Random Process Background

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Outline

1. Measurable Space
   - Measurable Space
   - Sigma Alebra
   - Topological Space

2. Stochatic Process

Space (1)

- A **space** consists of
  selected mathematical objects that are treated as points, and selected relationships between these points.
  - the nature of the points can vary widely:
    for example, the points can be
    - elements of a set
    - functions on another space
    - subspaces of another space
  - It is the relationships that define the nature of the space.

https://en.wikipedia.org/wiki/Space_(mathematics)

Space (2)

- While modern mathematics uses many types of **spaces**, such as
  - Euclidean spaces
  - linear spaces
  - topological spaces
  - Hilbert spaces
  - probability spaces
- it does not define the notion of **space** itself.

https://en.wikipedia.org/wiki/Space_(mathematics)

Space (3)

- a **space** is
  a set (or a universe) with some added structure
- It is not always clear
  whether a given mathematical object should be considered
  as a geometric **space**, or an algebraic **structure**
- A general definition of **structure** embraces
  all common types of **space**

https://en.wikipedia.org/wiki/Space_(mathematics)

Mathematical objects (1)

- A **mathematical object** is
  an abstract concept arising in mathematics.

- an **mathematical object** is anything
  that has been (or could be) formally defined,
  and with which one may do
  - deductive reasoning
  - mathematical proofs

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (2)

- Typically, a **mathematical object**
  - can be a value that can be assigned to a variable
  - therefore can be involved in formulas

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (3)

- Commonly encountered **mathematical objects** include
  - numbers
  - sets
  - functions
  - expressions
  - geometric objects
  - transformations of other mathematical objects
  - spaces

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (4)

- **Mathematical objects** can be very *complex*;
  - for example, the followings are considered as **mathematical objects** in **proof theory**.
    - theorems
    - proofs
    - theories

https://en.wikipedia.org/wiki/Mathematical_object

Structure (1)

- a **structure** is a set
  endowed with some *additional features* on the set
  - e.g. an *operation*
  - *relation*
  - *metric*
  - *topology*
- Often, the *additional features* are
  attached or related to the set,
  so as to provide it with some *additional*
  meaning or *significance*.

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

Structure (2)

- A partial list of possible **structures** are
  - measures
  - algebraic structures (groups, fields, etc.)
  - topologies
  - metric structures (geometries)
  - orders
  - events
  - equivalence relations
  - differential structures
  - categories.

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

Mathematical space (1)

- A **mathematical space** is, informally, a collection of mathematical objects under consideration.
- The **universe** of mathematical objects within a **space** are *precisely* defined entities whose rules of *interaction* come baked into the rules of the **space**.

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

Mathematical space (2)

- A **space** differs from a **mathematical set** in several important ways:
  - A **mathematical set** is also a collection of objects
  - but these objects are being pulled from a **space** (or **universe**) of objects where the rules and definitions have already been agreed upon

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

Mathematical space (3)

- A **space** differs from a **mathematical set** in several important ways:
  - A **mathematical set** has no internal structure,
  - whereas a **space** usually has some internal structure.

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

Mathematical space (4)

- having some internal structure
  could mean a variety of things,
  but typically it involves
  - *interactions* and *relationships*
    between elements of the **space**
  - *rules* on how to *create* and *define*
    new elements of the **space**

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

Measurable space (1)

- A **measurable space** is
  any **space** with a **sigma-algebra**
  which can then be equipped with a measure
  - collection of **subsets** of the **space**
    following certain **rules** with a way
    to assign sizes to those sets.

https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have

Measurable space (2)

- Intuitively,
  certain **sets** belonging to a **measurable space**
  can be given a size in a *consistent way*.

  consistent way means that certain axioms are met:
  - the empty set is given a size of zero
  - if a measurable set is contained inside another one, then its size is less than or equal to the size of the containing set
  - the size of a disjoint union of sets is the sum of the individual sets' sizes

https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have

Probability space

- A **probability space** is simply
  a **measurable space** equipped with a probability measure.
- A probability measure has the special property
  of giving the entire **space** a size of **1**.
  - this then implies that the size
    of any disjoint union of sets
    (the sum of the **sizes** of the sets)
    in the **probability space**
    is less than or equal to **1**

https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have

Outline

Sigma algebra (1)

- We term the **structures** which allow us to use **measure** to be **sigma algebras**
- the only requirements for **sigma algebras** (on a **set** X) are:
  - the $\{\}$ and X are in the **set**.
  - if A is in the **set**, $complement(A)$ is in the **set**.
  - for any **sets** E_i in the set, $\bigcup_i E_i$ is in the **set** (for countable i).

https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7

Sigma algebra (2)

- The most intuitive way to think about a **sigma algebra** is that it is the kind of **structure** we can do probability on.
  - for example, we can assign ratios of areas and length, so the measure on such a set X tells something about the probability of its subsets.
  - we can find the probability of subsets A and B because we know their ratios with respect to a set X ;
  - we also know that
    - (the measure of) their complements are defined, and
    - their unions and intersections are defined,
    - so we know how to find the probability of things in this set X.

https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7

Sigma algebra (3)

- The **sigma algebra** which contains the standard topology on R (that is, *all* open sets on R) is called the **Borel Sigma Algebra**, and the elements of this set are called **Borel sets**.
- What this gives us, is the set of sets on which outer measure gives our list of dreams. That is, if we take a **Borel set** and we check that length follows translation, additivity, and interval length, it will always hold.

https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7

Sigma algebra (4)

- The set of Lebesgue measurable sets is
  the set of **Borel sets**,
  along with (union) all the sets
  which differ from a Borel set
  by a set of measure 0.
- More intuitively, it is all the sets
  we can normally measure,
  plus a bunch of stuff
  that doesn't affect our ideas of area or volume
  (think about the border of the circle above).

https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7

Borel Sets (1-1)

- a **Borel set** is any set in a topological space
  that can be formed from open sets
  (or, equivalently, from closed sets)
  through the operations of
  - countable union,
  - countable intersection, and
  - relative complement.

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (1-2)

- For a **topological space X**,
  the collection of all Borel sets on X forms a σ-algebra,
  known as the **Borel algebra** or **Borel σ-algebra**.
- The B**orel algebra** on **X** is the smallest **σ-algebra** containing all open sets (or, equivalently, all closed sets).

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (1-3)

- **Borel sets** are important in measure theory,
  since any measure defined on the open sets of a space,
  or on the closed sets of a space,
  must also be defined on all Borel sets of that space.
- Any measure defined on the Borel sets
  is called a **Borel measure**.
- **Borel sets** and the associated **Borel hierarchy** also play a fundamental role in descriptive set theory.

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (2)

- **Borel sets** are those obtained from intervals by means of the operations allowed in a **σ-algebra**. So we may construct them in a (transfinite) "sequence" of steps:
- ... And again and again.

https://math.stackexchange.com/questions/220248/understanding-borel-sets

Borel Sets (3-1)

1. Start with finite unions of closed-open intervals.
   These sets are completely elementary, and they form an algebra.
2. Adjoin countable unions and intersections of elementary sets.
   What you get already includes open sets and closed sets, intersections of an open set and a closed set, and so on.
   Thus you obtain an algebra, that is still not a σ-algebra.

https://math.stackexchange.com/questions/220248/understanding-borel-sets

Borel Sets (3)

3. Again, adjoin countable unions and intersections to 2.
   Observe that you get a strictly larger class, since a countable intersection of countable unions of intervals is not necessarily included in 2.
   Explicit examples of sets in 3 but not in 2 include F_σ sets, like, say, the set of *rational numbers*.
4. And do the same again.

https://math.stackexchange.com/questions/220248/understanding-borel-sets

Borel Sets (4-1)

- And even after a sequence of steps we are not yet finished. Take, say, a countable union of a set constructed at step 1, a set constructed at step 2, and so on. This union may very well not have been constructed at any step yet. By axioms of σ-algebra, you should include it as well - if you want, as step ∞

https://math.stackexchange.com/questions/220248/understanding-borel-sets

Borel Sets (4-2)

- (or, technically, the first infinite ordinal, if you know what that means).
- And then continue in the same way until you reach the first uncountable ordinal. And only then will you finally obtain the generated σ -algebra.

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2. Stochatic Process

Topology

- **topology**
  from the Greek words
  τόπος, 'place, location',
  and λόγος, 'study'
  is concerned with the properties of a geometric object
  - that are *preserved* under continuous deformations, such as stretching, twisting, crumpling, and bending;
  - that is, without closing holes, opening holes, tearing, gluing, or passing through itself.

https://en.wikipedia.org/wiki/Topology

Topological space (1)

- a **topological space** is, roughly speaking,
  a geometrical space
  in which closeness is defined
  but cannot necessarily be measured
  by a numeric distance.

https://en.wikipedia.org/wiki/Borel_set

Topological space (2)

- More specifically, a **topological space** is
- a set whose elements are called points,
- along with an additional structure called a topology,
  - which can be defined as
  - a set of neighbourhoods for each point
  - that satisfy some axioms
  - formalizing the concept of closeness.

https://en.wikipedia.org/wiki/Borel_set

Topological space (3)

- There are several equivalent definitions of a topology, the most commonly used of which is the definition through open sets, which is easier than the others to manipulate.

https://en.wikipedia.org/wiki/Borel_set

Topological space (4)

- A **topological space** is the most general type of a mathematical space that allows for the definition of
  - limits,
  - continuity, and
  - connectedness.
- Common types of **topological spaces** include
  - Euclidean spaces,
  - metric spaces and
  - manifolds.

https://en.wikipedia.org/wiki/Borel_set

Topological space (5)

- Although very general,
  the concept of **topological spaces** is fundamental,
  and used in virtually every branch of modern mathematics.
- The study of **topological spaces** in their own right
  is called point-set topology or general topology.

https://en.wikipedia.org/wiki/Borel_set

Open set (1)

- an **open set** is a generalization of an **open interval** in the real line.
- a **metric space** is a set along with a distance defined between any two points
- in a **metric space**,
  an **open set** is a set that, along with every point P, contains all points that are sufficiently near to P
  - all points whose distance to P is less than some value depending on P

https://en.wikipedia.org/wiki/Open_set

Open set (2)

- More generally, an **open set** is
  a member of a given collection of subsets of a given set,
  a collection that has the property of containing
  - every union of its members
  - every finite intersection of its members
  - the empty set
  - the whole set itself

https://en.wikipedia.org/wiki/Open_set

Open set (2)

- A set in which such a collection is given
  is called a **topological space**, and
  the collection is called a **topology**.
- These conditions are very loose,
  and allow enormous flexibility in the choice of open sets.
- For example,
  - every subset can be open (the discrete topology), or
  - no subset can be open (the indiscrete topology) except
    - the space itself and
    - the empty set .

https://en.wikipedia.org/wiki/Open_set

Open set (3)

Example:

- The *circle* represents the set of points (x, y) satisfying $x^2 + y^2 = r^2$.
- The *disk* represents the set of points (x, y) satisfying $x^2 + y^2 < r^2$.
- The *circle* set is an **open set**,
- the *disk* set is its **boundary set**, and
- the union of the *circle* and *disk* sets is a **closed set**.

https://en.wikipedia.org/wiki/Open_set

Open set (4)

- A **set** is a collection of distinct objects.
- Given a **set** A, we say that a is an **element** of A if a is one of the distinct objects in A, and we write $a \in A$ to denote this
- Given two **sets** A and B, we say that A is a **subset** of B if every element of A is also an element of B write $A \subseteq B$ to denote this.

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Open set (5) Open Balls

- We give these definitions in general, for when one is working in $\mathbb{R}^n$ since they are really not all that different to define in $\mathbb{R}^n$ than in $\mathbb{R}^2$
- An **open ball** $B_r(\boldsymbol{a})$ in $\mathbb{R}^n$ centered at $\boldsymbol{a} = (a_1, \ldots a_n) \in \mathbb{R}^n$ with radius r is the set of all points $\boldsymbol{x} = (x_1, \ldots x_n) \in \mathbb{R}^n$ such that the distance between x and $\boldsymbol{a}$ is less than r
- In $\mathbb{R}^2$ an **open ball** is often called an **open disk**

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Open set (6) Interior points

- Suppose that $S \subseteq \mathbb{R}^n$.
- A **point** $\boldsymbol{p} \in S$ is an **interior point** of S if there exists an open ball $B_r(\boldsymbol{p}) \subseteq S$.
- Intuitively, $\boldsymbol{p}$ is an **interior point** of S if we can *squeeze* an entire open ball centered at $\boldsymbol{p}$ within S

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Open set (7) boundary points

- A point $\boldsymbol{p} \in \mathbb{R}^n$ is a **boundary point** of S if all open balls centered at $\boldsymbol{p}$ contain both points in S and points not in S .
- The **boundary** of S is the set ∂S that consists of all of the **boundary points** of S.

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Open set (8)

- (Open and Closed Sets)
- A set $O \subseteq \mathbb{R}^n$ is **open**
  if every point in O is an interior point.
- A set $C \subseteq \mathbb{R}^n$is **closed**
  if it contains all of its boundary points.

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https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Topologically distinguishable points (1)

- Intuitively, an open set provides a method to distinguish two points. For example, if about one of two points in a topological space, there exists an open set not containing the other (distinct) point, the two points are referred to as topologically distinguishable. In this manner, one may speak of whether two points, or more generally two subsets, of a topological space are "near" without concretely defining a distance. Therefore, topological spaces may be seen as a generalization of spaces equipped with a notion of distance, which are called metric spaces.

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Topologically distinguishable points (2)

- In the set of all real numbers, one has the natural Euclidean metric; that is, a function which measures the distance between two real numbers: $d(x, y) = |x - y|$. Therefore, given a real number x, one can speak of the set of all points close to that real number; that is, within ε of x. In essence, points within ε of x approximate x to an accuracy of degree ε. Note that $\varepsilon > 0$ always but as ε becomes smaller and smaller, one obtains points that approximate x to a higher and higher degree of accuracy. For example, if $x = 0$ and $\varepsilon = 1$, the points within ε of x are precisely the points of the interval $(-1, 1)$; that is, the set of all real numbers between -1 and 1. However, with $\varepsilon = 0.5$, the points within ε of x are precisely the points of $(-0.5, 0.5)$. Clearly, these points approximate x to a greater degree of accuracy than when $\varepsilon = 1$.

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Topologically distinguishable points (3)

- The previous discussion shows, for the case $x = 0$, that one may approximate x to higher and higher degrees of accuracy by defining ε to be smaller and smaller. In particular, sets of the form $(-\varepsilon, \varepsilon)$ give us a lot of information about points close to $x = 0$. Thus, rather than speaking of a concrete Euclidean metric, one may use sets to describe points close to x. This innovative idea has far-reaching consequences; in particular, by defining different collections of sets containing 0 (distinct from the sets $(-\varepsilon, \varepsilon)$), one may find different results regarding the distance between 0 and other real numbers. For example, if we were to define R as the only such set for "measuring distance", all points are close to 0 since there is only one possible degree of accuracy one may achieve in approximating 0: being a member of R. Thus, we find that in some sense, every real number is distance 0 away from 0. It

Topologically distinguishable points (4)

- In general, one refers to the family of sets containing 0, used to approximate 0, as a neighborhood basis; a member of this neighborhood basis is referred to as an open set. In fact, one may generalize these notions to an arbitrary set (X); rather than just the real numbers. In this case, given a point (x) of that set, one may define a collection of sets "around" (that is, containing) x, used to approximate x. Of course, this collection would have to satisfy certain properties (known as axioms) for otherwise we may not have a well-defined method to measure distance. For example, every point in X should approximate x to some degree of accuracy. Thus X should be in this family. Once we begin to define "smaller" sets containing x, we tend to approximate x to a greater degree of accuracy. Bearing this in mind, one may define the remaining axioms that the family of sets about x is required to satisfy.

Open)

- (Open and Closed Sets)
-

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Stochastic Process (1)

In probability theory and related fields,
a **stochastic** (/stoʊˈkæstɪk/) or **random** process is
a mathematical object usually defined as
a family of **random variables**.

The word stochastic in English was originally used as an adjective with the definition "pertaining to **conjecturing**", and stemming from a Greek word meaning "to aim at a mark, guess", and the Oxford English Dictionary gives the year 1662 as its earliest occurrence.

From Ancient Greek στοχαστικός (stokhastikós), from στοχάζομαι (stokházomai, “aim at a target, guess”), from στόχος (stókhos, “an aim, a guess”).

https://en.wikipedia.org/wiki/Stochastic
https://en.wiktionary.org/wiki/stochastic

Stochastic Process (2)

The definition of a **stochastic process** varies,
but a **stochastic process** is *traditionally* defined as
a collection of **random variables** indexed by some set.

The terms **random process** and **stochastic process**
are considered synonyms and are used interchangeably,
without the **index set** being precisely specified.

Both "**collection**", or "**family**" are used
while instead of "**index set**", sometimes
the terms "**parameter set**" or "**parameter space**" are used.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (3)

The term **random function** is also used to refer to
a **stochastic** or **random process**,
though sometimes it is only used
when the stochastic process takes real values.

This term is also used when the **index sets** are
mathematical spaces other than the **real line**,

while the terms **stochastic process** and **random process**
are usually used when the **index set** is interpreted as time,

and other terms are used such as **random field**
when the **index set** is n-dimensional **Euclidean space** $\mathbb{R}^n$
or a manifold

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (4)

A **stochastic process** can be denoted,
by $\{X(t)\}_{t\in T}$, $\{X_t\}_{t\in T}$, $\{X(t)\}$, $\{X_t\}$ or simply as X or $X(t)$,
although $X(t)$ is regarded as an abuse of function notation.

For example, $X(t)$ or X_t are used to refer to
the **random variable** with the **index** t,
and not the entire **stochastic process**.

If the **index set** is $T = [0, \infty)$, then one can write,
for example, $(X_t, t \geq 0)$ to denote the **stochastic process**.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (1)

A **stochastic process** is defined as
a collection of **random variables**
defined on a common **probability space** $(\Omega, \mathscr{F}, P)$,

- Ω is a **sample space**,
- $\mathscr{F}$ is a σ **-algebra**,
- P is a **probability measure**;
- the **random variables**, indexed by some set T,
- all take values in the same **mathematical space** S, which must be **measurable** with respect to some σ -algebra Σ

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (2)

In other words, for a given **probability space** $(\Omega, \mathscr{F}, P)$
and a **measurable space** (S, Σ),
a **stochastic process** is a
collection of S-valued **random variables**,
which can be written as:

$$\{X(t) : t \in T\}.$$

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (3)

Historically, in many problems from the natural sciences
a point $t \in T$ had the meaning of time,
so $X(t)$ is a **random variable** representing
a value observed at time t.

A **stochastic process** can also be written as $\{X(t, \omega) : t \in T\}$
to reflect that it is actually a function of two variables, $t \in T$ and $\omega \in \Omega$.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (4)

There are other ways to consider a stochastic process, with the above definition being considered the traditional one.

For example, a stochastic process can be interpreted or defined as a S^T-valued **random variable**, where S^T is the space of all the possible functions from the set T into the space S.

However this alternative definition as a "**function-valued random variable**" in general requires additional regularity assumptions to be **well-defined**.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (1)

The set T is called the **index set** or **parameter set** of the **stochastic process**.

Often this set is some subset of the real line, such as the natural numbers or an interval, giving the set T the interpretation of time.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (2)

In addition to these sets, the index set T can be
another set with a **total order** or
a more general set, such as the Cartesian plane R^2 or
n-dimensional **Euclidean space**,
where an element $t \in T$ can represent a point in space.

That said, many results and theorems are only possible
for **stochastic processes** with a **totally ordered index set**.

https://en.wikipedia.org/wiki/Stochastic_process

State space

The **mathematical space** S of a **stochastic process** is called its **state space**.

This mathematical space can be defined using integers, real lines, n-dimensional Euclidean spaces, complex planes, or more abstract mathematical spaces.

The **state space** is defined using elements that reflect the different values that the **stochastic process** can take.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (1)

A **sample function** is a single **outcome** of a **stochastic process**,
so it is formed by taking a single possible value
of each **random variable** of the **stochastic process**.

More precisely, if $\{X(t,\omega) : t \in T\}$ is a **stochastic process**,
then for any point $\omega \in \Omega$, the mapping $X(\cdot,\omega) : T \to S$,
is called a **sample function**, a **realization**, or,
particularly when T is interpreted as time,
a **sample path** of the **stochastic process** $\{X(t,\omega) : t \in T\}$.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (2)

This means that for a fixed $\omega \in \Omega$,
there exists a **sample function**
that maps the **index set** T to the **state space** S.

Other names for a **sample function** of a **stochastic process**
include **trajectory**, **path function** or **path**

https://en.wikipedia.org/wiki/Stochastic_process