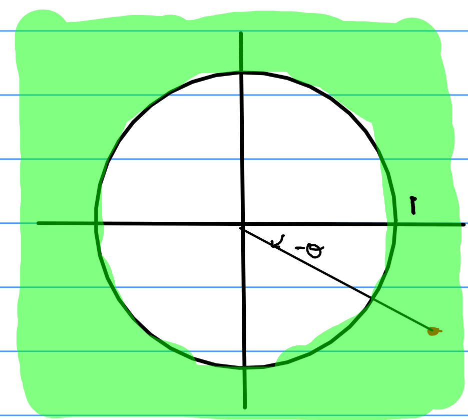
$z = \rho e^{j\theta}$

$z^{-1} = \frac{1}{\rho} e^{-j\theta}$









- inverse magnitude
- negative phase

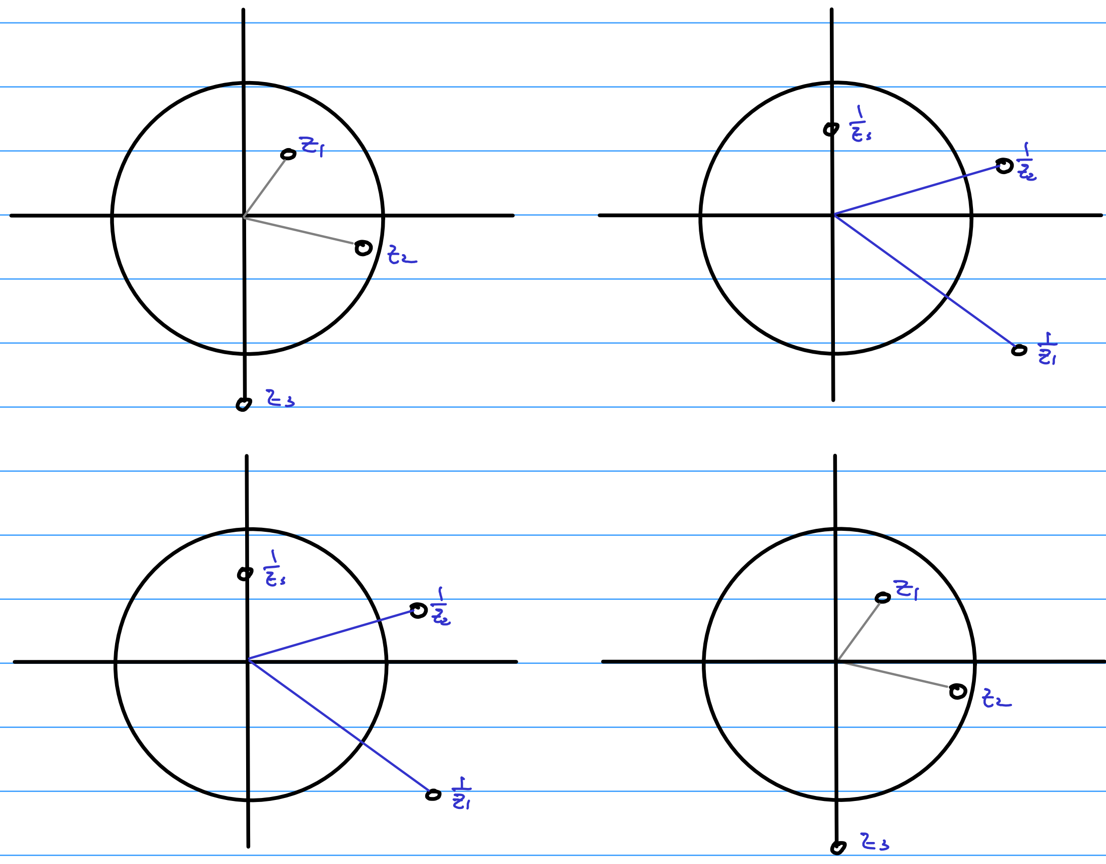
$$f(z) = \frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)(z-p_3)}$$

$$f(z^{-1}) = \frac{(\frac{1}{z}-z_1)(\frac{1}{z}-z_2)}{(\frac{1}{z}-p_1)(\frac{1}{z}-p_2)(\frac{1}{z}-p_3)}$$

$$= \frac{(1-z_1 z)(1-z_2 z)}{(1-p_1 z)(1-p_2 z)(1-p_3 z)}$$

$\frac{1}{z_1}, \frac{1}{z_2}$

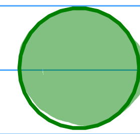
$\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}$



$g(z)$ with a simple pole

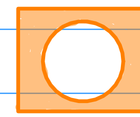
$b > 0$ assumed

$$g(z) = \frac{1}{1 - bz} = \boxed{\frac{b^{-1}}{b^{-1} - z}}$$ $|bz| < 1$



$|z| < \frac{1}{b}$

$$h(z) = \frac{1}{1 - \frac{b}{z}} = \boxed{\frac{z}{z - b}}$$ $\left|\frac{b}{z}\right| < 1$



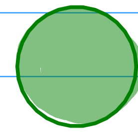
$|z| > b$

$$g(z^{-1}) = \frac{b^{-1}}{b^{-1} - z^{-1}} = \frac{z}{z - b} = h(z)$$

$$h(z^{-1}) = \frac{z^{-1}}{z^{-1} - b} = \frac{b^{-1}}{b^{-1} - z} = g(z)$$

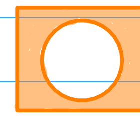
$$\boxed{\begin{aligned} g(z^{-1}) &= h(z) \\ h(z^{-1}) &= g(z) \end{aligned}}$$

$$g(z) = \boxed{\frac{b^{-1}}{b^{-1} - z}} = \frac{\bigcirc}{\bigcirc - z}$$



$|z| < \frac{1}{b}$

$$h(z) = \boxed{\frac{z}{z - b}} = \frac{z}{z - \square}$$



$|z| > b$

$$g(z^{-1}) = \frac{b^{-1}}{b^{-1} - z^{-1}} = \frac{z}{z - b} = h(z) \qquad \frac{\bigcirc}{\bigcirc - z^{-1}} = \frac{z}{z - \square}$$

$$h(z^{-1}) = \frac{z^{-1}}{z^{-1} - b} = \frac{b^{-1}}{b^{-1} - z} = g(z) \qquad \frac{z^{-1}}{z^{-1} - \square} = \frac{\bigcirc}{\bigcirc - z}$$

$$\bigcirc = b^{-1}$$
$$\square = b$$

$$\bigcirc^{-1} = \square$$
$$\square^{-1} = \bigcirc$$

Infinite Sum of G. P.

Simple pole $\Rightarrow$

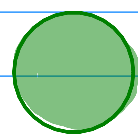
$$\frac{☁}{z-\square} \Rightarrow \frac{z}{z-\square} \Rightarrow \frac{1}{1-\frac{\square}{z}}$$ infinite sum of G.P

$$\frac{☁}{\triangle - z} \Rightarrow \frac{\bigcirc}{\bigcirc - z} \Rightarrow \frac{1}{1-\frac{z}{\bigcirc}}$$ infinite sum of G.P

Convergence Condition

$$\frac{b^{-1}}{b^{-1}-z} \Rightarrow$$ think this way $b^{-1}-z > 0$ $\quad \boxed{b^{-1} > |z|}$ $\quad$ pole b^{-1}

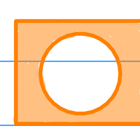
$$\boxed{\frac{b^{-1}}{b^{-1}-z}} = \frac{\bigcirc}{\bigcirc - z}$$



$|z| < \frac{1}{b}$ $\quad |z| < p$

$$\frac{z}{z-b} \Rightarrow$$ think this way $z - b > 0$ $\quad \boxed{|z| > b}$ $\quad$ pole b

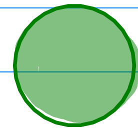
$$\boxed{\frac{z}{z-b}} = \frac{z}{z-\square}$$



$|z| > b$ $\quad |z| > p$

Two Sequences are involved (causal, anti-causal)

$\boxed{\dfrac{b^{-1}}{b^{-1} - z}} = \dfrac{\bigcirc}{\bigcirc - z}$



$|z| < \frac{1}{b}$ $\quad$ $|z| < p$

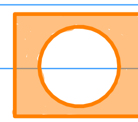
positive seq

① $(n \geq 0)$ $\quad (bz)^0 + (bz)^1 + (bz)^2 + \cdots = \sum_{n=0}^{\infty} b^n z^n$ $\quad$ L.S.

negative seq

② $(n \leq 0)$ $\quad (b^{-1}z^{-1})^0 + (b^{-1}z^{-1})^{-1} + (b^{-1}z^{-1})^{-2} + \cdots = \sum_{n=0}^{-\infty} b^{-n} z^{-n}$ $\quad$ Z.T.

$\boxed{\dfrac{z}{z - b}} = \dfrac{z}{z - \square}$



$|z| > b$ $\quad$ $|z| > p$

positive seq

① $(n \geq 0)$ $\quad (bz^{-1})^0 + (bz^{-1})^1 + (bz^{-1})^2 + \cdots = \sum_{n=0}^{\infty} b^n z^{-n}$ $\quad$ Z.T.

negative seq

② $(n \leq 0)$ $\quad (b^{-1}z)^0 + (b^{-1}z)^{-1} + (b^{-1}z)^{-2} + \cdots = \sum_{n=0}^{-\infty} b^{-n} z^{n}$ $\quad$ L.S.

$\boxed{n \geq 0}$ $\boxed{n \leq 0}$ $\boxed{\text{L.S.}}$ $\boxed{\text{Z.T.}}$

$$(\quad)^0 + (\quad)^1 + (\quad)^2 + \cdots \longrightarrow (n \geq 0)$$

$$(\quad)^0 + (\quad)^{-1} + (\quad)^{-2} + \cdots \longrightarrow (n \leq 0)$$

$$\sum \ \boxed{z^n} \longrightarrow \text{L.S.}$$

$$\sum \ \boxed{z^{-n}} \longrightarrow \text{Z.T.}$$

$(n \geq 0)$ $\quad$ $(n \leq 0)$

$$(b\,z) = (b^{-1} z^{-1})^{-1}$$

$$\boxed{\begin{array}{c}(b\,z)^{n} \\ b^{n} z^{n} \\ n=1,2,\cdots\end{array}} \qquad \boxed{\begin{array}{c}(b^{-1} z^{-1})^{n} \\ b^{-n} z^{-n} \\ n=-1,-2,\cdots\end{array}}$$

$(n \geq 0)$ $\quad$ $(n \leq 0)$

$$(b\,z^{-1}) = (b^{-1} z)^{-1}$$

$$\boxed{\begin{array}{c}(b\,z^{-1})^{n} \\ b^{n} z^{-n} \\ n=1,2,\cdots\end{array}} \qquad \boxed{\begin{array}{c}(b^{-1} z)^{n} \\ b^{-n} z^{n} \\ n=-1,-2,\cdots\end{array}}$$

$(n \geq 0) \qquad\qquad (n \leq 0)$

$$(b\,z) = (b^{-1} z^{-1})^{-1}$$

$$\boxed{\begin{array}{c}\displaystyle\sum_{n=1,2,\cdots} b^{n} z^{n} \\ |b\,z| \leq 1\end{array}} = \boxed{\begin{array}{c}\displaystyle\sum_{n=-1,-2,\cdots} b^{-n} z^{-n} \\ |b^{-1} z^{-1}|^{-1} \leq 1\end{array}}$$

$(n \geq 0) \qquad\qquad (n \leq 0)$

$$(b\,z^{-1}) = (b^{-1} z)^{-1}$$

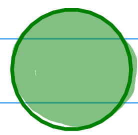
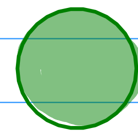
$$\boxed{\begin{array}{c}\displaystyle\sum_{n=1,2,\cdots} b^{n} z^{-n} \\ |b\,z^{-1}| \leq 1\end{array}} = \boxed{\begin{array}{c}\displaystyle\sum_{n=-1,-2,\cdots} b^{-n} z^{n} \\ |b^{-1} z|^{-1} \leq 1\end{array}}$$

$(n \geq 0)$ $\qquad\qquad$ $(n \leq 0)$

$b\,z \quad \Longleftarrow \quad (b^{-1} z^{-1})^{-1}$ $\qquad$ $n \geq 0$

$b^n z^n$ $\qquad\qquad$ $b^{-n} z^{-n}$ $\qquad$ $n \leq 0$

$\left|\frac{z}{b^{-1}}\right| < 1$ $\qquad\qquad$ $\left|\frac{z}{b^{-1}}\right| < 1$

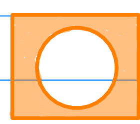
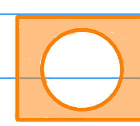
$\boxed{\frac{b^{-1}}{b^{-1} - z}}$  $\qquad$ $\boxed{\frac{b^{-1}}{b^{-1} - z}}$ 

$(n \geq 0)$ $\qquad\qquad$ $(n \leq 0)$

$b\,z^{-1} \quad \Longleftarrow \quad (b^{-1} z)^{-1}$ $\qquad$ $n \geq 0$

$b^n z^{-n}$ $\qquad\qquad$ $b^{-n} z^{n}$ $\qquad$ $n \leq 0$

$\left|\frac{b}{z}\right| < 1$ $\qquad\qquad$ $\left|\frac{b}{z}\right| < 1$

$\boxed{\frac{z}{z - b}}$  $\qquad$ $\boxed{\frac{z}{z - b}}$ 

	$\frac{\bigcirc}{\bigcirc - z}$	$\frac{z}{z - \square}$
	pole $p = \bigcirc$	pole $p = \square$
	c.r $\left(\frac{z}{\bigcirc}\right)$	c.r $\left(\frac{\square}{z}\right)$
	r.o.c $\lvert z\rvert < \bigcirc$	r.o.c $\lvert z\rvert > \square$
$(n \geq 0)$	$\sum_{n=0}^{\infty}\left(\frac{z}{\bigcirc}\right)^{n} = \sum_{n=0}^{\infty}\left(\frac{1}{\bigcirc}\right)^{n} z^{n}$	$\sum_{n=0}^{\infty}\left(\frac{\square}{z}\right)^{n} = \sum_{n=0}^{\infty} \square^{n} z^{-n}$
$(n \leq 0)$	$\sum_{n=0}^{-\infty}\left(\frac{z}{\bigcirc}\right)^{-n} = \sum_{n=0}^{-\infty} \bigcirc^{n} z^{-n}$	$\sum_{n=0}^{-\infty}\left(\frac{\square}{z}\right)^{-n} = \sum_{n=0}^{-\infty}\left(\frac{1}{\square}\right)^{n} z^{n}$
	L.S: $b^{n} z^{n}$ $(n \geq 0)$	Z.T: $b^{n} z^{-n}$ $(n \geq 0)$
	Z.T: $b^{-n} z^{-n}$ $(n \leq 0)$	L.S: $b^{-n} z^{n}$ $(n \leq 0)$
	$p = \bigcirc = b^{-1}$	$p = \square = b$

$$\sum_{n=0}^{\infty}(\quad)^{n} = \sum_{n=0}^{-\infty}(\quad)^{-n}$$

L.S.

$$\sum_{n=0}^{\infty}\left(\frac{z}{\bigcirc}\right)^{n} = \boxed{\sum_{n=0}^{\infty}}\left(\frac{1}{\bigcirc}\right)^{n}\boxed{z^{n}}$$

$$(n \geq 0)\quad \boxed{b^{n}}\boxed{z^{n}}$$

Z.T

$$\sum_{n=0}^{\infty}\left(\frac{\square}{z}\right)^{n} = \boxed{\sum_{n=0}^{\infty}}\ \square^{n}\ \boxed{z^{-n}}$$

$$(n \geq 0)\quad \boxed{b^{n}}\boxed{z^{-n}}$$

Z.T

$$\sum_{n=0}^{-\infty}\left(\frac{z}{\bigcirc}\right)^{-n} = \boxed{\sum_{n=0}^{-\infty}}\ \bigcirc^{n}\ \boxed{z^{-n}}$$

$$(n \leq 0)\quad \boxed{b^{-n}}\boxed{z^{n}}$$

L.S.

$$\sum_{n=0}^{-\infty}\left(\frac{\square}{z}\right)^{-n} = \boxed{\sum_{n=0}^{-\infty}}\left(\frac{1}{\square}\right)^{n}\boxed{z^{n}}$$

$$(n \leq 0)\quad \boxed{b^{-n}}\boxed{z^{n}}$$

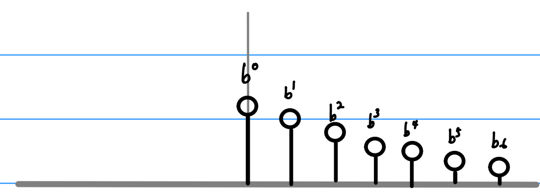
$$p = \bigcirc = b^{-1}$$

$$p = \square = b$$

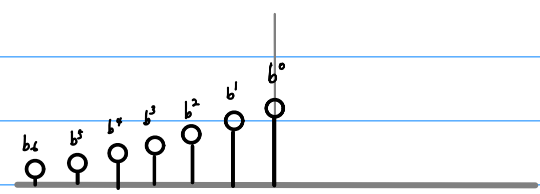
$\boxed{n \geq 0}$ $\boxed{b^n}$ & $\boxed{n \leq 0}$ $\boxed{b^{-n}}$ assumed

	$\dfrac{\bigcirc}{\bigcirc - z}$	$\dfrac{z}{z - \square}$
$(n \geq 0)$	L.S. $\left(\dfrac{z}{\bigcirc}\right)^{n} \Longleftrightarrow b^{n} z^{n}$ $(n \geq 0)$	Z.T $\left(\dfrac{\square}{z}\right)^{n} \Longleftrightarrow b^{n} z^{-n}$ $(n \geq 0)$
$(n \leq 0)$	Z.T $\left(\dfrac{z}{\bigcirc}\right)^{-n} \Longleftrightarrow b^{-n} z^{-n}$ $(n < 0)$	L.S. $\left(\dfrac{\square}{z}\right)^{-n} \Longleftrightarrow b^{-n} z^{n}$ $(n < 0)$
	$p = \bigcirc = b^{-1}$	$p = \square = b$

$(n \geq 0)$ $\quad n \quad$ $\boxed{b^n}$



$(n \leq 0)$ $\quad -n \quad$ $\boxed{b^{-n}}$



$$\left(\frac{z}{\bigcirc}\right)^{n},\quad \left(\frac{z}{\bigcirc}\right)^{-n},\quad \left(\frac{\square}{z}\right)^{n},\quad \left(\frac{\square}{z}\right)^{-n} \qquad \boxed{p}$$

$$\frac{z^{+}}{\bigcirc^{-}} \qquad \frac{z}{p} \longrightarrow \text{L.S.}$$

$(n \geq 0)$ $\quad p^{-n} z^{n}$ L.S

$(n \geq 0) \longrightarrow \boxed{b^{n}}\, z^{n}$ L.S.

$$p = \bigcirc = b^{-1}$$

$$\frac{\square^{+}}{z^{-}} \qquad \frac{p}{z} \longrightarrow \text{Z.T.}$$

$(n \geq 0)$ $\quad p^{n} z^{-n}$ Z.T.

$(n \geq 0) \longrightarrow \boxed{b^{n}}\, z^{-n}$ Z.T.

$$p = \square = b$$

$$\frac{\bigcirc^{+}}{z^{-}} \qquad \frac{p}{z} \longrightarrow \text{Z.T.}$$

$(n \leq 0)$ $\quad p^{n} z^{-n}$ Z.T.

$(n \leq 0) \longrightarrow \boxed{b^{-n}}\, z^{-n}$ Z.T.

$$p = \bigcirc = b^{-1}$$

$$\frac{z^{+}}{\square^{-}} \qquad \frac{z}{p} \longrightarrow \text{L.S.}$$

$(n \leq 0)$ $\quad p^{-n} z^{n}$ L.S.

$(n \leq 0) \longrightarrow \boxed{b^{-n}}\, z^{n}$ L.S.

$$= b$$

$$\frac{z^{+}}{p^{-}} \qquad \frac{p^{+}}{z^{-}}$$

L.S. $\qquad$ Z.T.

$$p = \bigcirc = \square^{-1}$$

$$p = \square = \bigcirc^{-1}$$

	L.S.	Z.T.
$(n \geqslant 0)$		
$(n \leqslant 0)$		

	L.S.	Z.T.
$(n \geqslant 0)$	$\dfrac{z^{+}}{\bigcirc^{-}}$ $\dfrac{z}{p}$ $(n \geqslant 0)$ $p^{-n} z^{n}$ L.S $(n \geqslant 0)$ $b^{n} z^{n}$ L.S. $p = \bigcirc = b^{-1}$	$\dfrac{\square^{+}}{z^{-}}$ $\dfrac{p}{z}$ $(n \geqslant 0)$ $p^{n} z^{-n}$ Z.T. $(n \geqslant 0)$ $b^{n} z^{-n}$ Z.T. $p = \square = b$
$(n \leqslant 0)$	$\dfrac{z^{+}}{\square^{-}}$ $\dfrac{z}{p}$ $(n \leqslant 0)$ $p^{-n} z^{n}$ L.S. $(n \leqslant 0)$ $b^{-n} z^{n}$ L.S. $p = \square = b$	$\dfrac{\bigcirc^{+}}{z^{-}}$ $\dfrac{p}{z}$ $(n \leqslant 0)$ $p^{n} z^{-n}$ Z.T. $(n \leqslant 0)$ $b^{-n} z^{-n}$ Z.T. $p = \bigcirc = b^{-1}$

	L.S. $\quad p^{-n}z^{n}$	Z.T. $\quad p^{n}z^{-n}$
$(n \geq 0)$	$\left(\frac{z}{\bigcirc}\right)^{n} \quad \left(\frac{1}{\bigcirc}\right)^{n}\boxed{z^{n}}$ $\boxed{\frac{\bigcirc}{\bigcirc - z}} \quad p^{-n}z^{n}$	$\left(\frac{\square}{z}\right)^{n} \quad \square^{n}\boxed{z^{-n}}$ $\boxed{\frac{z}{z - \square}} \quad p^{n}z^{-n}$
$(n \leq 0)$	$\left(\frac{\square}{z}\right)^{-n} \quad \left(\frac{1}{\square}\right)^{n}\boxed{z^{n}}$ $\boxed{\frac{z}{z - \square}} \quad p^{-n}z^{n}$	$\left(\frac{z}{\bigcirc}\right)^{-n} \quad \bigcirc^{n}\boxed{z^{-n}}$ $\boxed{\frac{\bigcirc}{\bigcirc - z}} \quad p^{n}z^{-n}$
	$(n \geq 0) \quad p = \bigcirc = b^{-1}$ $(n \leq 0) \quad p = \square = b$	$(n \geq 0) \quad p = \square = b$ $(n \leq 0) \quad p = \bigcirc = b^{-1}$

	L.S. $\bigcirc = \square^{-1}$ $\quad p^{-n} z^{n}$	Z.T. $\bigcirc = \square^{-1}$ $\quad p^{n} z^{-n}$
$(n \geq 0)$	$\left(\frac{z}{\square^{-1}}\right)^{n}$ $\quad \left(\frac{1}{\square^{-1}}\right)^{n} \boxed{z^{n}}$ $\boxed{\frac{\square^{-1}}{\square^{-1} - z}}$ $\quad p^{-n} z^{n}$	$\left(\frac{\square}{z}\right)^{n}$ $\quad \square^{n} \boxed{z^{-n}}$ $\boxed{\frac{z}{z - \square}}$ $\quad p^{n} z^{-n}$
$(n \leq 0)$	$\left(\frac{\square}{z}\right)^{-n}$ $\quad \left(\frac{1}{\square}\right)^{n} \boxed{z^{n}}$ $\boxed{\frac{z}{z - \square}}$ $\quad p^{-n} z^{n}$	$\left(\frac{z}{\square^{-1}}\right)^{-n}$ $\quad \square^{-n} \boxed{z^{-n}}$ $\boxed{\frac{\square^{-1}}{\square^{-1} - z}}$ $\quad p^{n} z^{-n}$
	$(n \geq 0)$ $\quad p = \bigcirc = b^{-1}$ $(n \leq 0)$ $\quad p = \square = b$	$(n \geq 0)$ $\quad p = \square = b$ $(n \leq 0)$ $\quad p = \bigcirc = b^{-1}$

b & p with Σ notations

	L.S.	Z.T.
$(n \geq 0)$	$\sum_{n=0}^{\infty}\left(\frac{z}{\bigcirc}\right)^{n} = \sum_{n=0}^{\infty} p^{-n} z^{n} = \sum_{n=0}^{\infty} b^{n} z^{n}$	$\sum_{n=0}^{\infty}\left(\frac{\square}{z}\right)^{n} = \sum_{n=0}^{\infty} p^{n} z^{-n} = \sum_{n=0}^{\infty} b^{n} z^{-n}$
$(n \leq 0)$	$\sum_{n=0}^{-\infty}\left(\frac{\square}{z}\right)^{-n} = \sum_{n=0}^{-\infty} p^{-n} z^{n} = \sum_{n=0}^{-\infty} b^{-n} z^{n}$	$\sum_{n=0}^{-\infty}\left(\frac{z}{\bigcirc}\right)^{-n} = \sum_{n=0}^{-\infty} p^{n} z^{-n} = \sum_{n=0}^{-\infty} b^{-n} z^{-n}$
	$(n \geq 0)$ $p = \bigcirc = b^{-1}$ $(n \leq 0)$ $p = \square = b$	$(n \geq 0)$ $p = \square = b$ $(n \leq 0)$ $p = \bigcirc = b^{-1}$

L.S.	Z.T.
L.S.	Z.T.

$(n \geq 0)$	$(n \geq 0)$
$(n \leq 0)$	$(n \leq 0)$

p^{-n}	p^{n}
p^{-n}	p^{n}

b^{n}	b^{n}
b^{-n}	b^{-n}

$p = \bigcirc$	$p = \square$
$p = \square$	$p = \bigcirc$

$\bigcirc = b^{-1}$	$\square = b$
$\square = b$	$\bigcirc = b^{-1}$

$\square^{n}$	$\square^{n}$
$\square^{-n}$	$\square^{-n}$

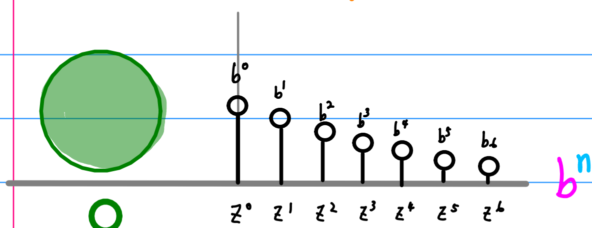
$\square^{n}$	$\square^{n}$
$\square^{-n}$	$\square^{-n}$

$\frac{\bigcirc}{\bigcirc - z}$	$\frac{z}{z - \square}$
$\frac{z}{z - \square}$	$\frac{\bigcirc}{\bigcirc - z}$

$\frac{\square^{-1}}{\square^{-1} - z}$	$\frac{z}{z - \square}$
$\frac{z}{z - \square}$	$\frac{\square^{-1}}{\square^{-1} - z}$

L.S.: $a_n z^n$ Z.T.: $x_n z^{-n}$

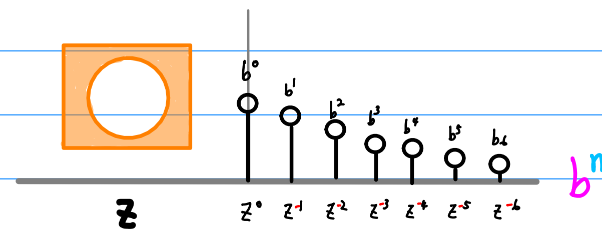
$p^{-n} z^{n}$



$$\frac{\bigcirc}{\bigcirc - z}$$

$$\frac{b^{-1}}{b^{-1} - z}$$

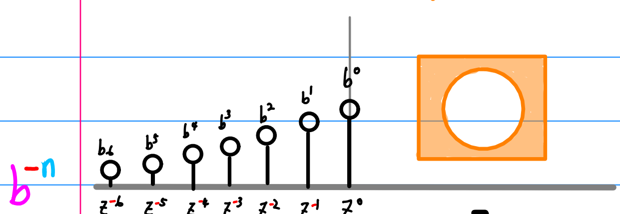
$p^{n} z^{-n}$



$$\frac{z}{z - \square}$$

$$\frac{z}{z - b}$$

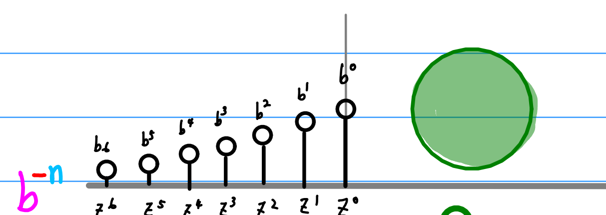
$p^{-n} z^{n}$



$$\frac{z}{z - \square}$$

$$\frac{z}{z - b}$$

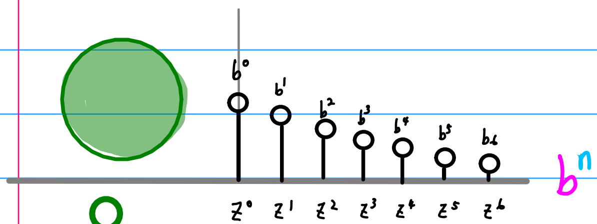
$p^{n} z^{-n}$



$$\frac{\bigcirc}{\bigcirc - z}$$

$$\frac{b^{-1}}{b^{-1} - z}$$

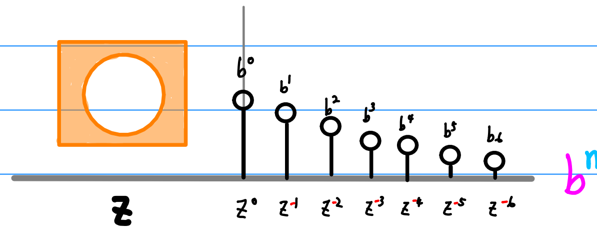
$b^n z^n$



$$\frac{\bigcirc}{\bigcirc - z}$$

$$\frac{b^{-1}}{b^{-1} - z}$$

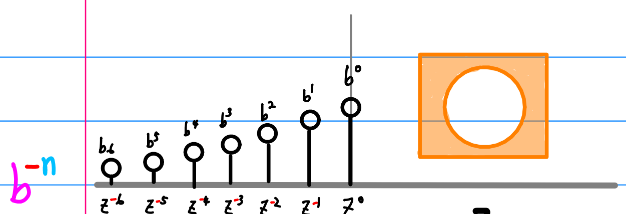
$b^n z^{-n}$



$$\frac{z}{z - \square}$$

$$\frac{z}{z - b}$$

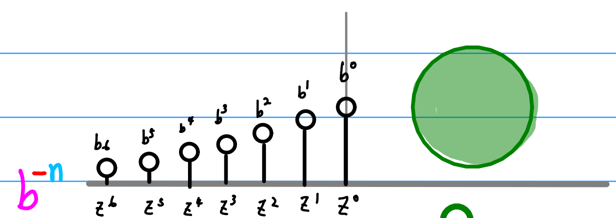
$b^{-n} z^n$



$$\frac{z}{z - \square}$$

$$\frac{z}{z - b}$$

$b^{-n} z^{-n}$



$$\frac{\bigcirc}{\bigcirc - z}$$

$$\frac{b^{-1}}{b^{-1} - z}$$

b^n & b^{-n}

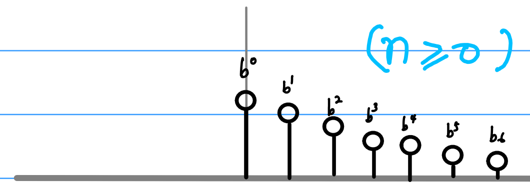
$0 < b < 1$ assumed

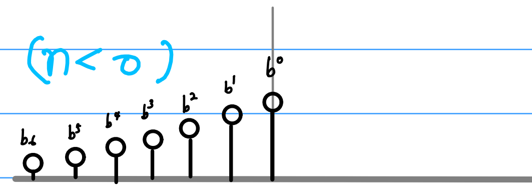
$(n \geq 0) \quad a_n = x_n = b^n$

$(n < 0) \quad a_n = x_n = b^{-n}$

$a_n = p^{-n} \quad (n \geq 0,\ n < 0)$

$x_n = p^n \quad (n \geq 0,\ n < 0)$





a_n Laurent Series Coefficient

x_n input to z-Transform

causal signal $\quad \begin{cases} x_n = 0 & n < 0 \\ x_n \neq 0 & n \geq 0 \end{cases}$

anti-causal signal $\quad \begin{cases} x_n \neq 0 & n < 0 \\ x_n = 0 & n \geq 0 \end{cases}$

p the simple pole of $f(z)$ or $X(z)$

$$\boxed{\frac{b^{-1}}{b^{-1} - z}} \qquad p = b^{-1}$$

$$\boxed{\frac{z}{z - b}} \qquad p = b$$

$Z.T.: x_n z^{-n}$ $\quad$ $L.S.: a_n z^n$

Z. T	z^{-n} $(n \geqslant 0)$	$b^n z^{-n}$ $(n \geqslant 0)$	+ −
	z^{-n} $(n<0)$	$b^{-n} z^{-n}$ $(n<0)$	− −
L. S	z^n $(n \geqslant 0)$	$b^n z^n$ $(n \geqslant 0)$	+ +
	z^n $(n<0)$	$b^{-n} z^n$ $(n<0)$	− +

$(n \geqslant 0)$	Z. T. z^{-n}	Z. T. $b^n z^{-n}$	+ −
	L. S. z^n	L. S. $b^n z^n$	+ +
$(n<0)$	Z. T. z^{-n}	Z. T. $b^{-n} z^{-n}$	− −
	L. S. z^n	L. S. $b^{-n} z^n$	− +

Laurent Series a_n	z-Transform x_n
$a_n = b^n \quad (n \geq 0)$	$x_n = b^n \quad (n \geq 0)$
$bz = \frac{z}{p} \qquad p = b^{-1}$	$bz^{-1} = \frac{p}{z} \qquad p = b$
$\left\lvert \frac{z}{p} \right\rvert < 1 \qquad \lvert z \rvert < p$	$\left\lvert \frac{p}{z} \right\rvert < 1 \qquad \lvert z \rvert > p$
$a_n = p^{-n}$	$x_n = p^n$

$a_n = b^{-n} \quad (n < 0)$	$x_n = b^{-n} \quad (n < 0)$
$b^{-1}z = \frac{z}{p} \qquad p = b$	$b^{-1}z^{-1} = \frac{p}{z} \qquad p = b^{-1}$
$\left\lvert \frac{z}{p} \right\rvert^{-1} < 1 \qquad \lvert z \rvert > p$	$\left\lvert \frac{p}{z} \right\rvert^{-1} < 1 \qquad \lvert z \rvert < p$
$a_n = p^{-n}$	$x_n = p^n$
$(n < 0) \rightarrow (k > 0)$	$(n < 0) \rightarrow (k > 0)$

Converging Geometric Series

$$\boxed{\begin{aligned} \left|\frac{z}{p}\right| &< 1 \\ |z| &< p \end{aligned}}$$

$$\left|\frac{p}{z}\right| > 1$$

$$\frac{1}{1-\frac{z}{p}} = \frac{p}{p-z}$$

think $p - z > 0$

$p > |z|$

$$\boxed{\begin{aligned} \left|\frac{p}{z}\right| &< 1 \\ p &< |z| \end{aligned}}$$

$$\left|\frac{z}{p}\right| > 1$$

$$\frac{1}{1-\frac{p}{z}} = \frac{z}{z-p}$$

think $z - p > 0$

$|z| > p$

z- Transform

$$\boxed{\frac{p}{z}} = \boxed{b z^{-1}}$$

causal

$(n \geq 0)$ $\quad p = b$

$$\boxed{\frac{p}{z}} = \boxed{b^{-1} z^{-1}} \rightarrow \frac{z}{p} = b z$$

anticausal

$(n < 0)$ $\quad p = b^{-1}$

Laurent Series

$$\boxed{\frac{z}{p}} = \boxed{b z} \qquad p = b^{-1}$$

$(n \geq 0)$

$$\boxed{\frac{z}{p}} = \boxed{b^{-1} z} \rightarrow \frac{p}{z} = b z^{-1}$$

$(n < 0)$ $\quad p = b$

Simple pole p & common ratio b

$$\left|\frac{z}{p}\right| < 1$$

$$|z| < p$$

$$\frac{1}{1-\frac{z}{p}} = \frac{p}{p-z}$$

$$\boxed{\frac{z}{p}} = b\,z \qquad p = b^{-1}$$

$$a_n = b^n \quad (n \geq 0)$$

$$\left|\frac{p}{z}\right| < 1$$

$$p < |z|$$

$$\frac{1}{1-\frac{p}{z}} = \frac{z}{z-p}$$

$$\boxed{\frac{p}{z}} = b\,z^{-1} \qquad p = b$$

$$a_n = b^{-n} \quad (n < 0)$$

$$\left|\frac{p}{z}\right| < 1$$

$$|z| > p$$

$$\frac{1}{1-\frac{p}{z}} = \frac{z}{z-p}$$

$$\boxed{\frac{p}{z}} = b\,z^{-1} \qquad p = b$$

$$x_n = b^n \quad (n \geq 0)$$

$$\left|\frac{z}{p}\right| < 1$$

$$|z| < p$$

$$\frac{1}{1-\frac{z}{p}} = \frac{p}{p-z}$$

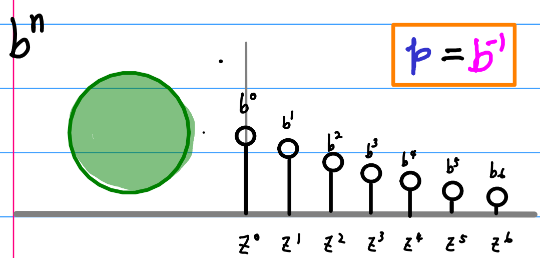
$$\boxed{\frac{z}{p}} = b\,z \qquad p = b^{-1}$$

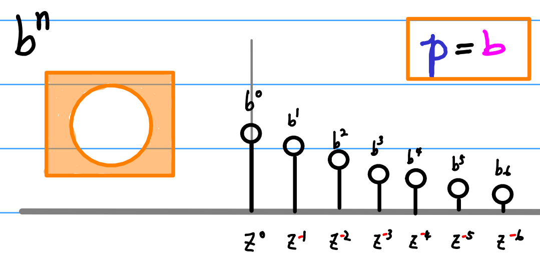
$$x_n = b^{-n} \quad (n < 0)$$

	L.S.	Z.T.
$n \geq 0$	$\left\lvert \frac{z}{p} \right\rvert < 1$	$\left\lvert \frac{p}{z} \right\rvert < 1$
	$\boxed{\frac{p}{p-z}}$	$\boxed{\frac{z}{z-p}}$
	$a_n = p^{-n} = b^n$	$x_n = p^n = b^n$
$n < 0$	$\left\lvert \frac{p}{z} \right\rvert < 1$	$\left\lvert \frac{z}{p} \right\rvert < 1$
	$\boxed{\frac{z}{z-p}}$	$\boxed{\frac{p}{p-z}}$
	$a_n = p^{-n} = b^{-n}$	$x_n = p^n = b^{-n}$

$(n \geq 0)$	$a_n = x_n = b^n$
$(n < 0)$	$a_n = x_n = b^{-n}$

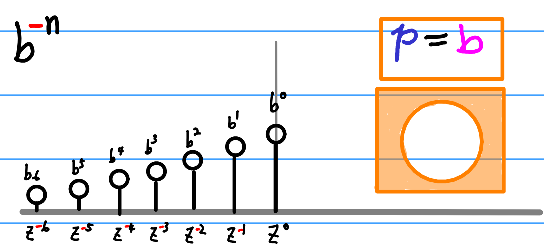
Laurent Series	$x_n = p^n$	$(n \geq 0,\ n < 0)$
z- Transform	$a_n = p^{-n}$	$(n \geq 0,\ n < 0)$

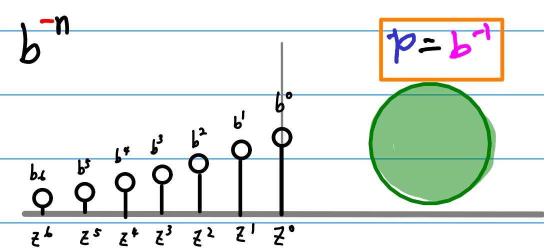




$$\boxed{a_n = p^{-n} \quad (n \geq 0)}$$

$$\boxed{x_n = p^n \quad (n \geq 0)}$$



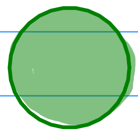
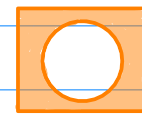
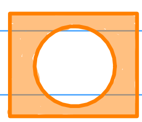
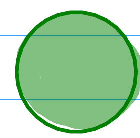


$$\boxed{a_n = p^{-n} \quad (n \leq 0)}$$

$$\boxed{x_n = p^n \quad (n \leq 0)}$$

$(n \geq 0)$	$a_n = x_n = b^n$
$(n < 0)$	$a_n = x_n = b^{-n}$

Laurent Series	$x_n = p^n$	$(n \geq 0,\ n < 0)$
z-Transform	$a_n = p^{-n}$	$(n \geq 0,\ n < 0)$

	Laurent Series	Z - Transform
$n \geq 0$	 $\lvert z \rvert < p$ $\left\lvert \frac{z}{p} \right\rvert < 1$ ① $z^1, z^2, z^3, \cdots$ ② $\frac{1}{1-\frac{z}{p}} = \frac{p}{p-z}$ ③ $a_n = p^{-n} = b^n \quad (p = b^{-1})$	 $\lvert z \rvert > p$ $\left\lvert \frac{p}{z} \right\rvert < 1$ ① $z^{-1}, z^{-2}, z^{-3}, \cdots$ ② $\frac{1}{1-\frac{p}{z}} = \frac{z}{z-p}$ ③ $x_n = p^n = b^n \quad (p = b)$
$n \leq 0$	 $\lvert z \rvert > p$ $\left\lvert \frac{p}{z} \right\rvert < 1$ ① $z^{-1}, z^{-2}, z^{-3}, \cdots$ ② $\frac{1}{1-\frac{p}{z}} = \frac{z}{z-p}$ ③ $a_n = p^{-n} = b^{-n} \quad (p = b)$	 $\lvert z \rvert < p$ $\left\lvert \frac{z}{p} \right\rvert < 1$ ① $z^1, z^2, z^3, \cdots$ anti-causal ② $\frac{1}{1-\frac{z}{p}} = \frac{p}{p-z}$ ③ $x_n = p^n = b^{-n} \quad (p = b^{-1})$

$$a_n = \left(\tfrac{1}{2}\right)^n \quad (n \geq 0)$$
$$= p^{-n} \quad (n \geq 0) \qquad p = 2$$
$$f(z) = \frac{2}{2 - z}$$

$$x_n = \left(\tfrac{1}{2}\right)^n \quad (n \geq 0)$$
$$= p^{n} \quad (n \geq 0) \qquad p = \tfrac{1}{2}$$
$$X(z) = \frac{z}{z - 0.5}$$

$$a_n = \left(\tfrac{1}{2}\right)^{-n} \quad (n \leq 0)$$
$$= p^{-n} \quad (n \leq 0) \qquad p = \tfrac{1}{2}$$
$$f(z) = \frac{z}{z - 0.5}$$

$$x_n = \left(\tfrac{1}{2}\right)^{-n} \quad (n \leq 0)$$
$$= p^{n} \quad (n \leq 0) \qquad p = 2$$
$$X(z) = \frac{2}{2 - z}$$

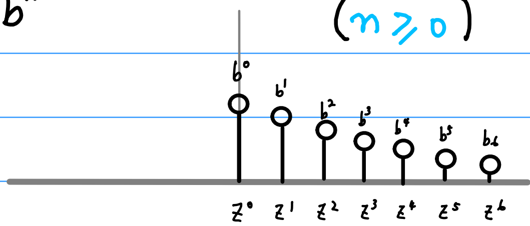
$$a_n = b^n \quad (n \geq 0)$$
$$= p^{-n} \quad (n \geq 0) \qquad p = b^{-1}$$
$$f(z) = \frac{b^{-1}}{b^{-1} - z}$$

$$x_n = b^n \quad (n \geq 0)$$
$$= p^{n} \quad (n \geq 0) \qquad p = b$$
$$X(z) = \frac{z}{z - b}$$

$$a_n = b^{-n} \quad (n \leq 0)$$
$$= p^{-n} \quad (n \leq 0) \qquad p = b$$
$$f(z) = \frac{z}{z - b}$$

$$x_n = b^{-n} \quad (n \leq 0)$$
$$= p^{n} \quad (n \leq 0) \qquad p = b^{-1}$$
$$X(z) = \frac{b^{-1}}{b^{-1} - z}$$

b^n $(n \geq 0)$

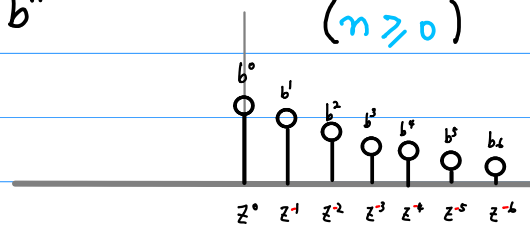


$$X(z^{-1}) = \frac{z^{-1}}{z^{-1} - 0.5} \qquad |z| < 2$$

$$f(z) = \frac{2}{2-z} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n$$

$$a_n = \left(\frac{1}{2}\right)^n = p^{-n} \qquad \boxed{p = 2}$$

b^n $(n \geq 0)$

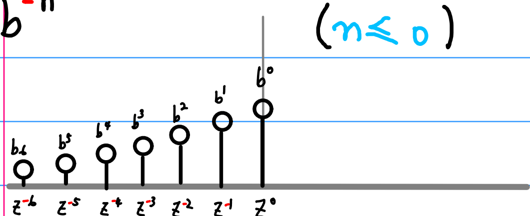


$$|z| > \frac{1}{2}$$

$$X(z) = \frac{z}{z - 0.5} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$x_n = \left(\frac{1}{2}\right)^n = p^n \qquad \boxed{p = \frac{1}{2}}$$

b^{-n} $(n \leq 0)$

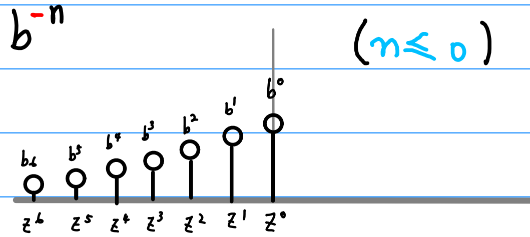


$$X(z^{-1}) = \frac{2}{2 - z^{-1}} \qquad |z| > \frac{1}{2}$$

$$f(z) = \frac{z}{z - 0.5} = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} z^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$a_n = \left(\frac{1}{2}\right)^{-n} = p^{n} \qquad \boxed{p = \frac{1}{2}}$$

b^{-n} $(n \leq 0)$

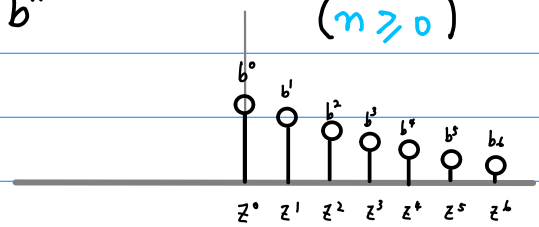


$$|z| < 2$$

$$X(z) = \frac{2}{2 - z} = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n$$

$$x_n = \left(\frac{1}{2}\right)^{-n} = p^n \qquad \boxed{p = 2}$$

b^n $(n \geq 0)$

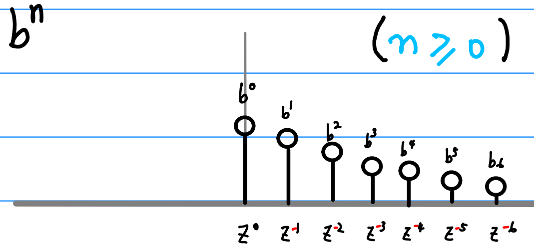


$$X(z^{-1}) = \frac{z^{-1}}{z^{-1} - b} \qquad |z| < b^{-1}$$

$$f(z) = \frac{b^{-1}}{b^{-1} - z} = \sum_{n=0}^{\infty} b^n z^n$$

$$a_n = b^n$$
$$= p^{-n} \qquad \boxed{p = b^{-1}}$$

b^n $(n \geq 0)$

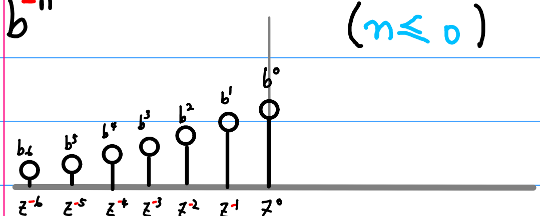


$$|z| > b$$

$$X(z) = \frac{z}{z - b} = \sum_{n=0}^{\infty} b^n z^{-n}$$

$$x_n = b^n$$
$$= p^n \qquad \boxed{p = b}$$

b^{-n} $(n \leq 0)$

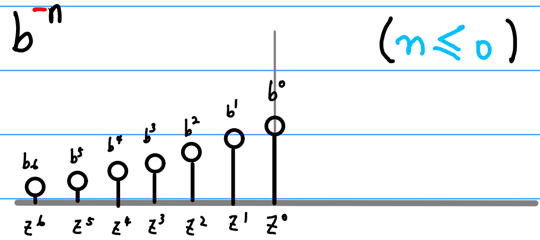


$$X(z^{-1}) = \frac{b^{-1}}{b^{-1} - z^{-1}} \qquad |z| > b$$

$$f(z) = \frac{z}{z - b} = \sum_{n=-\infty}^{0} b^{-n} z^n$$
$$= \sum_{n=0}^{\infty} b^n z^{-n}$$

$$a_n = b^{-n}$$
$$= p^{-n} \qquad \boxed{p = b}$$

b^{-n} $(n \leq 0)$



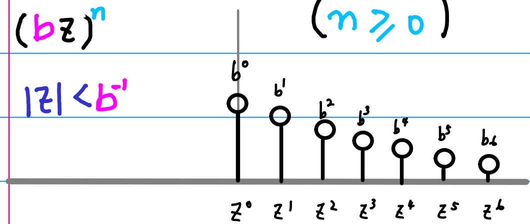
$$|z| < b^{-1}$$

$$X(z) = \frac{b^{-1}}{b^{-1} - z} = \sum_{n=-\infty}^{0} b^{-n} z^{-n}$$
$$= \sum_{n=0}^{\infty} b^n z^n$$

$$x_n = b^{-n}$$
$$= p^n \qquad \boxed{p = b^{-1}}$$

$(bz)^n$ $\quad (n \geq 0)$

$|z| < b^{-1}$

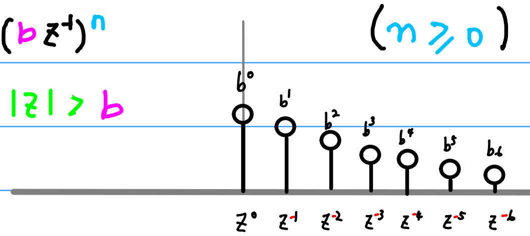


$$f(z) = \frac{1}{1 - bz} = \frac{b^{-1}}{b^{-1} - z}$$

$$a_n = b^n$$
$$= p^{-n} \qquad \boxed{p = b^{-1}}$$

$(bz^{-1})^n$ $\quad (n \geq 0)$

$|z| > b$

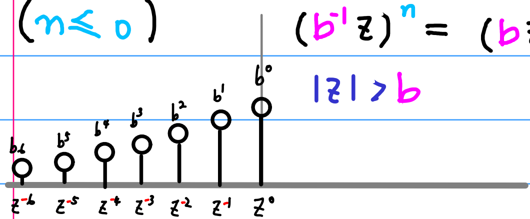


$$X(z) = \frac{1}{1 - b/z} = \frac{z}{z - b}$$

$$x_n = b^n$$
$$= p^n \qquad \boxed{p = b}$$

$(n \leq 0)$ $\quad (b^{-1}z)^n = (bz^{-1})^{-n}$

$|z| > b$

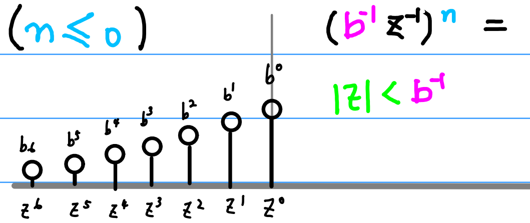


$$f(z) = \frac{1}{1 - (bz^{-1})} = \frac{z}{z - b}$$

$$a_n = b^{-n}$$
$$= p^{-n} \qquad \boxed{p = b}$$

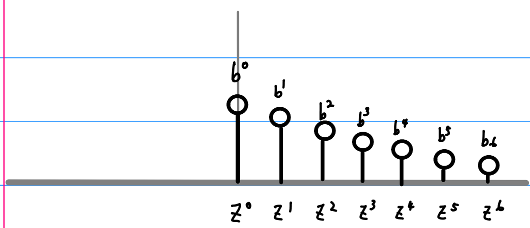
$(n \leq 0)$ $\quad (b^{-1}z^{-1})^n = (bz)^{-n}$

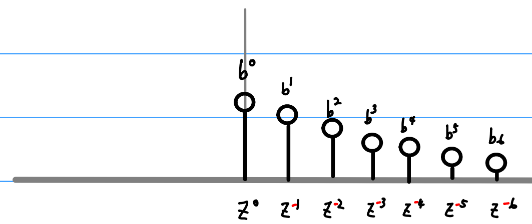
$|z| < b^{-1}$



$$X(z) = \frac{1}{1 - (bz)} = \frac{b^{-1}}{b^{-1} - z}$$

$$x_n = b^{-n}$$
$$= p^n \qquad \boxed{p = b^{-1}}$$



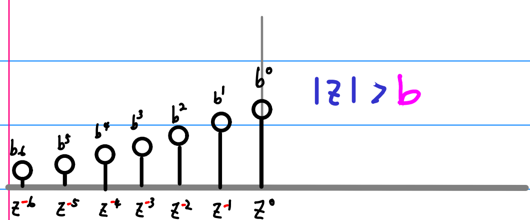


$$f(z) = \sum_{n=0}^{\infty} (bz)^n \qquad |bz| < 1$$

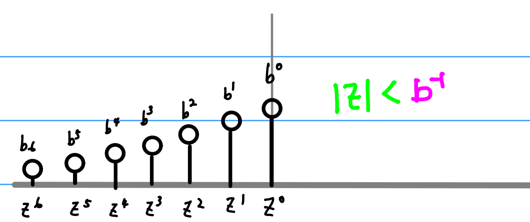
$$X(z) = \sum_{n=0}^{\infty} (bz^{-1})^n \qquad |bz^{-1}| < 1$$

$$a_n = b^n$$
$$= p^{-n} \qquad \boxed{p = b^{-1}}$$

$$x_n = b^n$$
$$= p^n \qquad \boxed{p = b}$$



$|z| > b$



$|z| < b^{-1}$

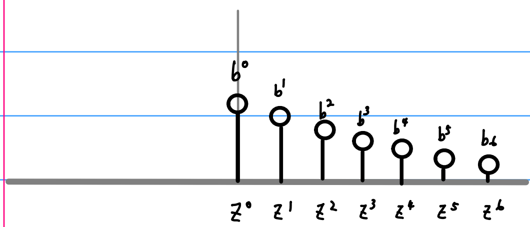
$$f(z) = \sum_{n=-\infty}^{0} (bz^{-1})^{-n} \qquad |bz^{-1}| < 1$$
$$= \sum_{n=0}^{\infty} (bz^{-1})^n$$

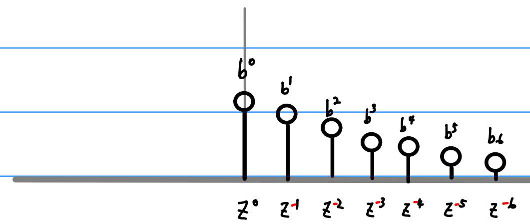
$$X(z) = \sum_{n=-\infty}^{0} (bz)^{-n} \qquad |bz| < 1$$
$$= \sum_{n=0}^{\infty} (bz)^n$$

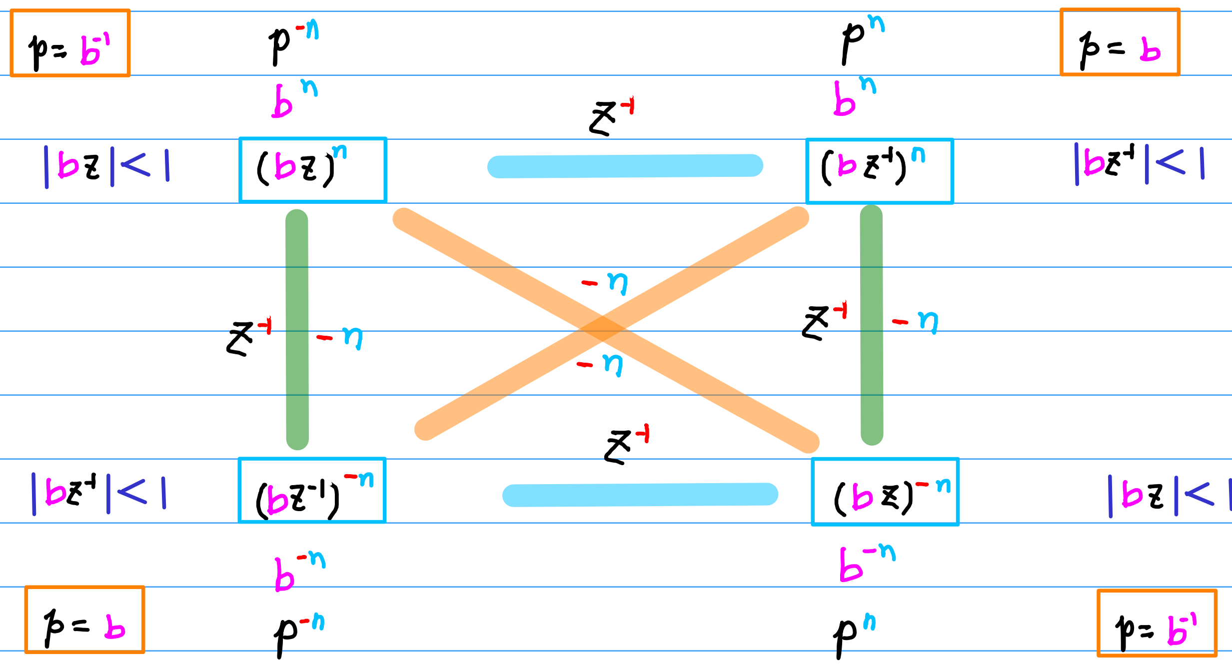
$$a_n = b^{-n}$$
$$= p^{-n} \qquad \boxed{p = b}$$

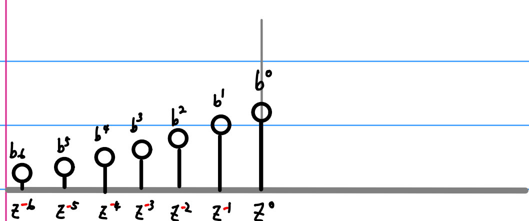
$$x_n = b^{-n}$$
$$= p^n \qquad \boxed{p = b^{-1}}$$

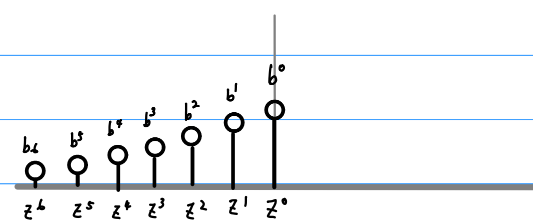
$$\frac{1}{1-(bz)} = \frac{b^{-1}}{b^{-1} - z} \qquad \frac{1}{1-(bz^{-1})} = \frac{z}{z - b}$$



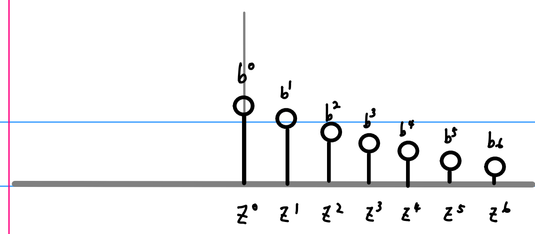


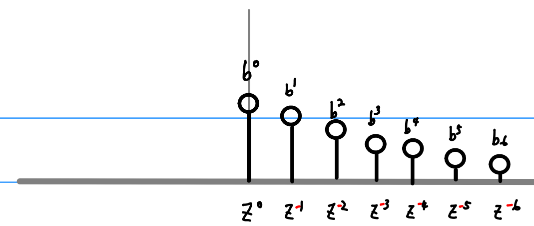






$$\frac{1}{1-(bz^{-1})} = \frac{z}{z - b} \qquad \frac{1}{1-(bz)} = \frac{b^{-1}}{b^{-1} - z}$$



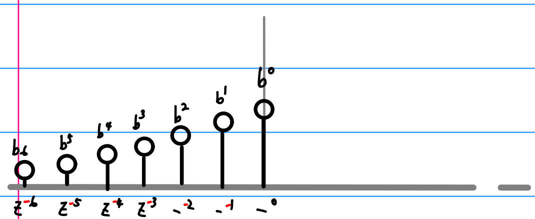


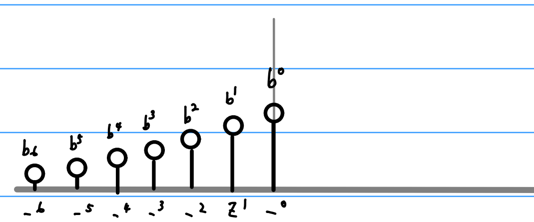
$$\boxed{\frac{1}{1-(bz)} = \frac{b^{-1}}{b^{-1}-z}}$$

$$\boxed{\frac{1}{1-(bz^{-1})} = \frac{z}{z-b}}$$

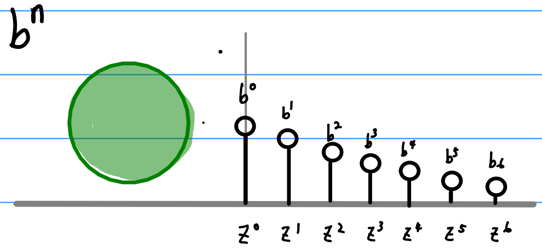
$$\boxed{\frac{1}{1-(bz^{-1})} = \frac{z}{z-b}}$$

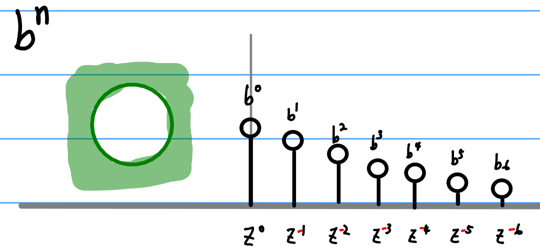
$$\boxed{\frac{1}{1-(bz)} = \frac{b^{-1}}{b^{-1}-z}}$$





$$x_n = a_{-n} \qquad X(z) = f(z)$$





$$f(z) = \frac{1}{1-(bz)} \qquad |z| < b^{-1}$$

$$X(z) = \frac{1}{1-(b/z)} \qquad |z| > b$$

$$\boxed{a_n = b^n \quad (n \geq 0)}$$

$$= p^{-n} \qquad (p = b^{-1})$$

$$\boxed{x_n = b^n \quad (n \geq 0)}$$

$$= p^{n} \qquad (p = b)$$

$$f(z) = \frac{1}{1-(b/z)} \qquad |z| > b$$

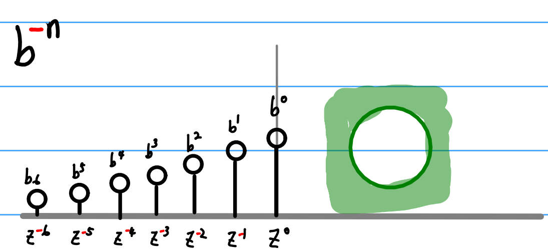
$$X(z) = \frac{1}{1-(bz)} \qquad |z| < b^{-1}$$

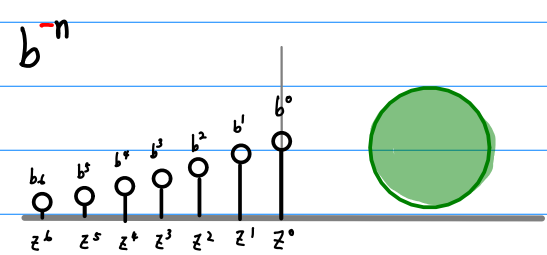
$$\boxed{a_n = b^{-n} \quad (n \leq 0)}$$

$$= p^{-n} \qquad (p = b)$$

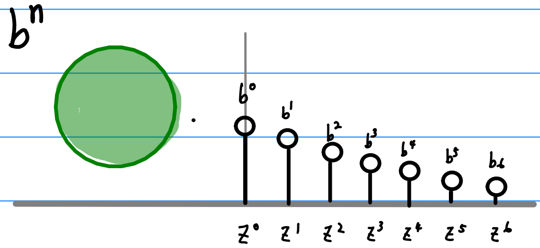
$$\boxed{x_n = b^{-n} \quad (n \leq 0)}$$

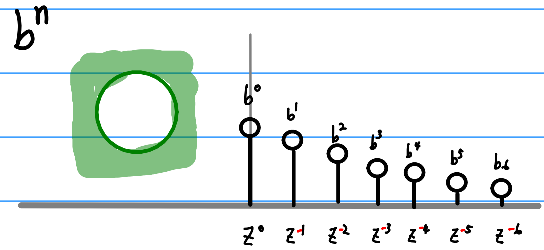
$$= p^{n} \qquad (p = b^{-1})$$





$$x_n = a_n \qquad X(z) = f(z^{-1})$$



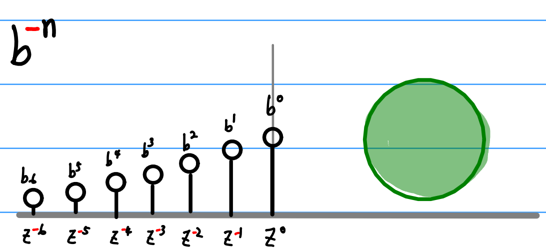


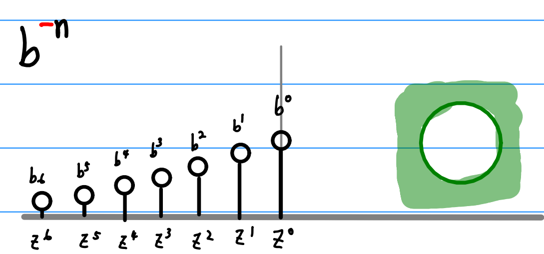
$$\boxed{f(z) = \frac{1}{1-(bz)} \quad |z| < b^{-1}} \longleftrightarrow \boxed{X(z) = \frac{1}{1-(b/z)} \quad |z| > b}$$

$a_n = b^n \quad (n \geqslant 0)$ $\qquad$ $x_n = b^n \quad (n \geqslant 0)$

$$\boxed{f(z) = \frac{1}{1-(b/z)} \quad |z| > b} \longleftrightarrow \boxed{X(z) = \frac{1}{1-(bz)} \quad |z| < b^{-1}}$$

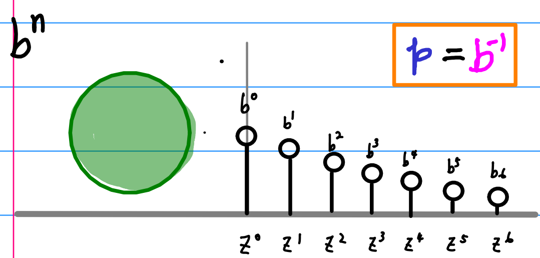
$a_n = b^{-n} \quad (n \leqslant 0)$ $\qquad$ $x_n = b^{-n} \quad (n \leqslant 0)$





$a_n = p^n$

$x_n = p^{-n}$

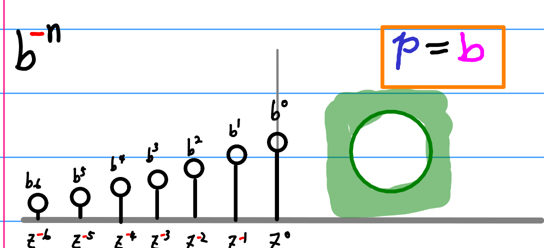


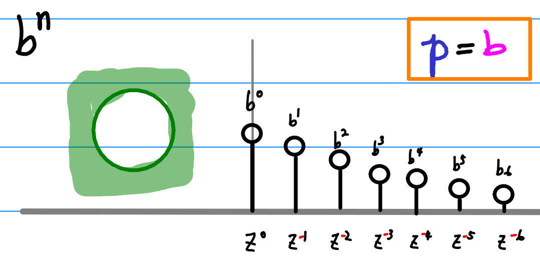
$$f(z) = \frac{1}{1-(bz)} \qquad |z| < b^{-1}$$

$$\boxed{a_n = p^{-n} \quad (n \geq 0)}$$

$$f(z) = \frac{1}{1-(b/z)} \qquad |z| > b$$

$$\boxed{a_n = p^{-n} \quad (n \leq 0)}$$



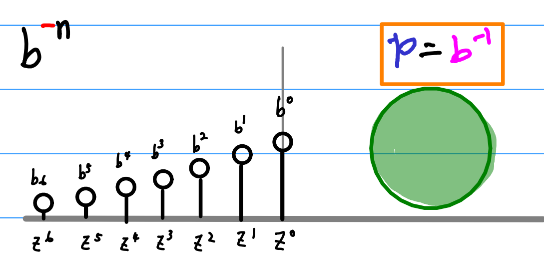


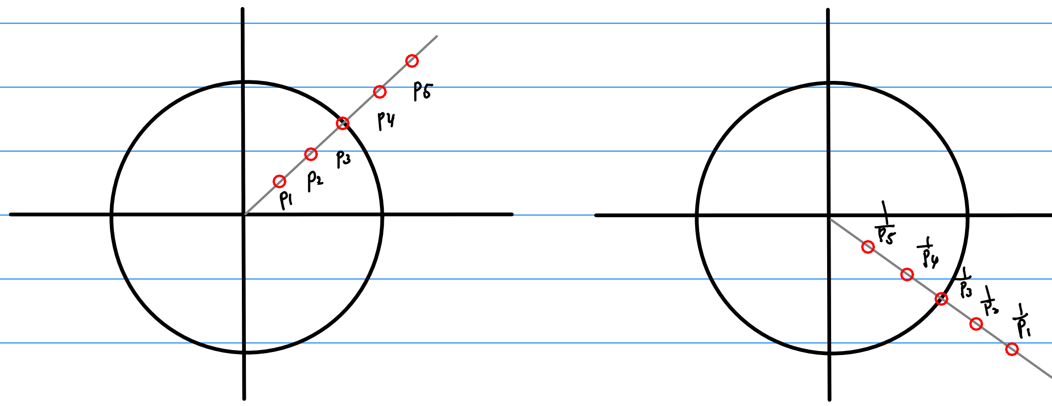
$$X(z) = \frac{1}{1-(b/z)} \qquad |z| > b$$

$$\boxed{x_n = p^n \quad (n \geq 0)}$$

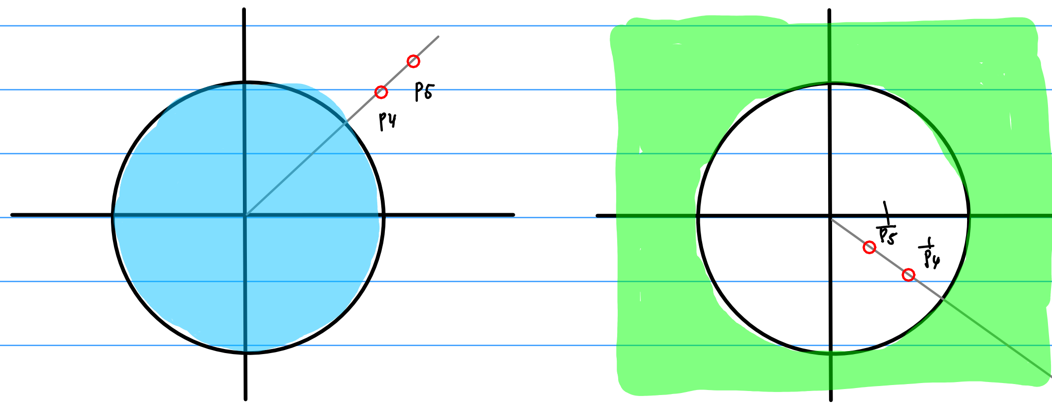
$$X(z) = \frac{1}{1-(bz)} \qquad |z| < b^{-1}$$

$$\boxed{x_n = p^n \quad (n \leq 0)}$$

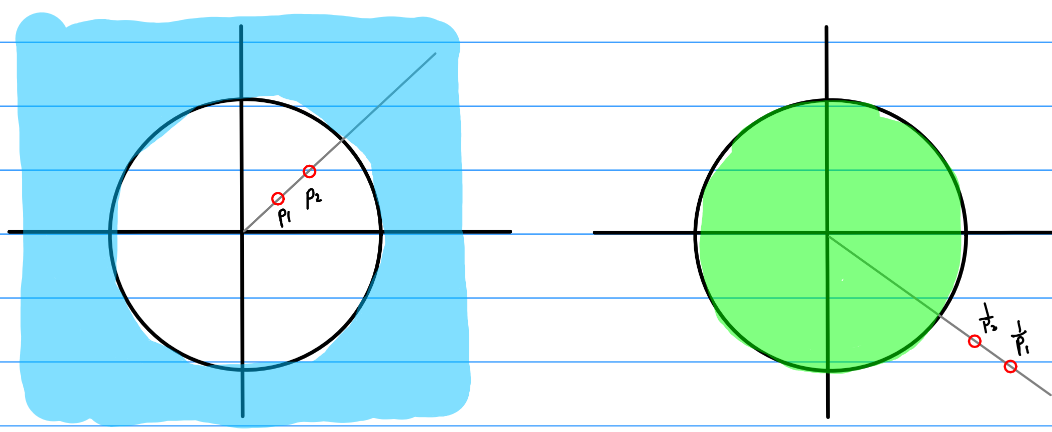




causal



anticausal



two-sided