Variable Block Adder (1C)

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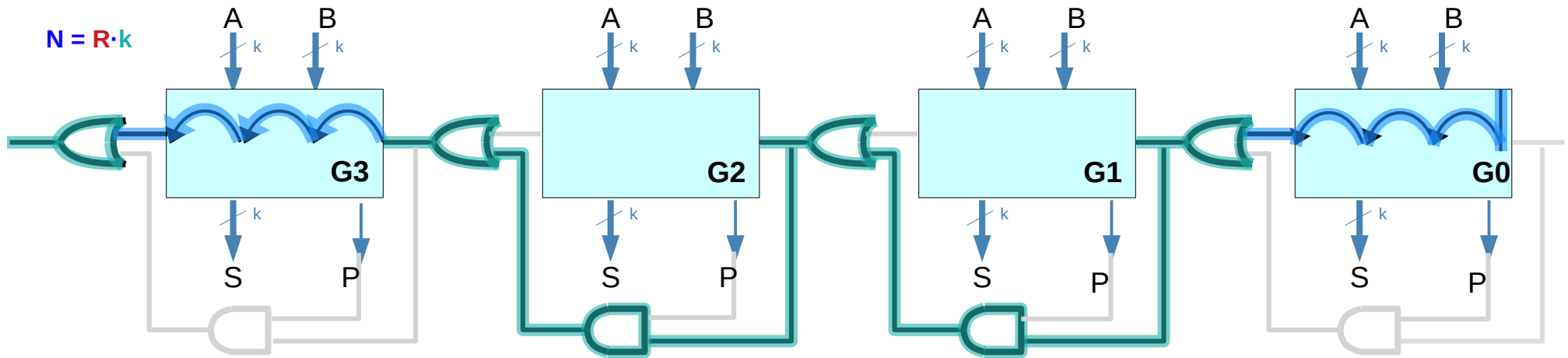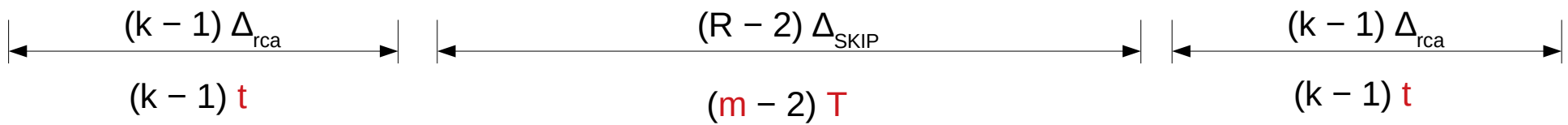Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Carry Skip Adder

$N = R \cdot k$

A B A B A B A B

k k k k k k k k

G3 G2 G1 G0

k k k k

S P S P S P S P

Fixed block size = k bits

$(k - 1) \Delta_{rca}$

$(R - 2) \Delta_{SKIP}$

$(k - 1) \Delta_{rca}$

$(k - 1)\ t$

$(m - 2)\ T$

$(k - 1)\ t$

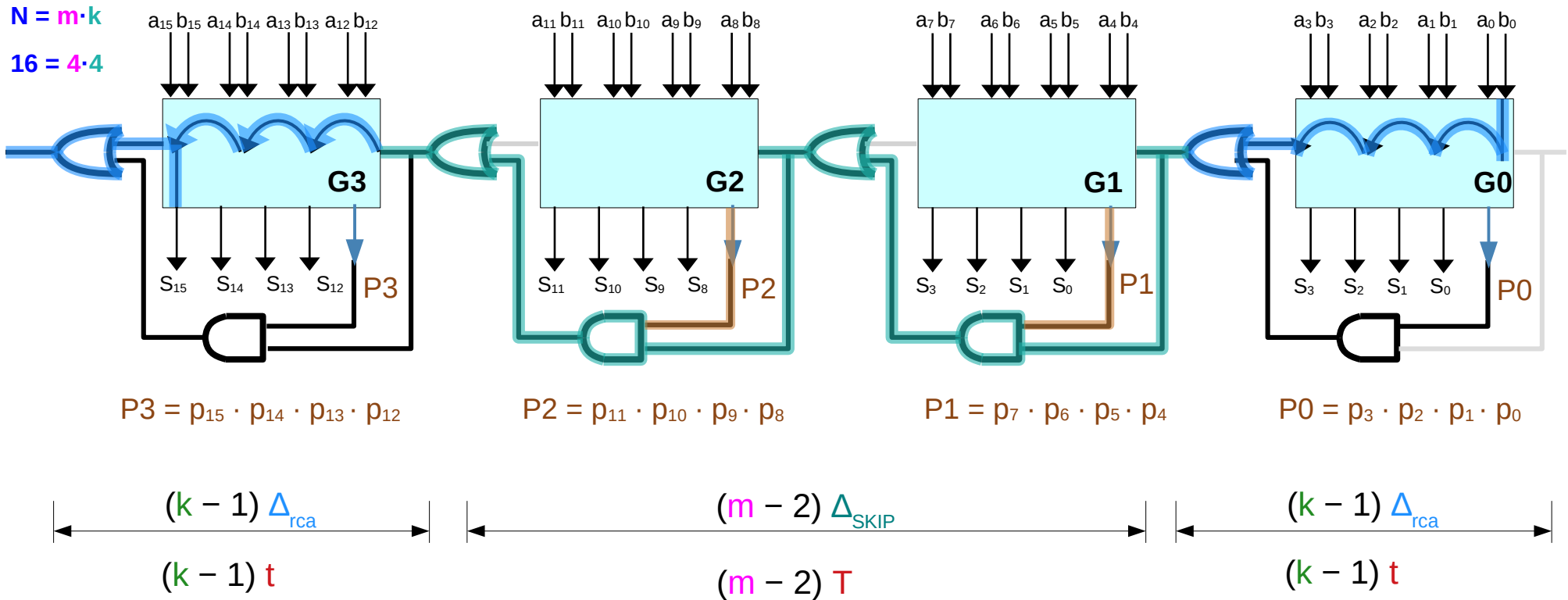**Variable block size = x_i bits for the i-th group**

$(x_i - 1)\ t$

$(m - 2)\ T$

$(x_j - 1)\ t$

t denote the time required for a carry signal to ripple across a bit
T denote the time required for the signal to skip over a group of bits
m denotes the optimal number of groups for an n-bit carry chain

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Carry Skip Adder – fixed block size

$N = m \cdot k$

$16 = 4 \cdot 4$

$a_{15}\, b_{15}\ a_{14}\, b_{14}\ a_{13}\, b_{13}\ a_{12}\, b_{12}$ $a_{11}\, b_{11}\ a_{10}\, b_{10}\ a_9\, b_9\ a_8\, b_8$ $a_7\, b_7\ a_6\, b_6\ a_5\, b_5\ a_4\, b_4$ $a_3\, b_3\ a_2\, b_2\ a_1\, b_1\ a_0\, b_0$

G3   **G2**   **G1**   **G0**

$S_{15}\ S_{14}\ S_{13}\ S_{12}$ P3 $S_{11}\ S_{10}\ S_9\ S_8$ P2 $S_3\ S_2\ S_1\ S_0$ P1 $S_3\ S_2\ S_1\ S_0$ P0

$P3 = p_{15} \cdot p_{14} \cdot p_{13} \cdot p_{12}$ $P2 = p_{11} \cdot p_{10} \cdot p_9 \cdot p_8$ $P1 = p_7 \cdot p_6 \cdot p_5 \cdot p_4$ $P0 = p_3 \cdot p_2 \cdot p_1 \cdot p_0$

$(k-1)\,\Delta_{rca}$ $(m-2)\,\Delta_{SKIP}$ $(k-1)\,\Delta_{rca}$

$(k-1)\,t$ $(m-2)\,T$ $(k-1)\,t$

t denote the time required for a carry signal to ripple across a bit
T denote the time required for the signal to skip over a group of bits
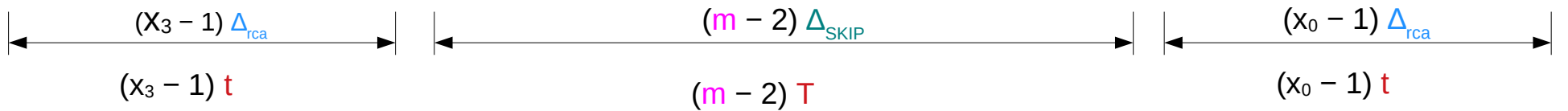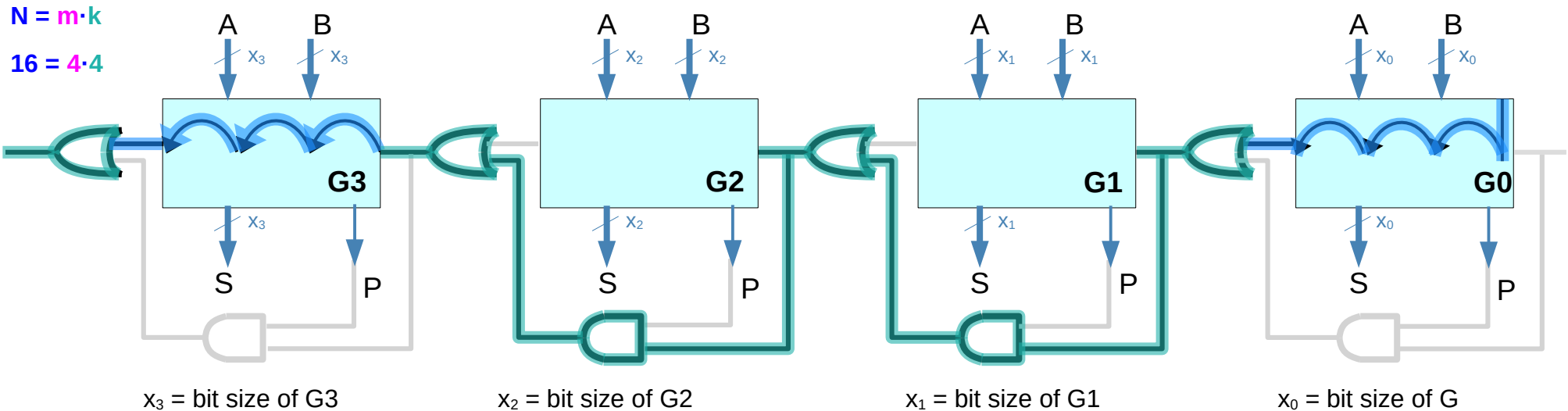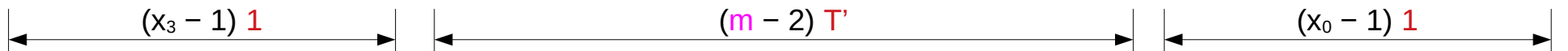m denotes the optimal number of groups for an n-bit carry chain

Fixed Block Size ➡ delay(P3) = delay(P2) = delay(P1) = delay(P0) = Fixed Delay

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

$N = m \cdot k$

$16 = 4 \cdot 4$

x_3 = bit size of G3

x_2 = bit size of G2

x_1 = bit size of G1

x_0 = bit size of G

$(X_3 - 1)\, \Delta_{rca}$

$(m - 2)\, \Delta_{SKIP}$

$(x_0 - 1)\, \Delta_{rca}$

$(x_3 - 1)\, t$

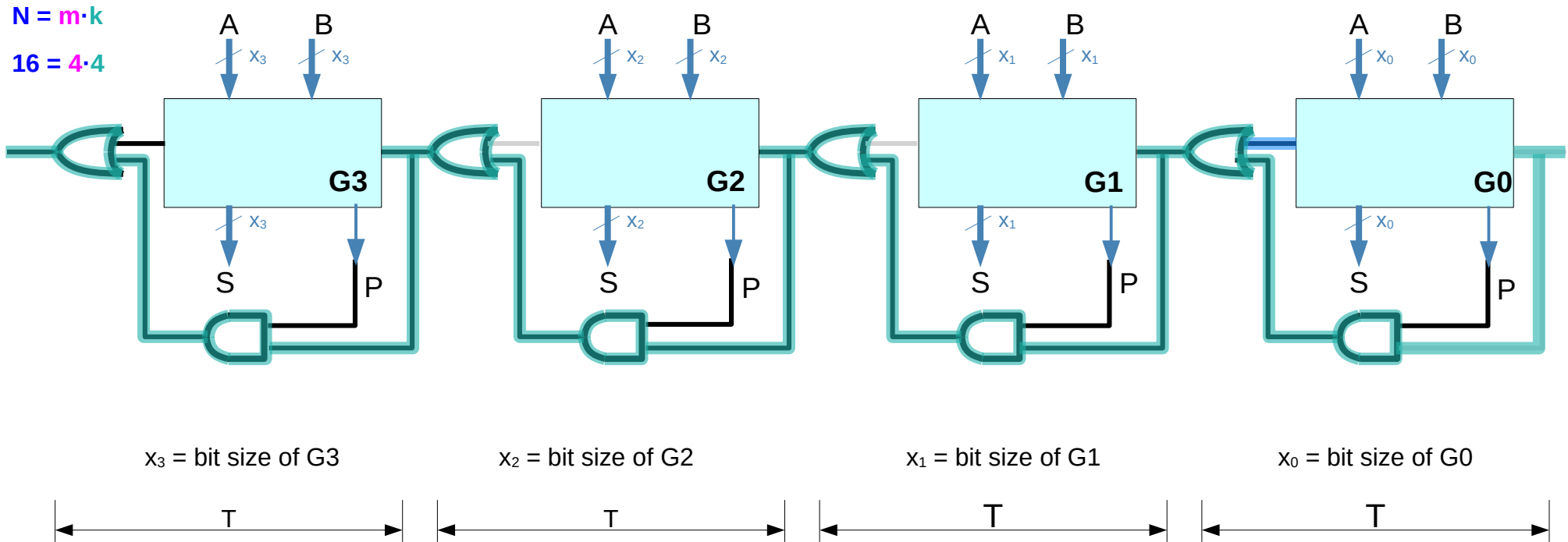$(m - 2)\, T$

$(x_0 - 1)\, t$

t denote the time required for a carry signal to ripple across a bit
T denote the time required for the signal to skip over a group of bits
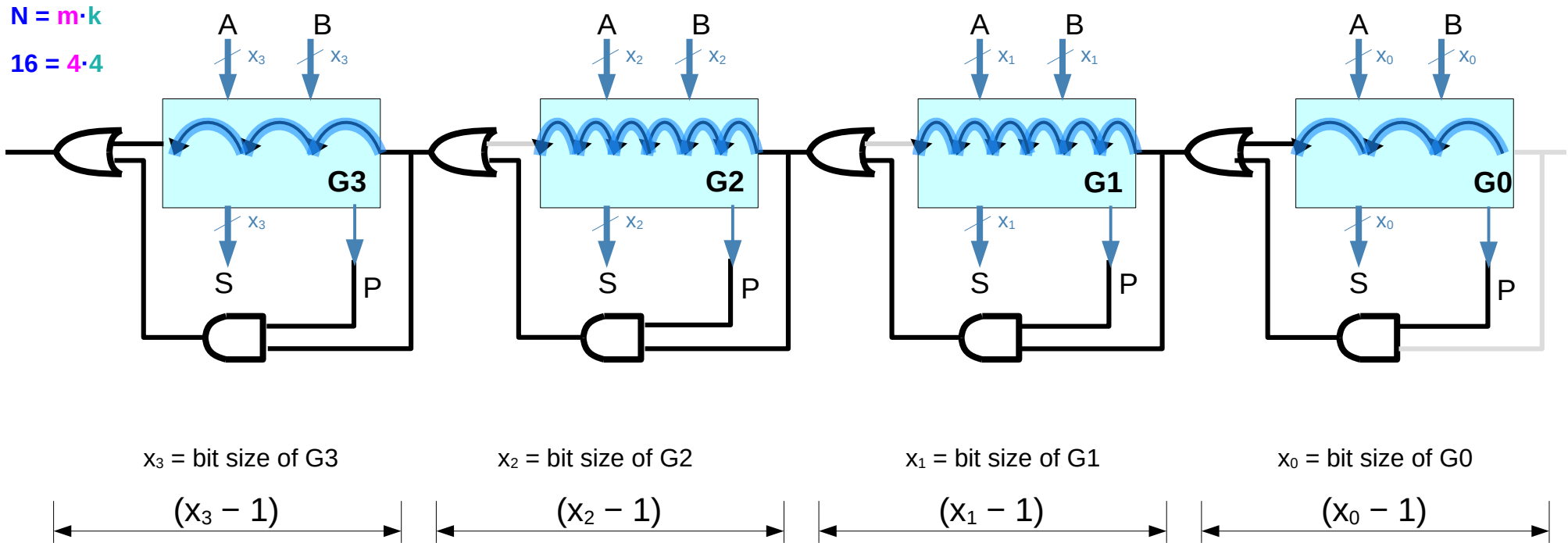m denotes the optimal number of groups for an n-bit carry chain

$(x_3 - 1)\, 1$

$(m - 2)\, T'$

$(x_0 - 1)\, 1$

T' normalized delay of T over t

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Variable Block Adder (1C)

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$N = m \cdot k$

$16 = 4 \cdot 4$

x_3 = bit size of G3 x_2 = bit size of G2 x_1 = bit size of G1 x_0 = bit size of G0

Carry Skip Delays

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Carry Skip Adder – maximum carry delay (3)

$N = m \cdot k$

$16 = 4 \cdot 4$

x_3 = bit size of G3 $\qquad$ x_2 = bit size of G2 $\qquad$ x_1 = bit size of G1 $\qquad$ x_0 = bit size of G0

$(x_3 - 1)$ $\qquad\qquad$ $(x_2 - 1)$ $\qquad\qquad$ $(x_1 - 1)$ $\qquad\qquad$ $(x_0 - 1)$

Carry Ripple delays

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Variable Block Adder (1C)

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Skip path delays and ripple delays

- n bits
- m groups
- x_i group size

lsb **msb**

$mT = 6T$

T T T T T T

skip delay1 iT

1 2 $i=3$ 4 5 $m=6$

bypass bypass
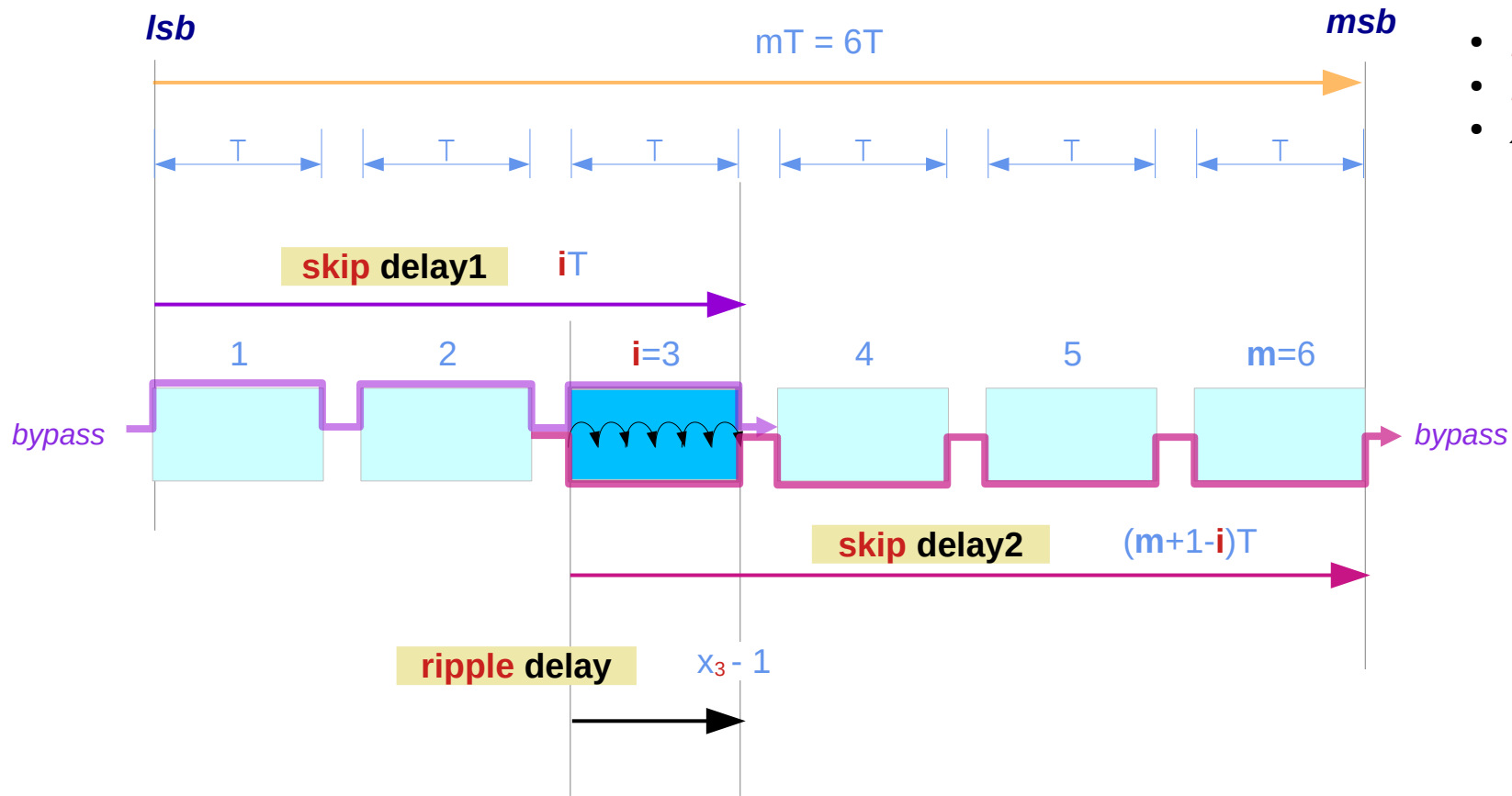
skip delay2 $(m+1-i)T$

ripple delay $x_3 - 1$

ripple delay $\leq$ **skip** delay1
ripple delay $\leq$ **skip** delay2

ripple delay $\leq$ min (**skip** delay1, **skip** delay2)

Overlapping Delay Paths

- x_i group size

$$ n = \sum_{i=1}^{m} x_i \leq \sum_{i=1}^{m} y_i $$

m=1   2   3   4   5   6

bypass

bypass

skip delay1   3T

terminating carry

ripple delay $x_3 - 1$

generating carry

skip delay2   (m+1-3)T

ripple delay $\leq$ skip delay1

ripple delay $\leq$ skip delay2

m = 2;
$y_1 = 1+T$
$y_2 = 1+T$

m = 3;
$y_1 = 1+T$
$y_2 = 1+2T$
$y_3 = 1+T$

m = 4;
$y_1 = 1+T$
$y_2 = 1+2T$
$y_3 = 1+2T$
$y_4 = 1+T$

m = 5;
$y_1 = 1+T$
$y_2 = 1+2T$
$y_3 = 1+3T$
$y_4 = 1+2T$
$y_5 = 1+T$

m = 6;
$y_1 = 1+T$
$y_2 = 1+2T$
$y_3 = 1+3T$
$y_4 = 1+3T$
$y_6 = 1+T$
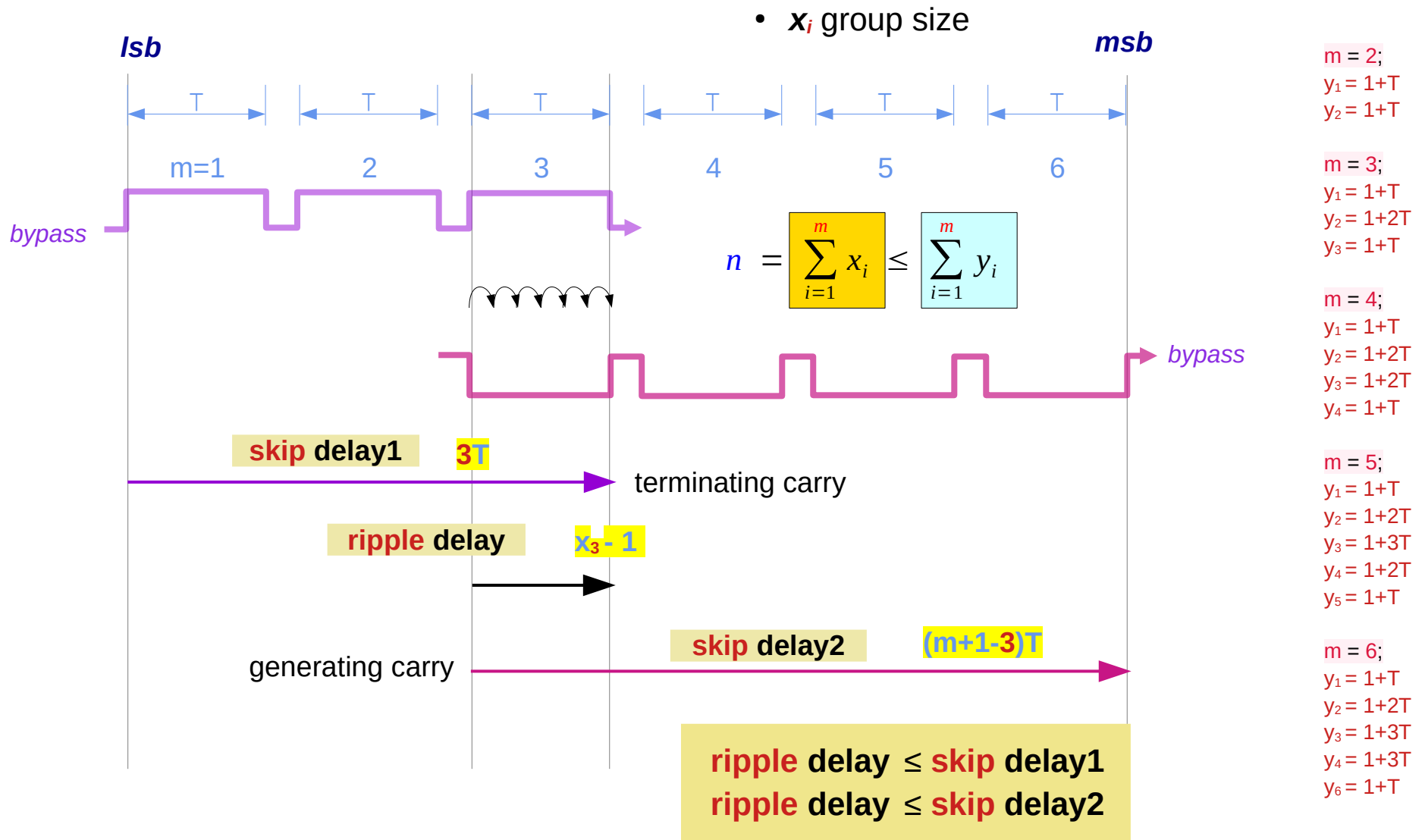
Minimum skip path delay ($y_i - 1$) of the ith group
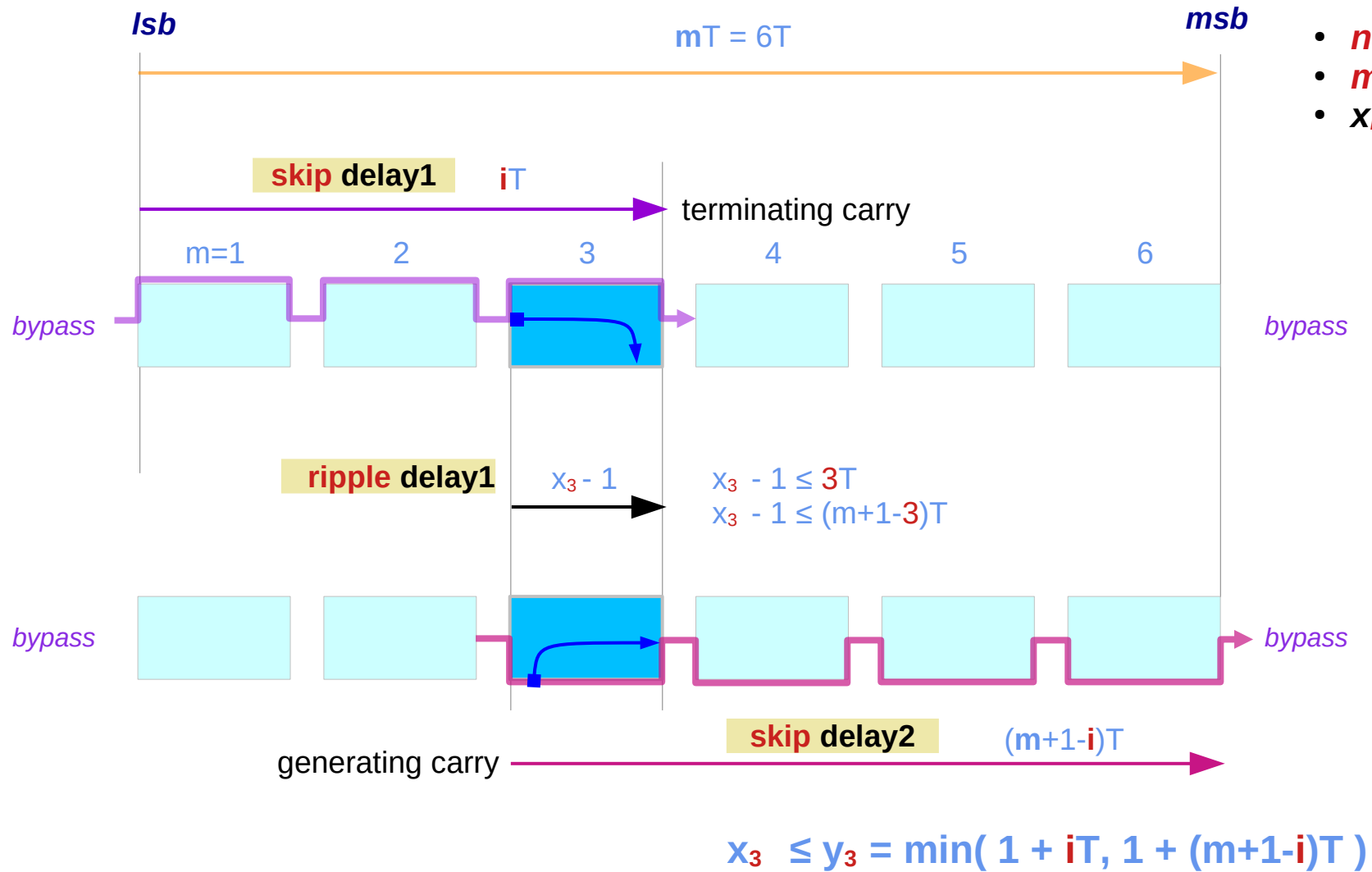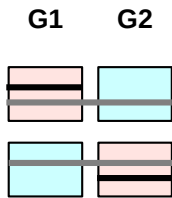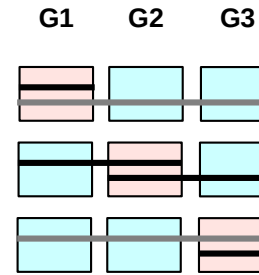
- n bits
- m groups
- x_i group size

lsb $mT = 6T$ msb

skip **delay1** iT

terminating carry

m=1   2   3   4   5   6

bypass   bypass

ripple **delay1** $x_3 - 1$ $x_3 - 1 \leq 3T$
$x_3 - 1 \leq (m+1-3)T$

bypass   bypass

generating carry   skip **delay2** $(m+1-i)T$

$$x_3 \leq y_3 = \min(1 + iT, 1 + (m+1-i)T)$$

Minimum skip path delay of the ith group (1)

$m = 2$;

$min(T, 2T) = T$

$min(2T, T) = T$

G1   **G2**

$m = 3$;

$min(T, 3T) = T$

$min(2T, 2T) = 2T$

$min(3T, T) = T$

G1   **G2**   **G3**

$m = 4$;

$min(T, 4T) = T$

$min(2T,3T) = 2T$

$min(3T,2T) = 2T$

$min(4T,T) = T$

G1   **G2**   **G3**   **G4**

$m = 5$;

$min(T, 5T) = T$

$min(2T, 4T) = 2T$

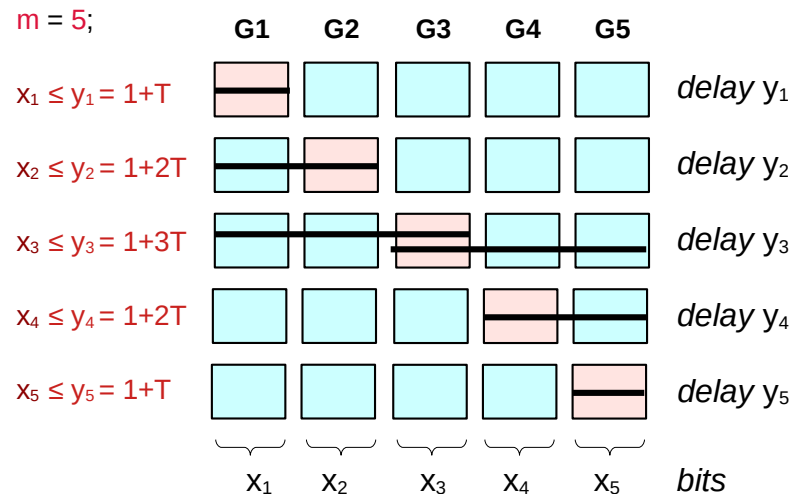$min(3T, 3T) = 3T$

$min(4T, 2T) = 2T$

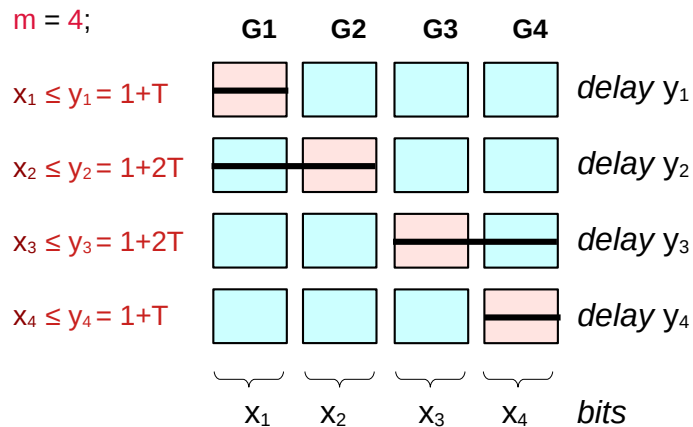$min(5T, T) = T$

G1   **G2**   **G3**   **G4**   **G5**

$$y_i = 1 + min\{iT, (m+1-i)T\}$$

➡
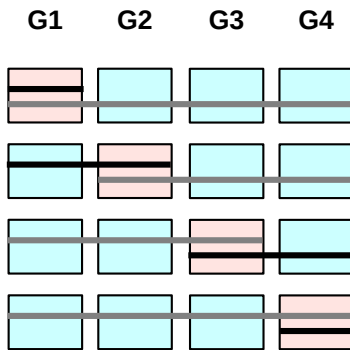
$$y_i = min\{1+iT, 1+(m+1-i)T\}, \quad i = 1, ..., m$$
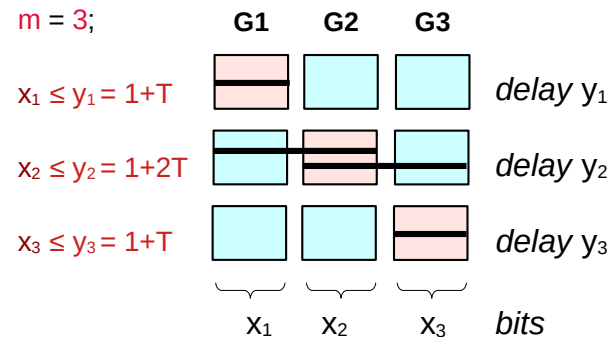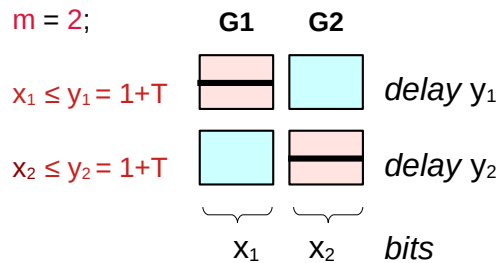
Minimum skip path delay of the ith group (2)

m = 6;

min(T, 6T) = T

min(2T, 5T) = 2T

min(3T, 4T) = 3T

min(4T, 3T) = 3T

min(5T, 2T) = 2T

min(6T, T) = T

| | G1 | G2 | G3 | G4 | G5 | G6 |

m = 7;

min(T, 7T) = T

min(2T, 6T) = 2T

min(3T, 5T) = 3T

min(4T, 4T) = 4T

min(5T, 3T) = 3T

min(6T, 2T) = 2T

min(7T, T) = T

| | G1 | G2 | G3 | G4 | G5 | G6 | G7 |
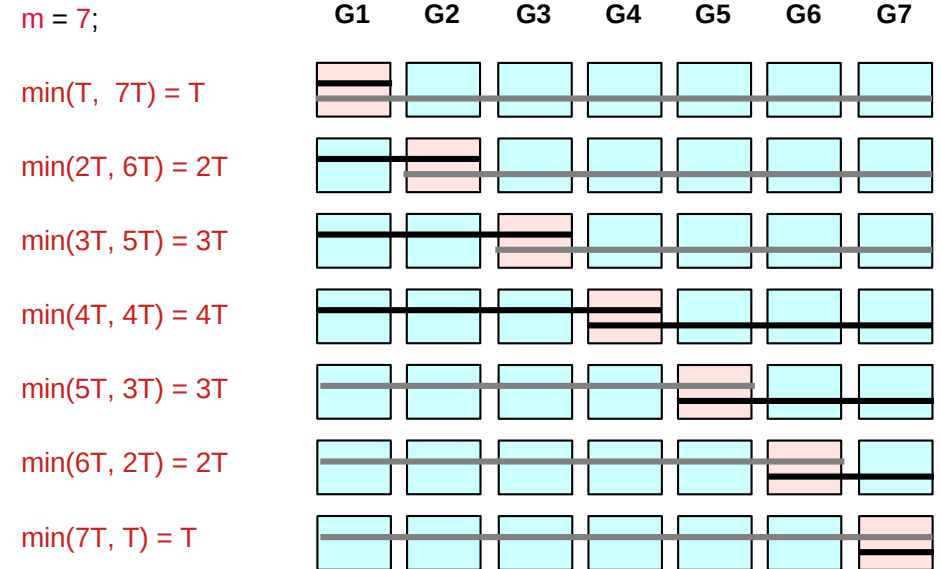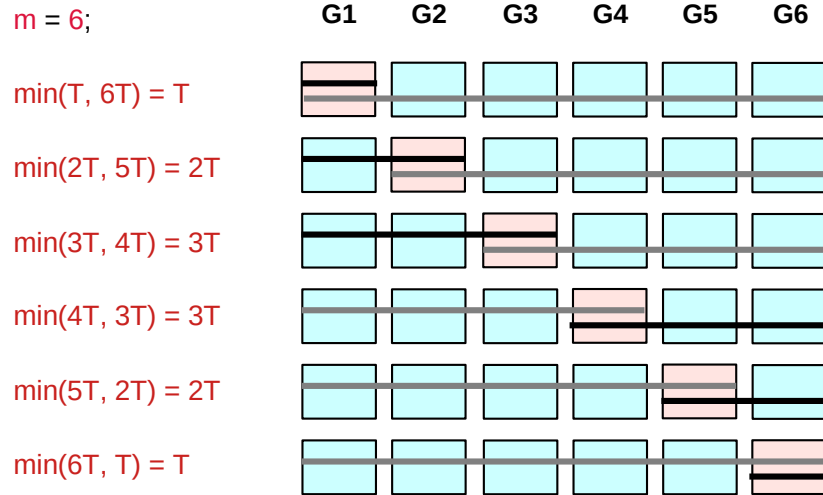
$$y_i = 1 + min\{iT, (m+1-i)T\}$$ ➡ $$y_i = min\{1+iT, 1+(m+1-i)T\}, \quad i = 1,...,m$$
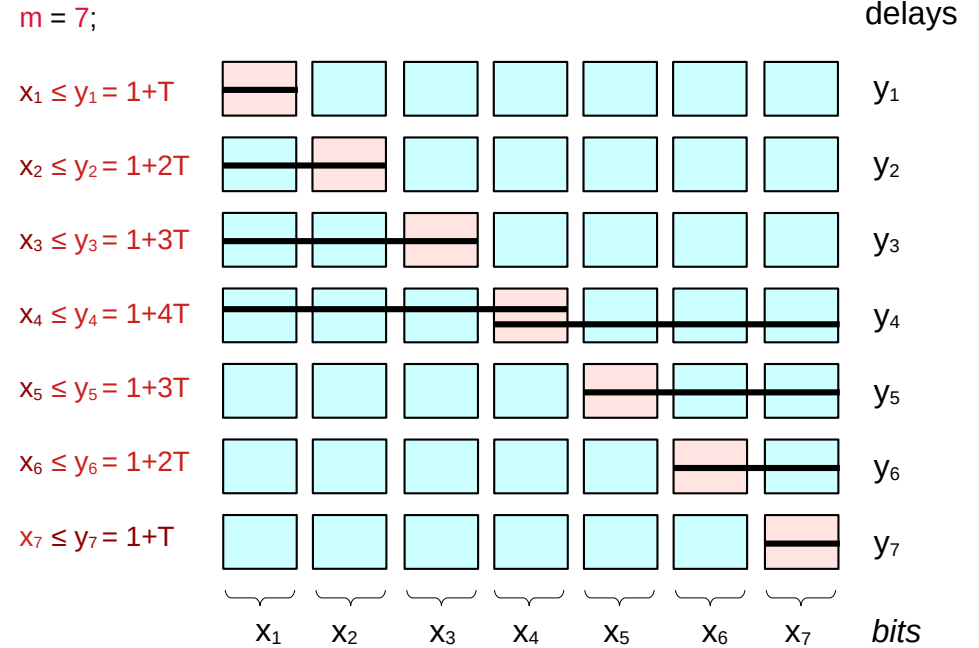
The ith group has x_i bits for a given m (1)

m = 2; **G1   G2**

$x_1 \leq y_1 = 1+T$ *delay* y_1

$x_2 \leq y_2 = 1+T$ *delay* y_2

x_1 x_2 *bits*

m = 3; **G1   G2   G3**

$x_1 \leq y_1 = 1+T$ *delay* y_1

$x_2 \leq y_2 = 1+2T$ *delay* y_2

$x_3 \leq y_3 = 1+T$ *delay* y_3

x_1 x_2 x_3 *bits*

m = 4; **G1   G2   G3   G4**

$x_1 \leq y_1 = 1+T$ *delay* y_1

$x_2 \leq y_2 = 1+2T$ *delay* y_2

$x_3 \leq y_3 = 1+2T$ *delay* y_3

$x_4 \leq y_4 = 1+T$ *delay* y_4

x_1 x_2 x_3 x_4 *bits*

m = 5; **G1   G2   G3   G4   G5**

$x_1 \leq y_1 = 1+T$ *delay* y_1

$x_2 \leq y_2 = 1+2T$ *delay* y_2

$x_3 \leq y_3 = 1+3T$ *delay* y_3

$x_4 \leq y_4 = 1+2T$ *delay* y_4

$x_5 \leq y_5 = 1+T$ *delay* y_5
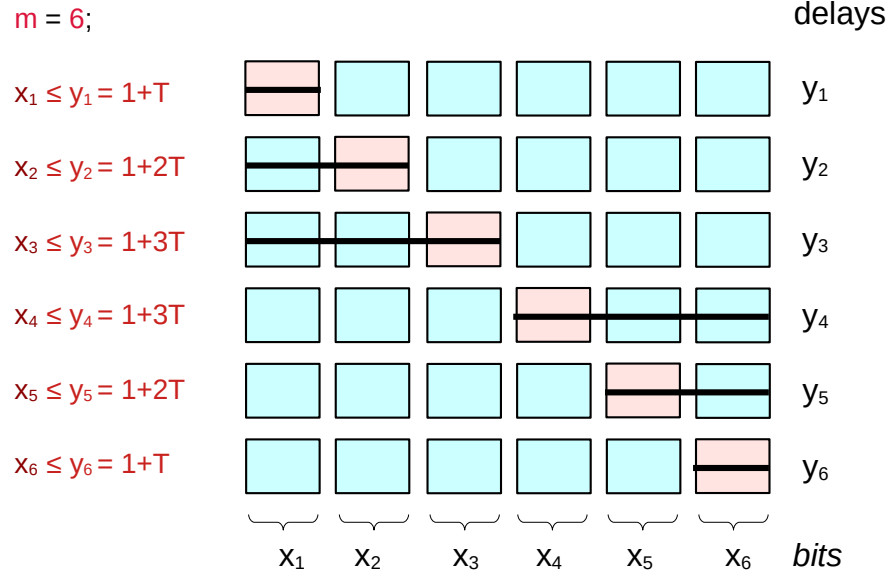
x_1 x_2 x_3 x_4 x_5 *bits*
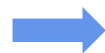
$$x_i - 1 \leq \min(\ iT, (m+1-i)T\ )$$

$$x_i \leq y_i = \min(\ 1 + iT, 1 + (m+1-i)T\ )$$

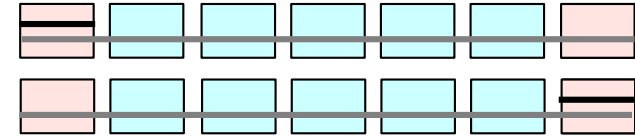# The ith group has x_i bits for a given m (2)

m = 6;

delays

$x_1 \leq y_1 = 1+T$ y_1

$x_2 \leq y_2 = 1+2T$ y_2

$x_3 \leq y_3 = 1+3T$ y_3

$x_4 \leq y_4 = 1+3T$ y_4

$x_5 \leq y_5 = 1+2T$ y_5

$x_6 \leq y_6 = 1+T$ y_6

x_1 x_2 x_3 x_4 x_5 x_6 *bits*

m = 7;

delays

$x_1 \leq y_1 = 1+T$ y_1

$x_2 \leq y_2 = 1+2T$ y_2

$x_3 \leq y_3 = 1+3T$ y_3

$x_4 \leq y_4 = 1+4T$ y_4

$x_5 \leq y_5 = 1+3T$ y_5

$x_6 \leq y_6 = 1+2T$ y_6

$x_7 \leq y_7 = 1+T$ y_7

x_1 x_2 x_3 x_4 x_5 x_6 x_7 *bits*

$$x_i - 1 \leq \min(\ iT, (m+1-i)T\ )$$

$$x_i \leq y_i = \min(\ 1 + iT, 1 + (m+1-i)T\ )$$

Symmetric histograms of y_i's

$y_1 = min\{1+1 \cdot T, 1+(m+1-1)T\} = 1 + T$

$y_m = min\{1+m \cdot T, 1+(m+1-m)T\} = 1 + T$

$x_1 \leq y_1 = 1+T \quad (bits)$

$x_m \leq y_m = 1+T \quad (bits)$

$y_2 = min\{1+2 \cdot T, 1+(m+1-2)T\} = 1 + 2T$

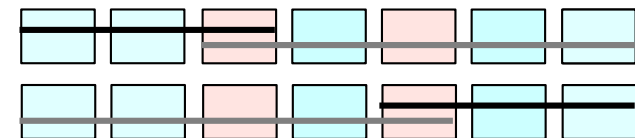$y_{m-1} = min\{1+(m-1) \cdot T, 1+(m+1-(m-1))T\} = 1 + 2T$

$x_2 \leq y_2 = 1+2T \quad (bits)$

$x_{m-1} \leq y_{m-1} = 1+2T \quad (bits)$
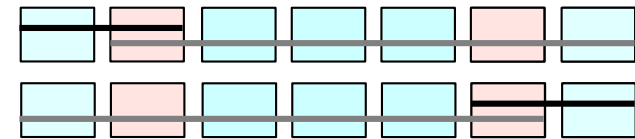
$y_3 = min\{1+3 \cdot T, 1+(m+1-3)T\} = 1 + 3T$

$y_{m-2} = min\{1+(m-2) \cdot T, 1+(m+1-(m-2))T\} = 1 + 3T$

$x_3 \leq y_3 = 1+3T \quad (bits)$

$x_{m-2} \leq y_{m-2} = 1+3T \quad (bits)$

Determining *m* the number of groups (1)

Method 1 – using a histogram

Let *m* be the <u>smallest</u> positive integer such that

$$n \le \sum_{i=1}^{m} y_i$$

m = 2;
while ($y_1 + \cdots + y_m$ < n) m = m+1 ;

$$y_i = min\{1+iT, 1+(m+1-i)T\}, \quad i = 1, \dots, m$$

m = 2; T = 3
y_1 = min {1+T, 1+2T}   = 1+T = 4
y_2 = min {1+2T, 1+T}   = 1+T = 4

m = 3; T = 3
y_1 = min {1+T, 1+3T}   = 1+T = 4
y_2 = min {1+2T, 1+2T} = 1+2T = 7
y_3 = min {1+3T, 1+T}   = 1+T = 4

m = 4; T = 3
y_1 = min {1+T, 1+4T}   = 1+T = 4
y_2 = min {1+2T, 1+3T} = 1+2T = 7
y_3 = min {1+3T, 1+2T} = 1+2T = 7
y_4 = min {1+4T, 1+T}   = 1+T = 4

m = 5; T = 3
y_1 = min {1+T, 1+5T}   = 1+T = 4
y_2 = min {1+2T, 1+4T} = 1+2T = 7
y_3 = min {1+3T, 1+3T} = 1+3T = 10
y_4 = min {1+4T, 1+2T} = 1+2T = 7
y_5 = min {1+5T, 1+T}   = 1+T = 4

m = 6; T = 3
y_1 = min {1+T, 1+6T}   = 1+T = 4
y_2 = min {1+2T, 1+5T} = 1+2T = 7
y_3 = min {1+3T, 1+4T} = 1+3T = 10
y_4 = min {1+4T, 1+3T} = 1+3T = 10
y_5 = min {1+5T, 1+2T} = 1+2T = 7
y_6 = min {1+6T, 1+T}   = 1+T = 4

m = 7; T = 3
y_1 = min {1+T, 1+7T} = 1+T = 4
y_2 = min {1+2T, 1+6T} = 1+2T= 7
y_3 = min {1+3T, 1+5T} = 1+3T= 10
y_4 = min {1+4T, 1+4T} = 1+4T= 13
y_5 = min {1+5T, 1+3T} = 1+3T= 10
y_6 = min {1+6T, 1+2T} = 1+2T= 7
y_7 = min {1+7T, 1+1T} = 1+T = 4

4+4 = 8
4+7+4 = 15
4+7+7+4 = 22
4+7+10+7+4 = 32
4+7+10+10+7+4 = 42
4+7+10+13+10+7+4 = 55

Determining *m* the number of groups (2)

Method 1 – using a histogram

Let *m* be the smallest positive integer such that

$$n \leq \left\lfloor \sum_{i=1}^{m} y_i \right\rfloor$$

$$\boxed{m = 2; \quad \text{while } (y_1 + \cdots + y_m < n) \ m = m+1 ;}$$

$$y_i = min\{1 + iT, 1 + (m+1-i)T\}, \quad i = 1, \dots, m$$

$n = 48$
$T = 3$ ⟹ $m = 7$
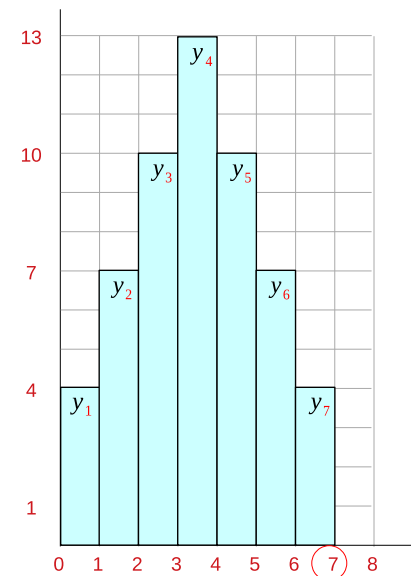
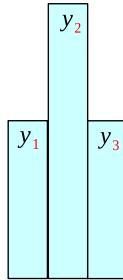| $m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$
$\sum_{i=1}^{2} y_i$	$\sum_{i=1}^{3} y_i$	$\sum_{i=1}^{4} y_i$	$\sum_{i=1}^{5} y_i$	$\sum_{i=1}^{6} y_i$	$\sum_{i=1}^{7} y_i$
$2 + 2 \cdot T$	$3 + 4 \cdot T$	$4 + 6 \cdot T$	$5 + 9 \cdot T$	$6 + 12 \cdot T$	$7 + 16 \cdot T$
$2 + 2 \cdot T < n$	$3 + 4 \cdot T < n$	$4 + 6 \cdot T < n$	$5 + 9 \cdot T < n$	$6 + 12 \cdot T < n$	$7 + 16 \cdot T > n$
$2 + 2 \cdot 3 < 48$	$3 + 4 \cdot 3 < 48$	$4 + 6 \cdot 3 < 48$	$5 + 9 \cdot 3 < 48$	$6 + 12 \cdot 3 < 48$	$7 + 16 \cdot 3 > 48$
$8 < 48$	$15 < 48$	$22 < 48$	$32 < 48$	$42 < 48$	$55 > 48$

$$m = 7$$

4+4 = 8 bits
4+7+4 = 15 bits
4+7+7+4 = 22 bits
4+7+10+7+4 = 32 bits
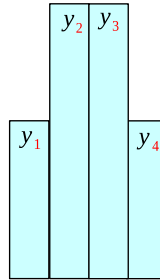4+7+10+10+7+4 = 42 bits
4+7+10+13+10+7+4 = 55 bits

no of groups
m = 2; T = 3
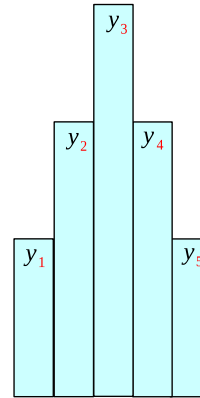y_1 = 1+T = 4
y_2 = 1+T = 4
        --
        8

no of groups
m = 3; T = 3
y_1 = 1+T = 4
y_2 = 1+2T = 7
y_3 = 1+T = 4
        --
        15
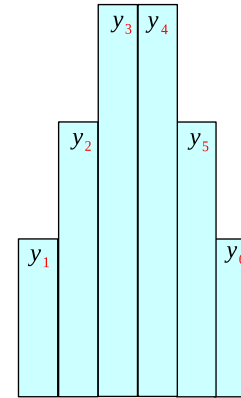
no of groups
m = 4; T = 3
y_1 = 1+T = 4
y_2 = 1+2T= 7
y_3 = 1+2T= 7
y_4 = 1+T = 4
        --
        22

no of groups
m = 5; T = 3
y_1 = 1+T = 4
y_2 = 1+2T = 7
y_3 = 1+3T = 10
y_4 = 1+2T = 7
y_5 = 1+T = 4
        --
        32

no of groups
m = 6; T = 3
y_1 = 1+T = 4
y_2 = 1+2T = 7
y_3 = 1+3T = 10
y_4 = 1+3T = 10
y_5 = 1+2T = 7
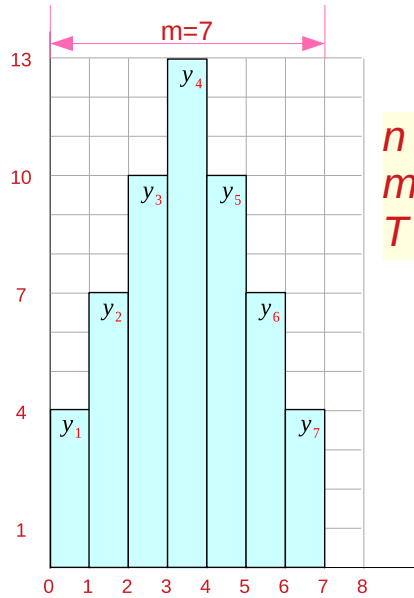y_6 = 1+T = 4
        --
        42
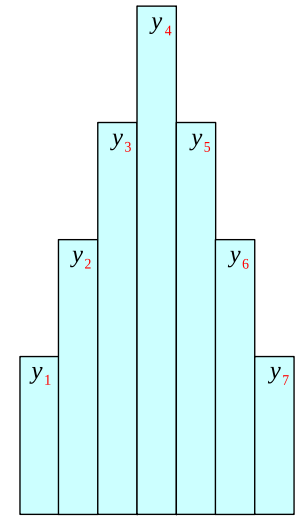
no of groups
m = 7; T = 3
y_1 = 1+T = 4
y_2 = 1+2T = 7
y_3 = 1+3T = 10
y_4 = 1+4T = 13
y_5 = 1+3T = 10
y_6 = 1+2T = 7
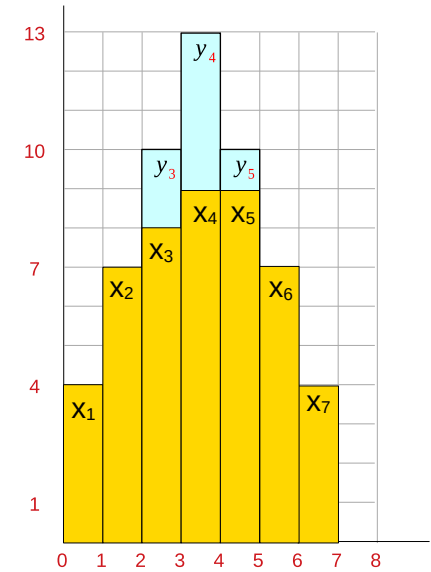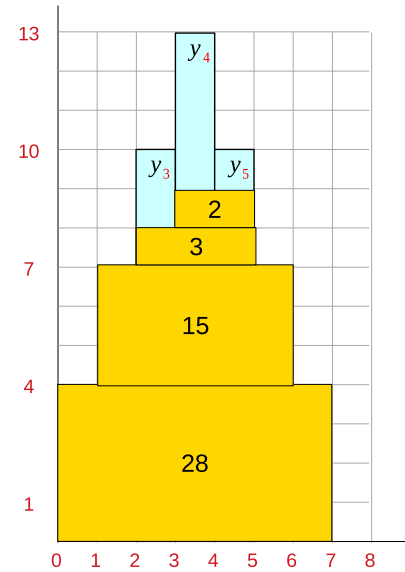y_7 = 1+T = 4
        --
        55

no of bits
n ≤ 8

no of bits
8 < n ≤ 15

no of bits
15 < n ≤ 22

no of bits
22 < n ≤ 32

no of bits
32 < n ≤ 42

no of bits
42 < n ≤ 55

n =48
T =3   →   *m* =7

Determining x_i the group size of the ith group

$n = 48$
$m = 7$
$T = 3$
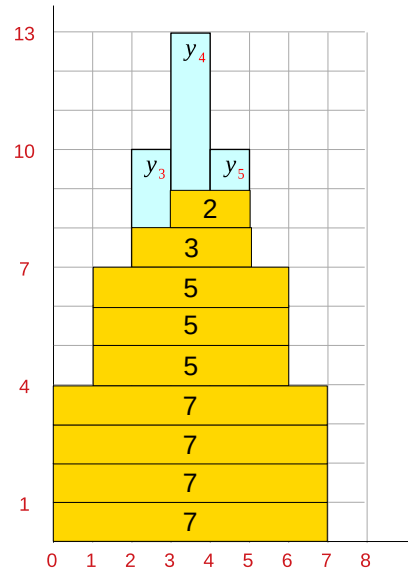
7	7
7	14
7	21
7	28
5	33
5	38
5	43
3	46
2	48

4 * 7 = 28	28
3 * 5 = 15	43 = 28 + 15
1 * 3 = 3	46 = 28 + 15 + 3
1 * 2 = 2	48 = 28 + 15 + 3 + 2

$x_1 = 4 \leq y_1 = 4$
$x_2 = 7 \leq y_2 = 7$
$x_3 = 8 < y_3 = 10$
$x_4 = 9 < y_4 = 13$
$x_5 = 9 < y_5 = 10$
$x_6 = 7 \leq y_6 = 7$
$x_7 = 4 \leq y_7 = 4$

$m = 7$ groups

Procedure

(I) Let *m* be the smallest positive integer such that

$$n \le \boxed{m + \frac{1}{2}mT + \frac{1}{4}m^2T + \left(1 - (-1)^m\right)\frac{1}{8}T} = \boxed{\sum_{i=1}^{m} y_i}$$

- total *n* =48 bits
- *m* =7 groups
- *i*-th group has x_i bits (size)
- constant skip delay $T = T(x_i) = 3$

(II) Let

$$\boxed{y_i = min\{1 + iT, 1 + (m+1-i)T\}, \quad i = 1, \ldots, m}$$

and construct a histogram whose *i-th* column has height y_i

for example, for T=3, and n=48, we have m=7

(III) It is easily verified that the area of the histogram in (II) is

$$\boxed{\sum_{i=1}^{m} y_i} = \boxed{m + \frac{1}{2}mT + \frac{1}{4}m^2T + \left(1 - (-1)^m\right)\frac{1}{8}T} \ge n$$

so these are at least *n* <u>unit squares</u> in the histogram
starting with the first row, shade in *n* of the squares, <u>row by row</u>
Let x_i denote the number of shaded squares in column *i* of the histogram,

$i = 1, \ldots, m$

$$n = \boxed{\sum_{i=1}^{m} x_i} \le \boxed{\sum_{i=1}^{m} y_i}$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Procedure - determining *m* the number of groups
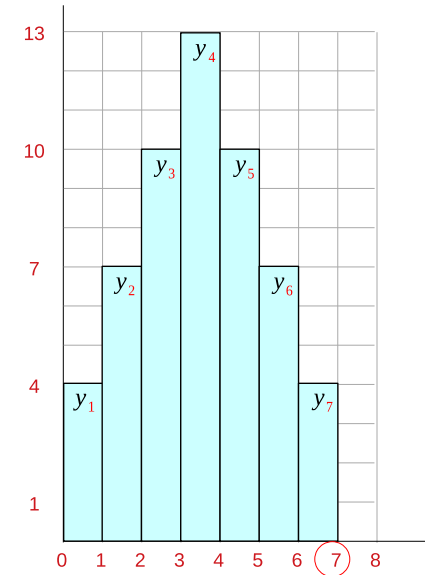
Method 1 – using a histogram

Let m be the smallest positive integer such that

$$n \leq \sum_{i=1}^{m} y_i$$

m = 2;
while (y₁+···+yₘ < n) m = m+1 ;

$$y_i = min\{1+iT, 1+(m+1-i)T\}, \quad i = 1,...,m$$

Method 2 – using a *closed formula*

Let m be the smallest positive integer such that

$$n \leq m + \frac{1}{2}mT + \frac{1}{4}m^2T + \left(1-(-1)^m\right)\frac{1}{8}T$$

$m = 2k$

$\frac{m}{2} = k$

$$y_i = min\{1+iT, \quad 1+(m+1-i)T\}, \quad i = 1,...,m \qquad m \; \tfrac{1}{2}k(k+1)$$

$y_1 = min\{1+1\cdot T, \quad 1+(m-0)\cdot T\} \quad 0 \leq x_1 \leq 1+1\cdot T$
$y_2 = min\{1+2\cdot T, \quad 1+(m-1)\cdot T\} \quad 0 \leq x_2 \leq 1+2\cdot T$
$y_3 = min\{1+3\cdot T, \quad 1+(m-2)\cdot T\} \quad 0 \leq x_3 \leq 1+3\cdot T$

$y_k = min\{1+k\cdot T, \quad 1+(k+1)\cdot T\} \quad 0 \leq x_k \leq 1+k\cdot T$
$y_{k+1} = min\{1+(k+1)\cdot T, \quad 1+k\cdot T\} \quad 0 \leq x_{k+1} \leq 1+k\cdot T$

$y_{m-2} = min\{1+(m-2)\cdot T, \quad 1+3\cdot T\} \quad 0 \leq x_{m-2} \leq 1+3\cdot T$
$y_{m-1} = min\{1+(m-1)\cdot T, \quad 1+2\cdot T\} \quad 0 \leq x_{m-1} \leq 1+2\cdot T$
$y_{m-0} = min\{1+(m-0)\cdot T, \quad 1+1\cdot T\} \quad 0 \leq x_{m-0} \leq 1+1\cdot T$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers
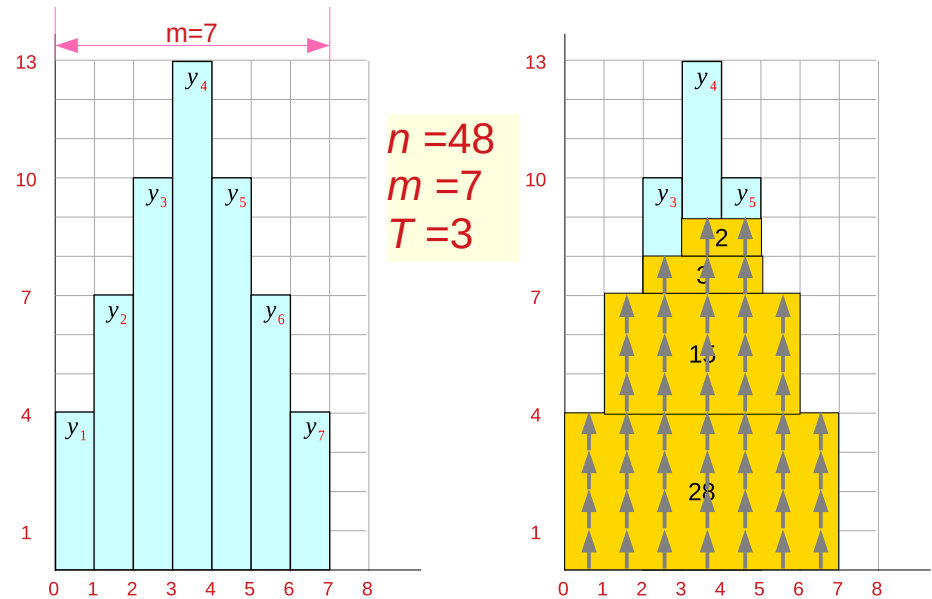
$$0 \leq x_i \leq y_i, i = 1,...,m \qquad \tfrac{1}{2}k(k+1)$$

construct a histogram
whose *i-th* column has height y_i

so these y_i's are at least n unit squares
in the histogram, starting with the first row,
shade in n of the squares, row by row

let x_i denote the number of shaded squares
in column i of the histogram,
$\quad i = 1, \ldots, m$

$n = 48$
$m = 7$
$T = 3$

$$0 \leq x_i \leq y_i, \qquad i = 1, \ldots, m$$

$$n = \sum_{i=1}^{m} x_i \leq \sum_{i=1}^{m} y_i$$

$$n = \sum_{i=1}^{7} x_i$$

$n = 4+7+ 8+ 9+ 9+7+4 = 48$
$\quad \leq 4+7+10+13+10+7+4 = 55$

N = m·k

16 = 4·4

A   B

x_3 x_3

G3

x_3

S   P

x_3 = bit size of G3

T
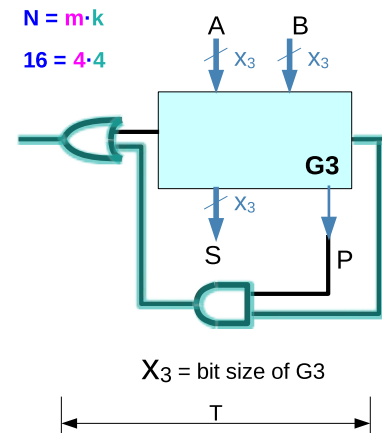
m = 7;		T = 3
$x_1 = 4$	$\leq$	$y_1 = 4$
$x_2 = 7$	$\leq$	$y_2 = 7$
$x_3 = 8$	$<$	$y_3 = 10$
$x_4 = 9$	$<$	$y_4 = 13$
$x_5 = 9$	$<$	$y_5 = 10$
$x_6 = 7$	$\leq$	$y_6 = 7$
$x_7 = 4$	$\leq$	$y_7 = 4$

Procedure

(I) Let m be the smallest positive integer

$$n \leq \sum_{i=1}^{m} y_i \qquad i = 1, \ldots, m$$

(II) Let

$$y_i = min\{1+iT, 1+(m+1-i)T\}$$

(III) Let x_i, $i = 1, \ldots, m$

starting with the first row, <u>row by row</u>

$$n = \sum_{i=1}^{m} x_i \leq \sum_{i=1}^{m} y_i$$

Variable block size = x_i bits for the i-th group

the scheme (i), (ii), (iii)
gives the **max propagation time** mT

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

find the <u>smallest</u> m

$$n \leq \sum_{i=1}^{m} y_i = \sum_{i=1}^{m} min\{1+iT, 1+(m+1-i)T\}$$

$m = 2$;
while ($y_1 + \cdots + y_m$ < n) m = m+1 ;

m = 2;	m = 4;	m = 5;	m = 6;	m = 7;
$y_1 = 1+T$	$y_1 = 1+T$	$y_1 = 1+T$	$y_1 = 1+T$	$y_1 = 1+T$
$y_2 = 1+T$	$y_2 = 1+2T$	$y_2 = 1+2T$	$y_2 = 1+2T$	$y_2 = 1+2T$
	$y_3 = 1+2T$	$y_3 = 1+3T$	$y_3 = 1+3T$	$y_3 = 1+3T$
m = 3;	$y_4 = 1+T$	$y_4 = 1+2T$	$y_4 = 1+3T$	$y_4 = 1+4T$
$y_1 = 1+T$		$y_5 = 1+T$	$y_5 = 1+2T$	$y_5 = 1+3T$
$y_2 = 1+2T$			$y_6 = 1+T$	$y_6 = 1+2T$
$y_3 = 1+T$				$y_7 = 1+T$

Propagation Time P

find the <u>smallest</u> *m*

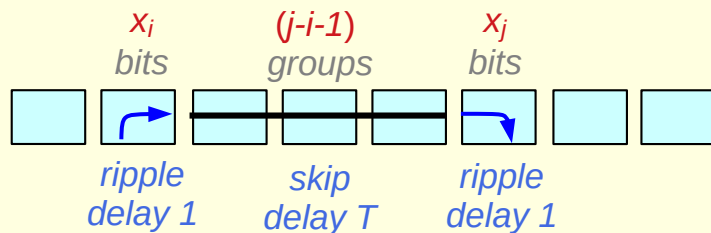$$n \ \leq \ \sum_{i=1}^{m} y_i \ = \ \sum_{i=1}^{m} min\{1+iT, 1+(m+1-i)T\}$$

(m) = 2;
while (y$_1$+···+y$_m$ < n) (m) = m+1 ;

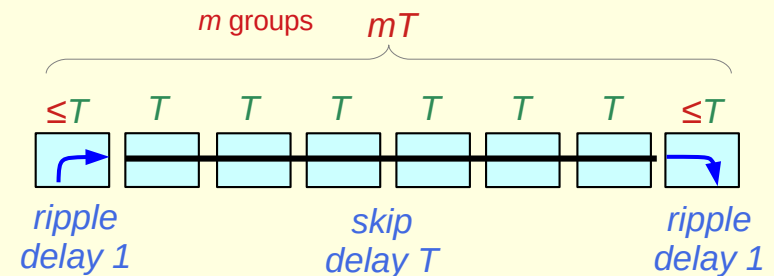the scheme (i), (ii), (iii)
gives the max propagation time *mT*

➡ propagation time of a carry signal ≤ *mT*
the maximum propagation time = *mT*

propagation time

x_i
bits

(j-i-1)
groups

x_j
bits

*ripple
delay 1*

*skip
delay T*

*ripple
delay 1*

$$P \ = \ P_{i,j} \qquad \forall i, \ \forall j \quad 1 \leq i, \ j \leq m$$

maximum propagation time

m groups

mT

≤T T T T T T T ≤T
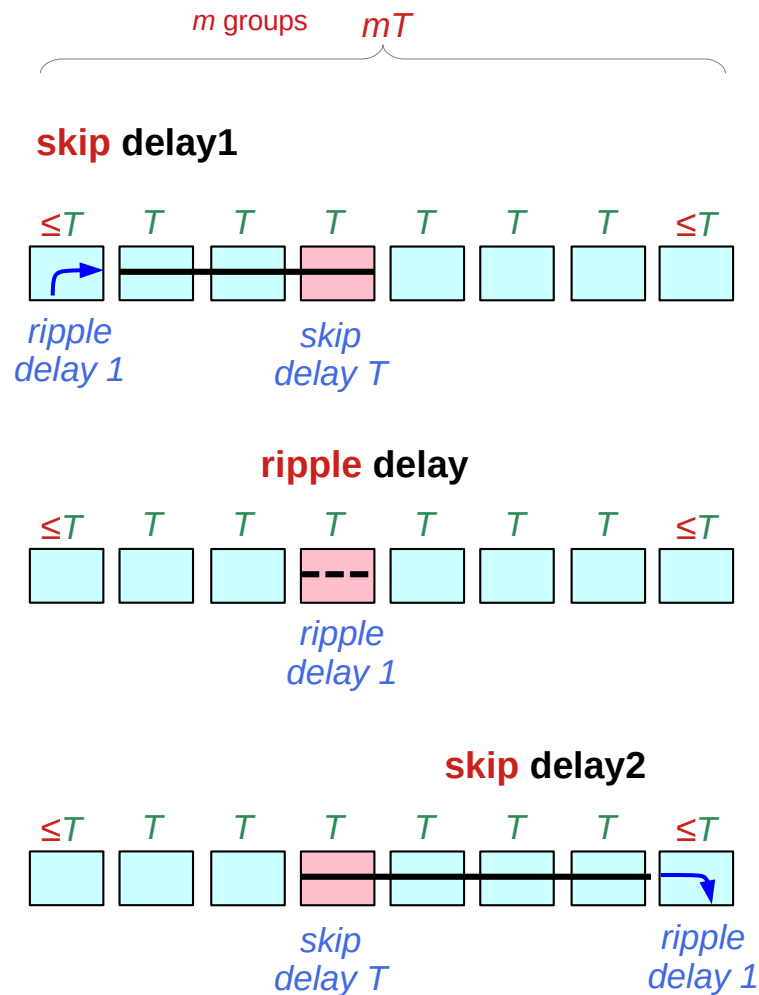
*ripple
delay 1*

*skip
delay T*

*ripple
delay 1*

$$P_{max} \ = \ P_{1,m} \ \leq \ mT$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum propagation time P

skip delay1

$\leq T$ T T T T T T $\leq T$

ripple delay 1

skip delay T

ripple delay

$\leq T$ T T T T T T $\leq T$

ripple delay 1

skip delay2

$\leq T$ T T T T T T $\leq T$

skip delay T

ripple delay 1

the scheme (i), (ii), (iii)
gives the max prop time *mT*

skip delay1	iT	generating carry
ripple delay	$x_i - 1$	
skip delay2	$(m+1-i)T$	terminating carry

$x_i - 1 \leq iT$
$x_i - 1 \leq (m+1-i)T$

$x_i \leq 1 + iT$
$x_i \leq 1 + (m+1-i)T$

$x_i \leq \min \{1 + iT, 1 + (m+1-i)T\}$
$x_i \leq y_i$

$y_i = \min \{1 + iT, 1 + (m+1-i)T\}$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum propagation time P

$$y_i = min\{1+iT, 1+(m+1-i)T\}, \quad i = 1, \dots, m$$

the scheme (i), (ii), (iii)
gives the max prop time mT

$$y_1 = min\{1+1\cdot T, 1+(m+1-1)T\} = 1 + T$$

$$y_m = min\{1+m\cdot T, 1+(m+1-m)T\} = 1 + T$$
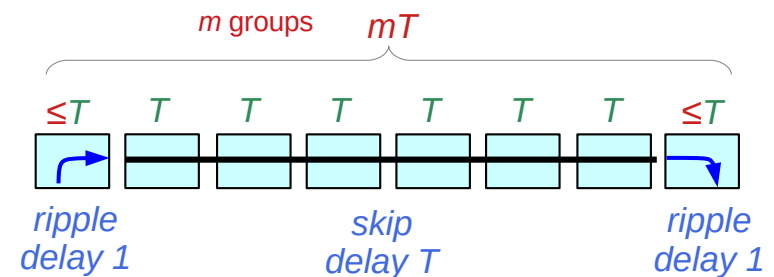
$$x_1 \leq y_1 = 1+T$$

$$x_m \leq y_m = 1+T$$

maximum propagation time

$x_1 - 1 \leq 1T$ $\quad\quad$ $x_1 - 1 \leq T$
$x_1 - 1 \leq (m+1-1)T$ $\quad$ $x_1 - 1 \leq mT$

$x_m - 1 \leq mT$ $\quad\quad$ $x_m - 1 \leq mT$
$x_m - 1 \leq (m+1-m)T$ $\quad$ $x_m - 1 \leq T$

$$P_{max} = P_{1,m} \leq mT$$

$$P = P_{i,j} \leq mT$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers
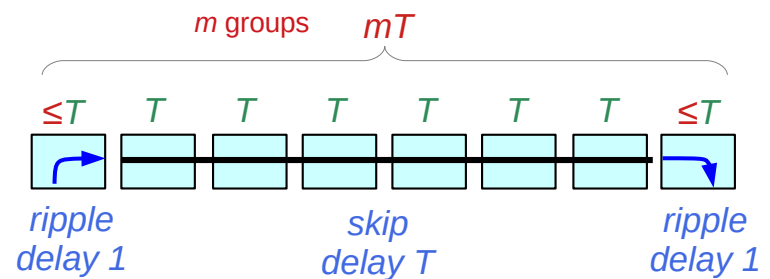
Maximum propagation time P

Lemma 1 When the bits of a carry skip adder are <u>grouped</u> according to the scheme (i)-(iii), the maximum propagation time of a carry signal is mT

the scheme (i), (ii), (iii) gives the max prop time mT

The carry generated at the 2nd bit position and terminating at the $(n-1)^{th}$ bit position clearly has propagation time mT.

We must show that *any other* carry signal has propagation time <u>smaller</u> than or equal to mT

maximum propagation time

$$ P_{max} = P_{1,m} \leq mT $$

> propagation time of a carry signal $\leq mT$
> the maximum propagation time = mT

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum delay and optimal group size

the maximum propagation time
$\propto$ the number of groups

$D \propto m$

- *not an optimal optimal division*

  - *larger number of groups →*
  - *larger delays →*

- when group size m is not *optimal*

  then there is an optimal group size = r

  - the maximum delay
    with the group size m $D_m = mT$

  - the maximum delay
    with the group size r $D_r = rT$

  - *r must be smaller than m $r \leq m$*

$$D_r < D_m$$

$$\Longrightarrow \quad rT < mT$$

$$\Longrightarrow \quad r < m$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers
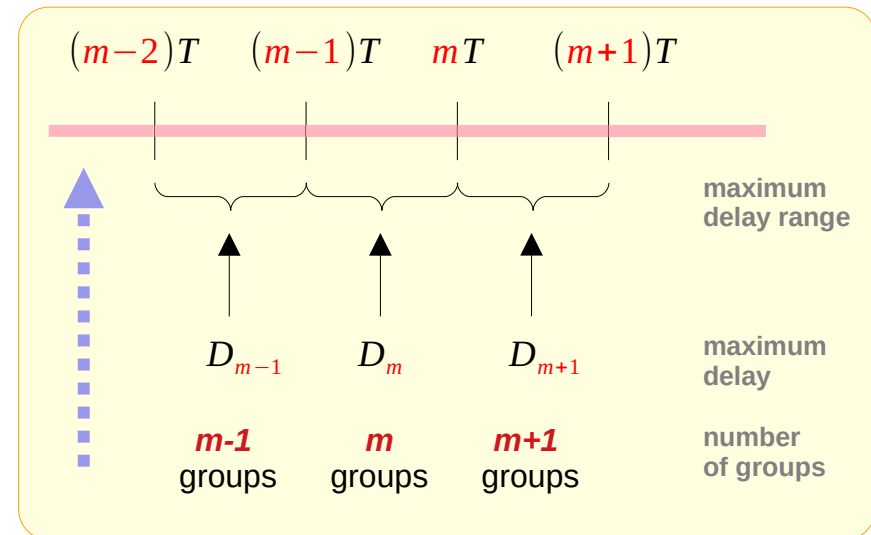
Maximum delay of a carry signal

Lemma 2   Let D denote the maximum delay of a carry signal
in a n bit carry skip adder with group sizes chosen optimally. Then

$$(m-1)T \;\leq\; D \;\leq\; mT$$

- **n** bits
- **r** groups

Since we have exhibited a <u>division</u> of the carry chain into groups
In such a way that the maximum delay of a carry signal is mT
We clearly have $D \;\leq\; mT$

the **maximum delay** = D
the **optimal group size** = m

$$(m-1)T \;\leq\; D \;\leq\; mT$$

$(m-2)T \quad (m-1)T \quad mT \quad (m+1)T$

maximum delay range

$D_{m-1} \qquad D_m \qquad D_{m+1}$

maximum delay

m-1
groups

m
groups

m+1
groups

number of groups

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum delay of a carry signal

$$(m-1)T \le D \le mT$$

Assume there are **r** groups

then 2 cases : **even r, odd r**

for each of these 2 cases

prove $mT - D < T + 1$

➡ $\boxed{mT - D \le T}$

➡ $(m-1)T \le D$

P: **the propagation delay of any carry signal path ≤ mT**
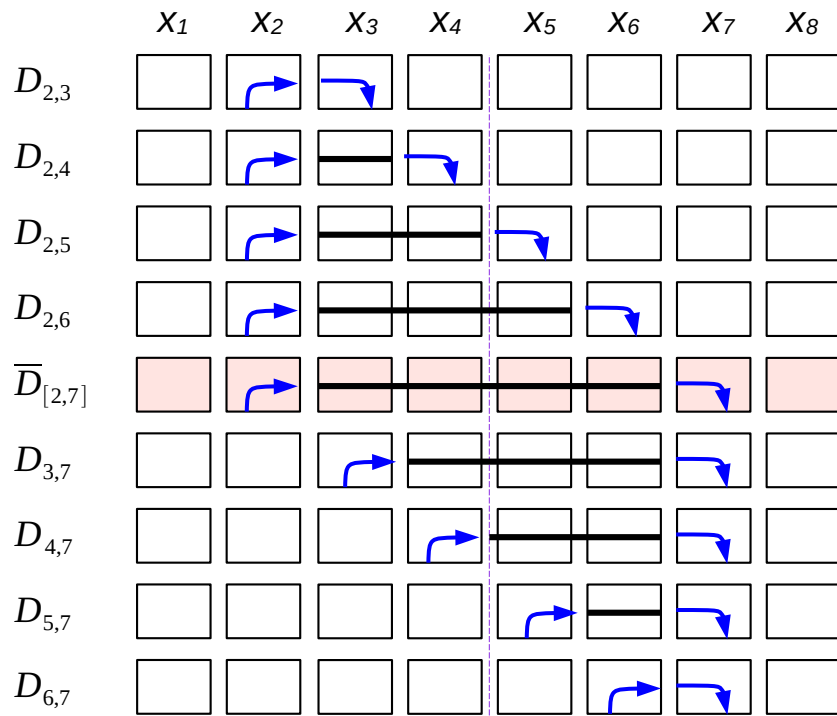
upper bound

D: **the max of P**

$\boxed{\textbf{diff (mT, D) ≤ T}}$

➡ **diff (mT, *max* P) ≤ T**

lower bound

(*m-1*)*T* ≤ **D**

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers
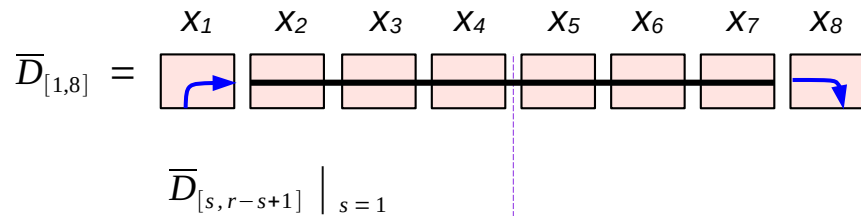
Maximum delays of carry signals (r = 2k)

$\overline{D}_{[2,7]}$ = the maximum delay of carry signals $\leq D$
generated in the i-th group and
terminated in the j-th group
such that $2 \leq i, j \leq 7$

$$\overline{D}_{[2,7]} = max \begin{Bmatrix} D_{2,3}, & D_{2,4}, & D_{2,5}, & D_{2,6}, \\ & D_{2,7}, & & \\ D_{3,7}, & D_{4,7}, & D_{5,7}, & D_{6,7} \end{Bmatrix}$$
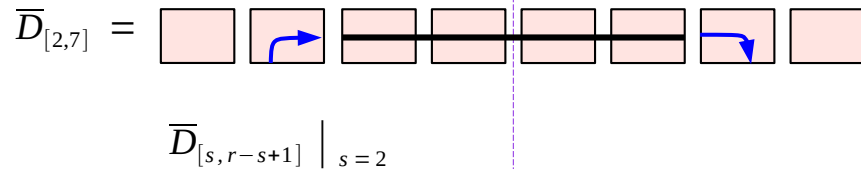
$$\overline{D}_{[2,7]} = \overline{D}_{[2,8-2+1]} = \overline{D}_{[s,8-s+1]}, \quad s = 2$$

$\overline{D}_{[s,r-s+1]}$ = the maximum delay of carry signals
generated in the i-th group and
terminated in the j-th group
such that $s \leq i, j \leq r-s+1$

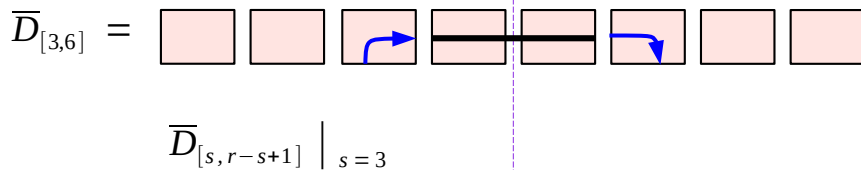Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum delays of carry signals (**r = 2k**)

$\overline{D}_{[1,8]} = \ $
$\overline{D}_{[s,r-s+1]} \Big|_{s=1}$

$\overline{D}_{[2,7]} = \ $
$\overline{D}_{[s,r-s+1]} \Big|_{s=2}$

$\overline{D}_{[3,6]} = \ $
$\overline{D}_{[s,r-s+1]} \Big|_{s=3}$

$\overline{D}_{[4,5]} = \ $
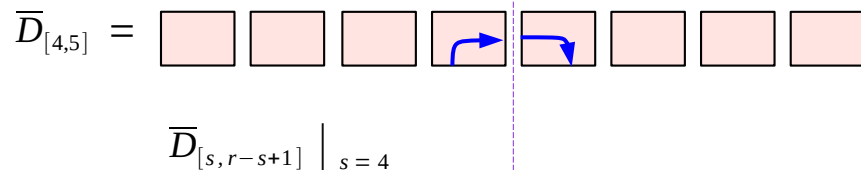$\overline{D}_{[s,r-s+1]} \Big|_{s=4}$

$\overline{D}_{all} = \ $

$\overline{D}_{[1,8]} = \ $ *The maximum delay of carry signals* $\leq D$
*generated in the i-th group or
terminated in the j-th group
such that 1 ≤ i, j ≤ 8*

$\overline{D}_{[2,7]} = \ $ *The maximum delay of carry signals* $\leq D$
*generated in the i-th group or
terminated in the j-th group
such that 2 ≤ i, j ≤ 7*

$\overline{D}_{[3,6]} = \ $ *The maximum delay of carry signals* $\leq D$
*generated in the i-th group or
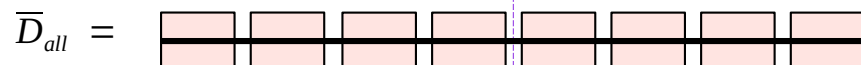terminated in the j-th group
such that 3 ≤ i, j ≤ 6*

$\overline{D}_{[4,5]} = \ $ *The maximum delay of carry signals* $\leq D$
*generated in the i-th group or
terminated in the j-th group
such that 4 ≤ i, j ≤ 5*

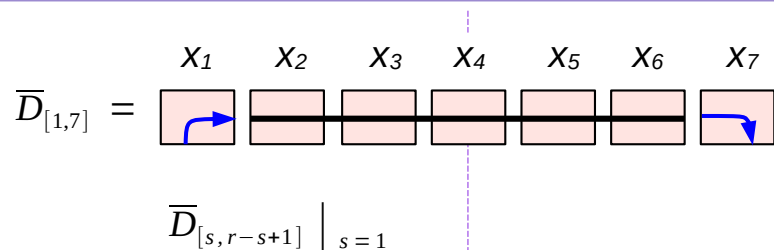$\overline{D}_{all} = \ $ *All skip delay* $\leq D$

$$D = max\{\overline{D}_{[1,8]},\ \overline{D}_{[2,7]},\ \overline{D}_{[3,6]},\ \overline{D}_{[4,5]}\}$$
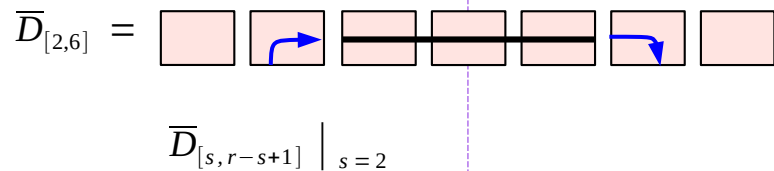
**Max delay of
all carry signals**

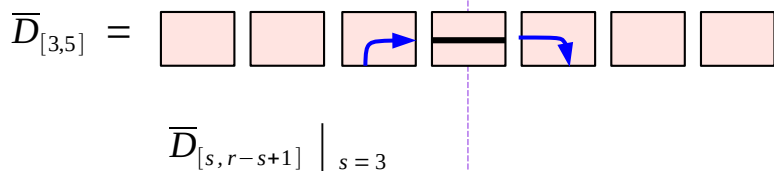Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum delays of carry signals (**r = 2k+1**)

$\overline{D}_{[1,7]} =$

X_1 X_2 X_3 X_4 X_5 X_6 X_7

$\overline{D}_{[s,r-s+1]} \Big|_{s=1}$

$\overline{D}_{[2,6]} =$

$\overline{D}_{[s,r-s+1]} \Big|_{s=2}$

$\overline{D}_{[3,5]} =$

$\overline{D}_{[s,r-s+1]} \Big|_{s=3}$

$\overline{D}_{all} =$

$\overline{D}_{all} =$

$\overline{D}_{[1,7]} =$ *The maximum delay of carry signals* $\leq$ D
  generated in the <mark>*i-th*</mark> *group or*
  terminated in the <mark>*j-th*</mark> *group*
  such that $1 \leq$ <mark>*i, j*</mark> ≤ 8

$\overline{D}_{[2,6]} =$ *The maximum delay of carry signals* $\leq$ D
  generated in the <mark>*i-th*</mark> *group or*
  terminated in the <mark>*j-th*</mark> *group*
  such that $2 \leq$ <mark>*i, j*</mark> ≤ 7

$\overline{D}_{[3,65]} =$ *The maximum delay of carry signals* $\leq$ D
  generated in the <mark>*i-th*</mark> *group or*
  terminated in the <mark>*j-th*</mark> *group*
  such that $3 \leq$ <mark>*i, j*</mark> ≤ 6

$\overline{D}_{all} =$ *All skip delay*

$\widetilde{D}_{all} =$ *Comparable to all skip delay* $\leq$ D

**Max delay of
all carry signals**

$D = max\{\overline{D}_{[1,8]}, \ \overline{D}_{[2,7]}, \ \overline{D}_{[3,6]}, \ \overline{D}_{[4,5]}\}$

Maximum delays of carry signals (r = 2k)

$$\overline{D}_{[1,8]} = $$

x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8

$$D = \max_{s=1}^{r/2} \overline{D}_{[s,r-s+1]}$$

$$= \max_{s=1}^{k} \overline{D}_{[s,2k+1-s]}$$

$$= \max_{s=1}^{4} \overline{D}_{[s,9-s]}$$

**Max delay of
all carry signals**

$$\overline{D}_{[1,r]} \leq D$$
$$\overline{D}_{[2,r-1]} \leq D$$
$$\vdots \qquad \vdots$$
$$\overline{D}_{[k,k+1]} \leq D$$

$$\overline{D}_{all} \leq D$$

$$\big(m-1\big)T \leq D$$

Lower bound of D

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum delays of carry signals (r = 2k+1)

$\overline{D}_{[1,8]} = $ x_1 x_2 x_3 x_4 x_5 x_6 x_7
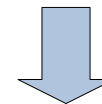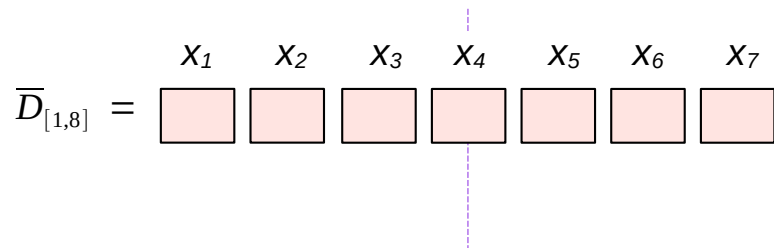
$$D = \max_{s=1}^{floor(r/2)} \overline{D}_{[s,r-s+1]}$$

$$= \max_{s=1}^{k} \overline{D}_{[s,2k+2-s]}$$

$$= \max_{s=1}^{3} \overline{D}_{[s,8-s]}$$

**Max delay of
all carry signals**

$$\overline{D}_{[1,r]} \leq D$$
$$\overline{D}_{[2,r-1]} \leq D$$
$$\vdots \qquad \vdots$$
$$\overline{D}_{[k,k+1]} \leq D$$

$$\widetilde{D}_{all} \quad \leq D$$

$$(m-1)T \leq D$$

Lower bound of D

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers
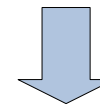
Example delays of carry signals (**r = 2k**) (1)

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Example delays of carry signals (r = 2k) (2)

$$D_{2,4} \leq \overline{D}_{[2,7]}$$

$$D_{3,5} \leq \overline{D}_{[3,6]}$$

$$D_{4,6} \leq \overline{D}_{[3,6]}$$

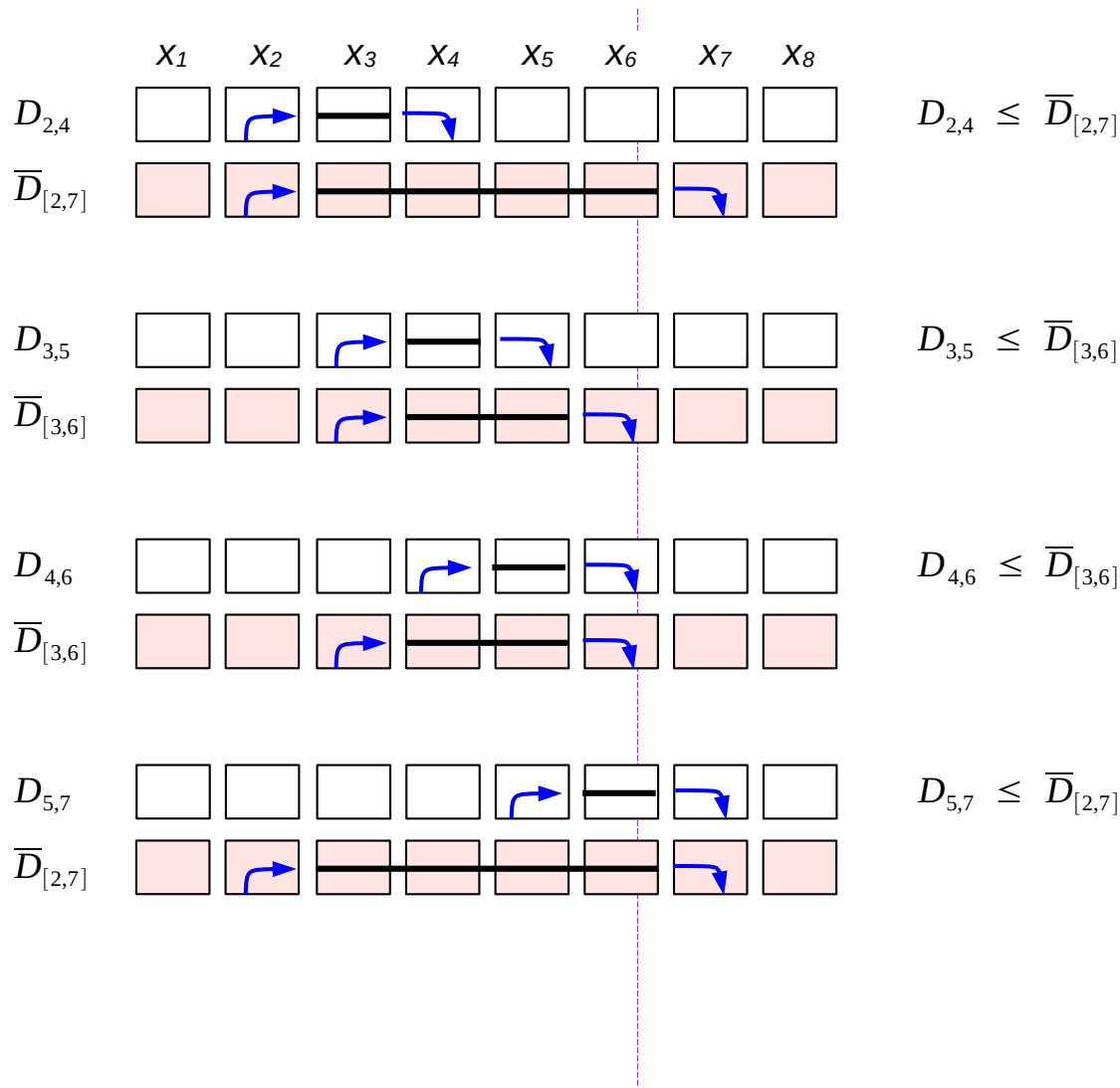$$D_{5,7} \leq \overline{D}_{[2,7]}$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (1-1)

Theorem 1

The scheme 2(i) – 2(iii) given above
for dividing the bits of a carry skip adder into groups
is optimal for 2 ≤ T ≤ 7

dividing the bits into groups by the scheme 2(i) – 2(iii)
gives m groups

propagation time of a carry signal $\leq mT$
the maximum propagation time = mT

the maximum delay = D
the optimal group size = m

$$(m-1)T \leq D \leq mT$$

(I) Let m be the smallest positive integer such that

$$n \leq m + \frac{1}{2}mT + \frac{1}{4}m^2T + (1-(-1)^m)\frac{1}{8}T$$

(II) Let $y_i = min\{1+iT, 1+(m+1-i)T\},$
$$i = 1,...,m$$

and construct a histogram
whose i-th column has height y_i

(III) the area of the histogram in (II) is

$$m + \frac{1}{2}mT + \frac{1}{4}m^2T + (1-(-1)^m)\frac{1}{8}T \geq n$$

so these are at least n unit squares
in the histogram starting with the first row,
shade in n of the squares, row by row
Let x_i denote the number of shaded squares
in column i of the histogram,
$i = 1, ..., m$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (1-2)

Assume

- the scheme by 2(i) – 2(iii) (*m* groups) is <u>not</u> optimal

- let **D** be the maximum delay
  corresponding to an optimal division of the bits into groups

- there are **r** groups in the optimal division.

Since a carry in signal to the least significant bit group
can skip over each group

we have $rT \leq D \leq mT$ so $r \leq m$

> if *m* is <u>not</u> optimal, <u>but</u> *r* is
> then $mT \geq rT$ (smaller delay rT)
> *thus* $m \geq r$ (smaller *r* exists)

m groups
- *not optimal division*
- **D** = maximum delay
- **m**T skip delay

r groups
- optimal division
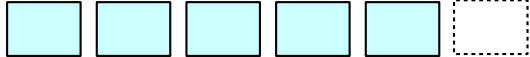- **r**T skip delay

skip delay
$rT \leq D \leq mT$

$r \leq m$

D = *max delay is assumed*
To be greater than all skip
delay **rT** *of the optimal division*

Optimal division into groups (1-2)

If the optimal division gives *m* groups

m groups □ □ □ □ □ □   *mT*

(*m-1*) groups □ □ □ □ □ ⬚   (*m-1*)*T*

$$D \leq mT$$

$$(m-1)T \leq D$$

- when optimal group size = m
  the maximum delay $D_m \leq mT$

- when optimal group size = (m-1)
  the maximum delay $D_{m-1} \leq (m-1)T$

Normally, by 2(i) – 2(iii) (*m* groups) is optimal
and its maximum delay *D* is less than all skip delay *mT*

$$D \leq mT$$

To prove this, first, negate that
- *m is <u>not</u> by the optimal division, but r is*
- *D is <u>greater</u> than all skip delay of the optimal division*

D = maximum delay

$rT \leq D \leq mT$

$r \leq m$ ➡ r < (m-1)

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (1-2)

$rT \leq D \leq mT$ so $r \leq m$

Optimal division : r groups
     $D' \leq$ all skip delay rT (r groups)

Non-optimal division : m groups
     $D \leq$ all skip delay mT (m groups)
     too many partitions m $r \leq m$

*Assume max delay D is greater
than all skip delay rT of the optimal division*

D = maximum delay

$rT \leq D \leq mT$

$r \leq m$ ➡ $r < (m-1)$

*if m is <u>not</u> optimal, <u>but</u> r is
then $mT \geq rT$ (smaller delay rT)
thus $m \geq r$ (smaller r exists)*

*D is max delay for m groups
D' is max delay for r groups
then $D' \leq rT \leq D \leq mT$*

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (2)

we have $rT \leq D \leq mT$ so $r \leq m$

If $r = m$

then $D = mT$ ➡ $D = rT$ $rT = D$

If $r = m\text{-}1$, $(r < m)$

$D \geq (m\text{-}1)T$ ➡ $D \geq (m\text{-}1)T = rT$ $rT \leq D$

if $r < m\text{-}1$, $(r < m)$ $r = m\text{-}2, m\text{-}3, m\text{-}3, \ldots$

$D \geq (m\text{-}1)T$ ➡ $D \geq (m\text{-}1)T > rT$ $rT < D$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (3)

we have $rT \leq D \leq mT$ so $r \leq m$

If $r = m$ then $D = mT$
and the **theorem** holds by **lemma** 1

When the bits of a carry skip adder are
grouped according to the scheme (i)-(iii),
the maximum propagation time
of a carry signal is mT

$(m\text{-}1)T \leq D \leq mT$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Lemma 1 *When the bits of a carry skip adder are <u>grouped</u> according to the scheme (i)-(iii), the maximum propagation time of a carry signal is mT*

Theorem 1 *The scheme 2(i) – 2(iii) given above for dividing the bits of a carry skip adder into groups is optimal for 2 ≤ T ≤ 7*

(5) $r = 2k$ $\qquad X = 4 - T^2$

$$mT - D \leq T + \frac{-8(T/n) + 4}{\sqrt{4(T/n) + 8(T/n^2)} + \sqrt{4(T/n) + 4/n^2}}$$

$r = 2k + 1$ $\qquad X = 4$

$$mT - D \leq T + \frac{(T - 2)^2/n}{\sqrt{4(T/n) + 4(T/n^2)} + \sqrt{4(T/n) + (T/n)^2 + 4/n^2}}$$

m groups – not optimal division
r groups – optimal division

D = maximum delay

$rT \leq D \leq mT$

$r \leq m$

Optimal division into groups (3)

If $r = m\text{-}1$, ($r < m$)
m and r have different parities and
it follows from (5)
that $mT - D \leq T$ for $2 \leq T \leq 7$

so that $D \geq (m\text{-}1)T$
since $r = m\text{-}1,$
$D \geq (m\text{-}1)T = rT$ $rT \leq D$

This means that a signal which
skips over each of the r groups (rT)
has delay less than the maximum D.

$rT \leq D \leq mT$

m is <u>not</u> optimal division
r is optimal division

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

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Lemma 1 *When the bits of a carry skip adder are <u>grouped</u> according to the scheme (i)-(iii), the maximum propagation time of a carry signal is mT*

Theorem 1 *The scheme 2(i) – 2(iii) given above for dividing the bits of a carry skip adder into groups is optimal for $2 \leq T \leq 7$*

(5) $r = 2k$ $X = 4 - T^2$

$$mT - D \leq T + \frac{-8(T/n)+4}{\sqrt{4(T/n)+8(T/n^2)} + \sqrt{4(T/n)+4/n^2}}$$

$r = 2k+1$ $X = 4$

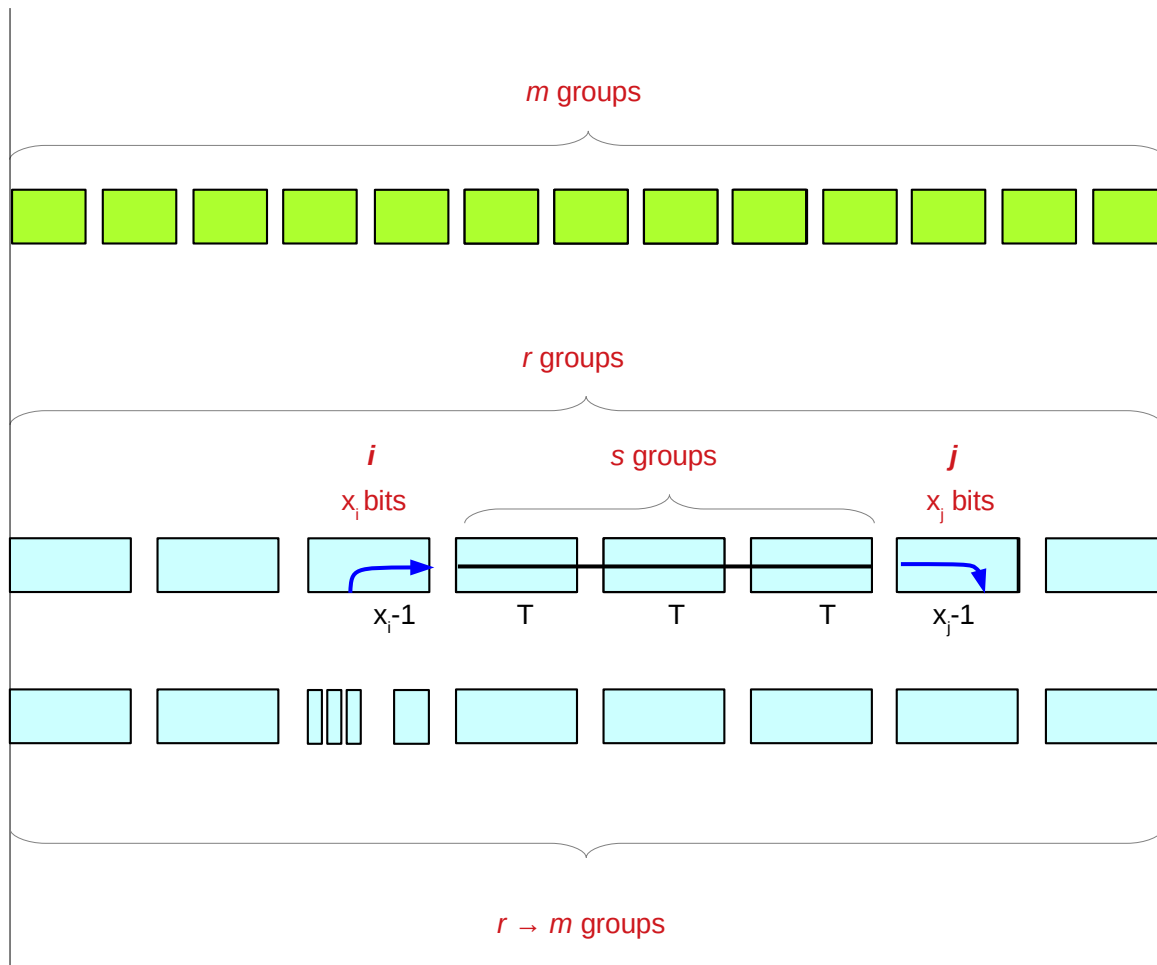$$mT - D \leq T + \frac{(T-2)^2/n}{\sqrt{4(T/n)+4(T/n^2)} + \sqrt{4(T/n)+(T/n)^2+4/n^2}}$$

m groups – not optimal division
r groups – optimal division

D = maximum delay

$rT \leq D \leq mT$

$r \leq m$

Optimal division into groups (4)

Similarly,
if <mark>*r < m-1*</mark>, (*r < m*)

(m-1)T ≤ D

since *r < m-1,*
rT < (m-1)T ≤ D

so that a signal which skips over each group
has delay *rT < D*.

rT < D ≤ mT

m is <u>not</u> optimal division
r is optimal division

m groups – not optimal division
r groups – optimal division

D = maximum delay

rT ≤ D ≤ mT

r ≤ m

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (5)

m groups

r groups

i
x_i bits

s groups

j
x_j bits

x_i-1   T   T   T x_j-1

r → m groups

m groups – not optimal division
r groups – optimal division

D = maximum delay

$rT \leq D \leq mT$

$r \leq m$

if m is not optimal, but r is

(((r +1) +1) +1) … ⟶ m
contradiction!   r must be m

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers
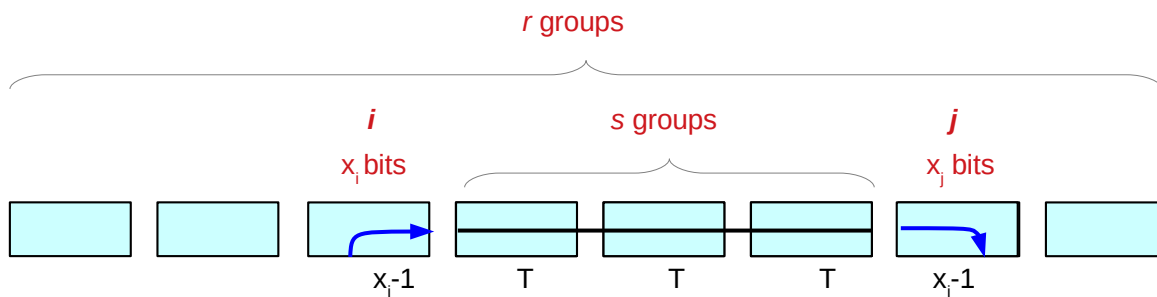
Optimal division into groups (5)

It follows that a signal with delay D

must <u>start</u> in a group i,
<u>ripple</u> to the <u>end</u> of group i,

then <u>skip</u> over $s < r$ groups and

either <u>terminate</u>, or <u>ripple</u> through
the first few bits of a group $j > i$.

m groups – not optimal division
r groups – optimal division
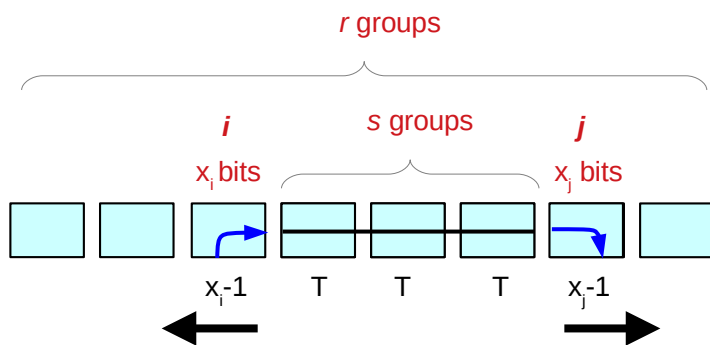
D = maximum delay

$rT \leq D \leq mT$

$r \leq m$

r groups

i

x_i bits

s groups

j

x_j bits

x_i-1   T   T   T x_j-1

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (6-1)

Let x_i and x_j denote
the lengths of the *i-th* and *j-th* groups respectively.

Assume that *i* is chosen as <u>small</u> as possible
and *j* as <u>large</u> as possible. (longer path)

r groups

i   *s* groups   *j*

x_i bits x_j bits
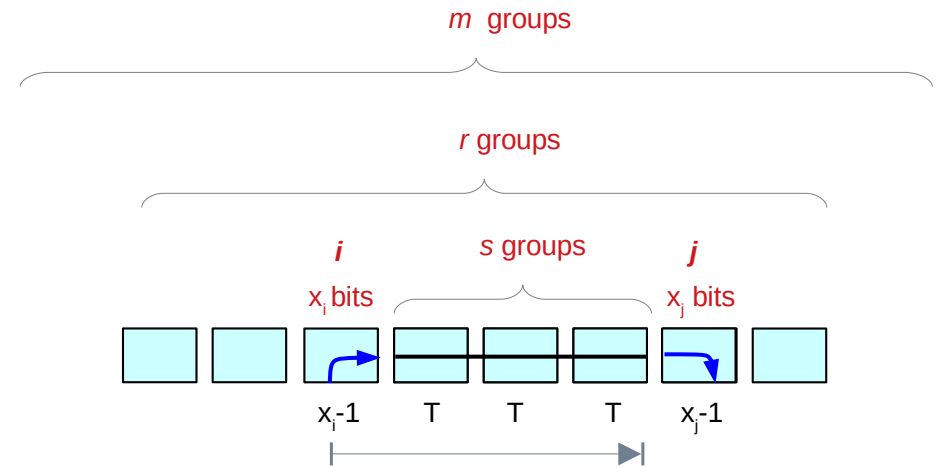
x_i-1   T   T   T x_j-1

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (6-2)

A signal <u>originating</u> in group i,
<u>rippling</u> to the end of this group i and
then skipping over the next s group
has delay $(x_i - 1) + sT$

m groups

r groups

i   s groups   j

x_i bits x_j bits

x_i-1   T   T   T x_j-1

$$D \le (x_i - 1) + sT$$
$$\le (x_i - 1) + (r - 1)T$$
$$\le (x_i - 1) + (m - 2)T.$$

$s < r$ groups ➡ $s \le (r\text{-}1)$

$r < m$ groups ➡ $r \le (m\text{-}1)$

if m is <u>not</u> optimal, <u>but</u> r is

$$D \le (x_i - 1) + sT$$
$$< (x_i - 1) + rT$$
$$< (x_i - 1) + mT.$$

$s < r < m$

$s \le (r - 1) < (m - 1)$

$s \le (r - 1) \le (m - 2)$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (6-3)

$$D \leq (x_i - 1) + sT$$
$$\leq (x_i - 1) + (r-1)T$$
$$\leq (x_i - 1) + (m-2)T$$

$$(m-1)T \leq D$$
$$D \leq (x_i - 1) + (m-2)T$$

$$(m-1)T \leq D \leq (x_i - 1) + (m-2)T$$

$$(m-1)T \leq (x_i - 1) + (m-2)T$$

Since $D \geq (m-1)T$
this implies that $x_i \geq T+1$

$$T \leq (x_i - 1)$$

$$T + 1 \leq x_i$$

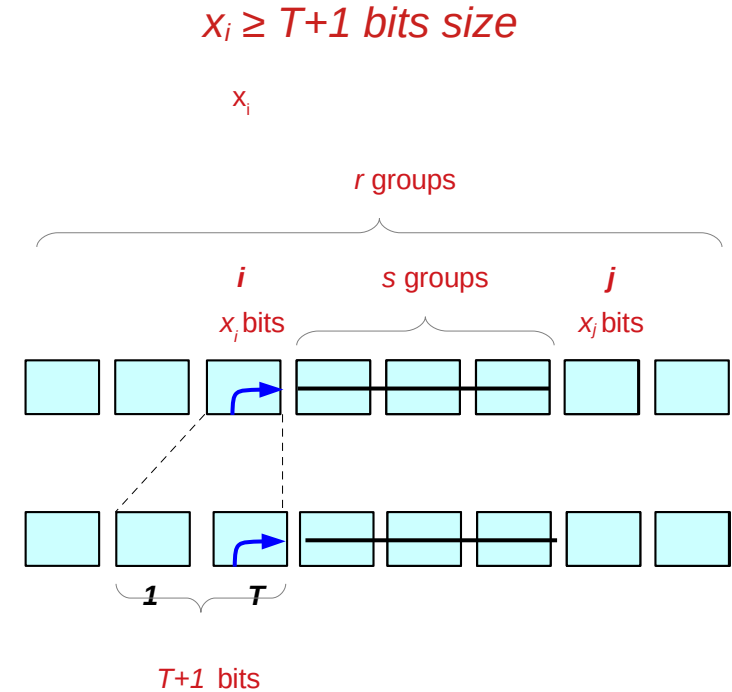Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (7)

Divide group *i* into two groups
such that the group containing the msb has size *T*.

Since the *i*-th group is the first group
in which a signal having maximum delay can <u>originate</u>,

this subdivision does <u>not</u> <u>increase</u>
the delay of any carry signal of maximum delay

However, it increases the <u>number</u> of groups by **1**

$x_i \geq T+1$ *bits size*

x_i

r groups

i   *s* groups   *j*

x_i bits x_j bits

T+1 bits

$$D \leq (x_i - 1) + s\,T$$
$$\leq (x_i - 1) + (r-1)\,T$$
$$\leq (x_i - 1) + (m-2)\,T .$$

$$(m-1)\,T \ < \ D$$
$$D \ < \ (x_i - 1) + (m-2)\,T$$
$$x_i \ \geq \ (T+1)$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (8-1)

Suppose now that a carry signal
originates in a group *i*, ripples to its end,
skips over $s \leq r\text{-}2$ groups and
finally ripples through the first few bits of a group *j*
and terminates.

We then have

$$D \leq (x_i - 1) + sT + (x_j - 1)$$
$$\leq x_i + x_j - 2 + (m - 3)T$$

So that either $x_i \geq T+1$ or $x_j \geq T+1$

s < r groups	*s < r* groups
s ≤ (r-1) groups	*s ≤ (r-2)* groups
r < m groups	*r < m* groups
r ≤ (m-1) groups	*r ≤ (m-2)* groups
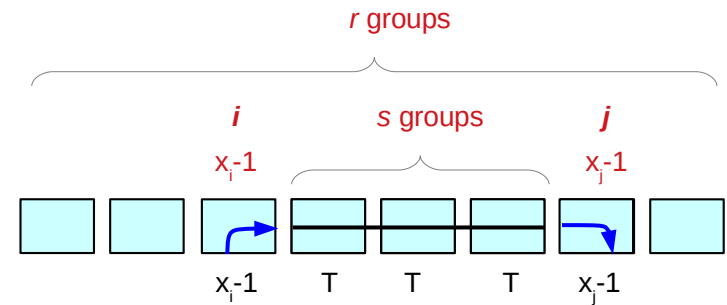
Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (8-2)

$s < r < m$

$s \leq (r-1) < (m-1)$
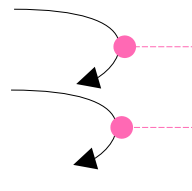
$s \leq (r-1) \leq (m-2)$

$s \leq (r-2) \leq (m-3)$
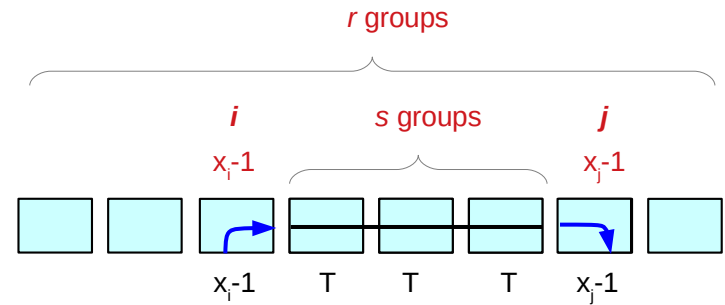
$s \leq (r-2)$

$$D \leq (x_i - 1) + sT + (x_j - 1)$$

$$\leq x_i + x_j - 2 + (r-2)T$$

$$\leq x_i + x_j - 2 + (m-3)T$$

$(m-1)T < D$

$$D < (x_i - 1) + (m-2)T \quad \Longleftrightarrow \quad x_i \geq (T+1)$$

$$D < (x_j - 1) + (m-2)T \quad \Longleftrightarrow \quad x_j \geq (T+1)$$

So that either $x_i \geq T+1$ or $x_j \geq T+1$

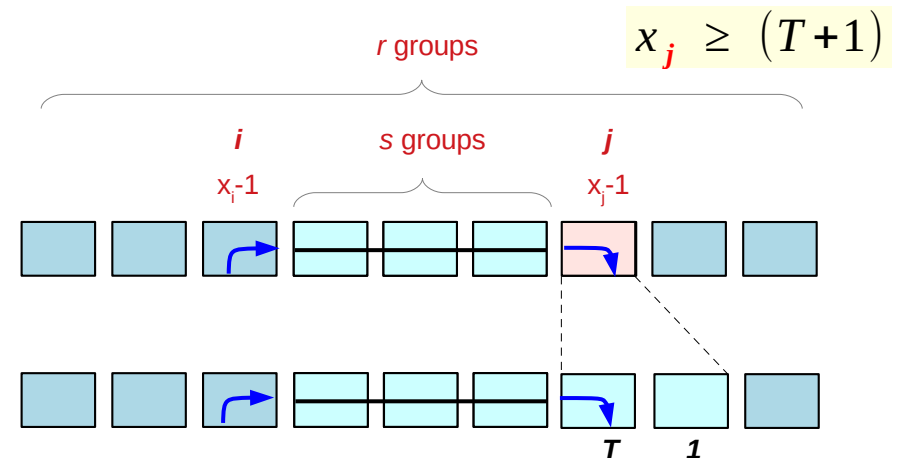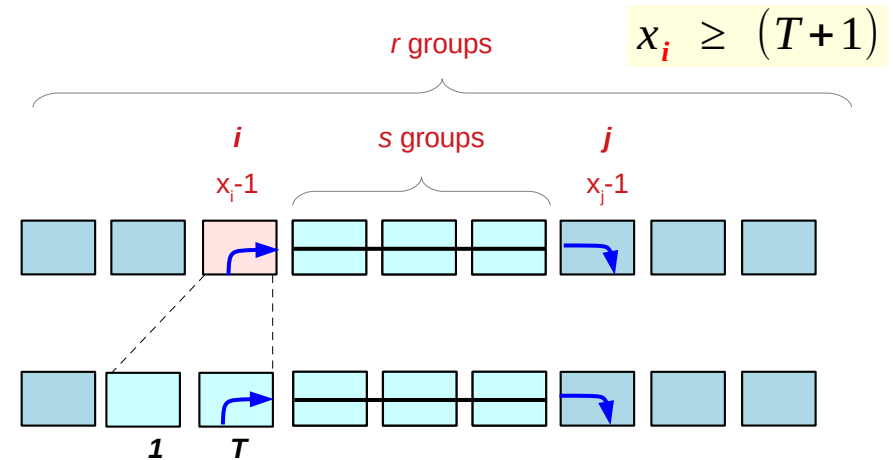Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (9)

So that either $x_i \geq T+1$ or $x_j \geq T+1$

This means that we can subdivide
one of the groups i, j
without increasing D
not both of them

Continuing in this way, we can always increase
the number r of group in an optimal division
of a carry chain by 1
without increasing D if $r < m$

This means that we can arrive
at an optimal division of the carry chain
into m groups.

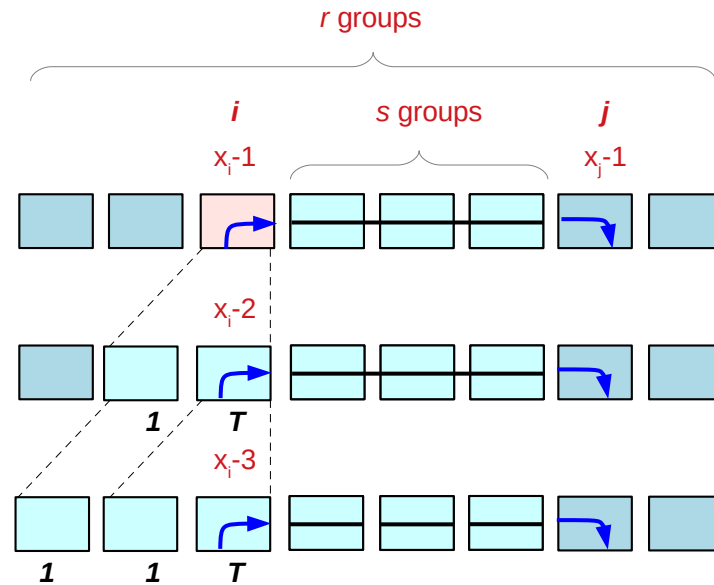$$x_i \geq (T+1)$$

r groups

i s groups j

x_i-1 x_j-1

1   T

$$x_j \geq (T+1)$$

r groups

i s groups j

x_i-1 x_j-1

T   1

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (9)

$$x_i \geq (T+1) > (T+2) \cdots$$

i   *s* groups   *j*

$x_i - 1$ $x_j - 1$

$$x_i \geq (T+1)$$

$x_i - 2$

$$x_i \geq (T+2)$$

1   **T**

$x_i - 3$

1   **1**   **T**

$$D \leq (x_i - 1) + sT$$
$$\leq (x_i - 1) + (r-1)T$$
$$\leq (x_i - 1) + (m-2)T$$

$$(m-1)T \leq D \leq (x_i - 1) + (m-2)T$$

if m is not optimal, but r is

$(((r + 1) + 1) + 1) \ldots \longrightarrow m$

contradiction!   m must be r

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (9)

$$x_j \geq (T+1) > (T+2) \cdots$$

r groups

i s groups j

x_i-1 x_j-1

x_j-2

T 1

x_j-3

T 1 1

$$x_j \geq (T+1)$$

$$x_j \geq (T+2)$$

if m is <u>not</u> optimal, <u>but</u> r is

(((r +1) +1) +1) … ⟶ m

contradiction!   m must be r

$$D \leq (x_i-1) + sT$$
$$\leq (x_i-1) + (r-1)T$$
$$\leq (x_i-1) + (m-2)T$$

$$(m-1)T \leq D \leq (x_i-1) + (m-2)T$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Variable Block Adder (1C)

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Young Won Lim
6/3/24

Optimal division into groups (9)
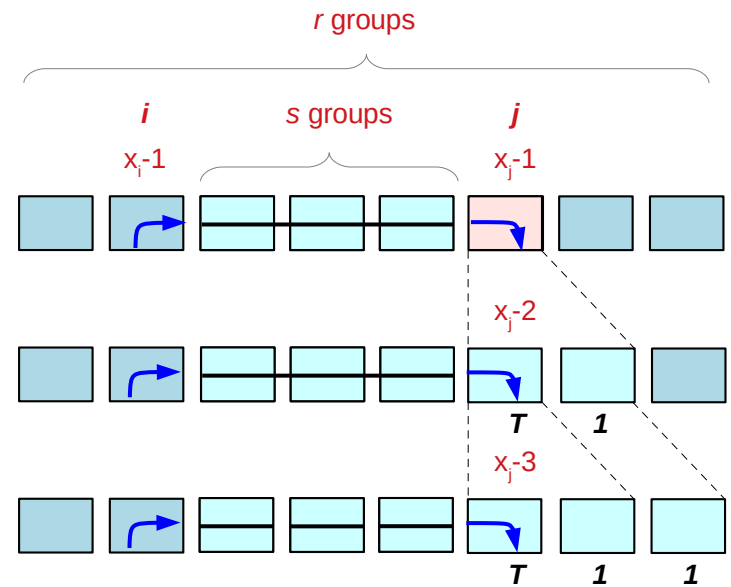
$$x_i \geq (T+1) > (T+2) \cdots$$
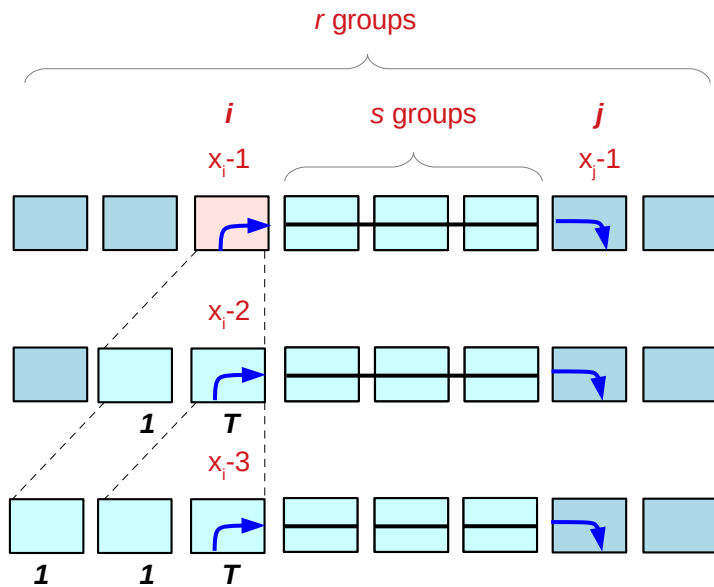
$$x_j \geq (T+1) > (T+2) \cdots$$

if m is <u>not</u> optimal, <u>but</u> r is

$(((r+1)+1)+1) \ldots \longrightarrow m$

contradiction!   m must be r

if m is <u>not</u> optimal, <u>but</u> r is

$(((r+1)+1)+1) \ldots \longrightarrow m$

contradiction!   m must be r

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (9)

Normally, by 2(i) – 2(iii) (*m* groups) is optimal
and its maximum delay D is less than all skip delay mT

$$D \le mT$$

To prove this, first, negate that
- *m is <u>not</u> by the optimal division, but r is*
- *D is <u>greater</u> than all skip delay of the optimal division*

Assume
- the scheme by 2(i) – 2(iii)
  (*m* groups) is <u>not</u> optimal
- let **D** be the maximum delay
   corresponding to an optimal division
- there are **r** groups in the optimal division.

  $(...(((r+1)+1)+1) \ldots +1) \to m : optimal$

if m is <u>not</u> optimal, <u>but</u> r is

$(((r +1) +1) +1) \ldots \longrightarrow m$

contradiction!   m must be r

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Optimal division into groups (11)

We must then have $D \geq mT$
which, together with **Lemma 2**,
Implies $D = mT$

This completes the proof of the theorem

m groups – not optimal division
r groups – optimal division

D = maximum delay

$rT \leq D \leq mT$

$r \leq m$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Lemma 2
Let D denote the maximum delay
of a carry signal in a *n* bit carry skip adder
with group sizes chosen optimally.

$$(m-1)T \ \leq \ D \ \leq \ mT$$

Theorem 1

The scheme 2(i) – 2(iii) given above
for dividing the bits of a carry skip adder into groups
is optimal for $2 \leq T \leq 7$