## Stationarity

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#### First-Order Stationary Processes Correlation and Covariance Functions

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Based on Probability, Random Variables and Random Signal Principles, P.Z. Peebles, Jr. and B. Shi

### Outline

First-Order Stationary Processes

2 Correlation and Covariance Functions

# First Order Stationary N Gaussian random variables

#### Definition

if the first order density function does not change with a shift in time origin

$$f_X(x_1;t_1) = f_X(x_1;t_1 + \Delta)$$

must be true for any time  $t_1$  and any real number  $\triangle$  if X(t) is to be a first-order stationary

# Consequences of stationarity N Gaussian random variables

### Definition

 $f_X(x, t_1)$  is independent of  $t_1$  the process mean value is a constant

$$m_X(t) = \overline{X} = constant$$

# the process mean value N Gaussian random variables

$$m_X(t) = \overline{X} = constant$$

$$m_X(t_1) = \int_{-\infty}^{\infty} x f_X(x; t_1) dx$$

$$m_X(t_2) = \int_{-\infty}^{\infty} x f_X(x; t_2) dx$$

$$m_{\mathbf{X}}(\mathbf{t_1}) = m_{\mathbf{X}}(\mathbf{t_1} + \Delta)$$

## Second-Order Stationary Process

N Gaussian random variables

#### Definition

if the second order density function does not change with a shift in time origin

$$f_X(x_1,x_2;t_1,t_2) = f_X(x_1,x_2;t_1+\Delta,t_2+\Delta)$$

must be true for any time  $t_1$ ,  $t_2$  and any real number  $\triangle$  if X(t) is to be a second-order stationary

Auto-correlation function

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

# N<sup>th</sup>-order Stationary Processes

#### Definition

if the second order density function does not change with a shift in time origin

$$f_X(x_1,\dots,x_N;t_1,\dots,t_N)=f_X(x_1,\dots,x_N;t_1+\Delta,\dots,t_N+\Delta)$$

must be true for any time $t_1,...,t_N$  and any real number  $\Delta$  if X(t) is to be a second-order stationary

# Wide Sense Stationary Process N Gaussian random variables

$$m_X(t) = \overline{X} = constant$$

$$E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

## The properties of autocorrelation functions (1)

### N Gaussian random variables

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$

$$R_{XX}(-\tau) = R_{XX}(\tau)$$

$$R_{XX}(0) = E\left[X^2(t)\right]$$

$$P[|X(t+\tau)-X(t)|>\varepsilon]=\frac{2}{\varepsilon^2}(R_{XX}(0)-R_{XX}(\tau))$$

# The properties of autocorrelation functions (2) N Gaussian random variables

### Definition

if  $X(t) = \overline{X} + N(t)$  where N(t) is WSS, is zero-mean, and has autocorrelation function  $R_{NN}(\tau) \to 0$  as  $|\tau| \to \infty$ , then

$$\lim_{|\tau|\to\infty}R_{XX}(\tau)=\overline{X}^2$$

if X(t) is mean square periodic, i.e, there exists a  $T \neq 0$  such that  $E\left[\left(X(t+T)-X(t)\right)^2\right]=0$  for all t, then  $R_{XX}(t)$  will have a periodic component with the same period  $R_{XX}(\tau)$  cannot have an arbitrary shape

## Crosscorrelation functions (1)

N Gaussian random variables

#### Definition

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau)$$

if

$$R_{XY}(t,t+\tau)=0$$

then X(t) and Y(t) are called orthogonal processes

## Crosscorrelation functions (2)

N Gaussian random variables

#### Definition

if X(t) and Y(t) are statistically independent

$$R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)] = m_X(t)m_Y(t+\tau)$$

if X(t) and Y(t) are stistically independent and are at least WSS,

$$R_{XY}(\tau) = \overline{XY}$$

which is constant

# The properties of crosscorrelation functions (1) N Gaussian random variables

$$R_{XY}(\tau) = R_{XY}(-\tau)$$

$$|R_{XY}(\tau)| = \sqrt{R_{XX}(0)R_{YY}(0)}$$

$$|R_{XY}(\tau)| \leq \frac{1}{2} \left[ R_{XX}(0) + R_{YY}(0) \right]$$

# The properties of crosscorrelation functions (2) N Gaussian random variables

### Definition

$$R_{YX}(-\tau) = E[Y(t)X(t-\tau)] = E[Y(s+\tau)X(s)] = R_{XY}(\tau)$$

$$E\left[\left\{\frac{\mathbf{Y}(\mathbf{t}+\mathbf{\tau})+\alpha \mathbf{X}(\mathbf{t})\right\}^{2}\right]\geq 0$$

the geometric mean of two positive numbers cannot exceed their arithmetic mean

# The properties of crosscorrelation functions (3) N Gaussian random variables

$$|R_{XY}(\tau)| \leq \frac{1}{2} \left[ R_{XX}(0) + R_{YY}(0) \right]$$

$$\sqrt{R_{XX}(0)R_{YX}(0)} \le \frac{1}{2} [R_{XX}(0) + R_{YY}(0)]$$

### Covariance Functions

N Gaussian random variables

$$C_{XX}(t,t+\tau) = E\left[\left\{X(t) - m_X(t)\right\} \left\{X(t+\tau) - m_X(t+\tau)\right\}\right]$$

$$C_{XY}(t,t+\tau) = E\left[\left\{X(t) - m_X(t)\right\} \left\{Y(t+\tau) - m_Y(t+\tau)\right\}\right]$$

$$C_{XX}(t,t+\tau) = R_{XX}(t,t+\tau) - m_X(t)m_X(t+\tau)$$

$$C_{XY}(t,t+\tau) = R_{XY}(t,t+\tau) - m_X(t)m_Y(t+\tau)$$
at least jointly WSS

$$C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2$$

$$C_{XY}(\tau) = R_{XY}(\tau) - \overline{XY}$$

## The properties of covariance functions

#### N Gaussian random variables

#### Definition

For a WSS process, variance does not depend on time and if au=0

$$C_{XX}(0) = R_{XX}(0) - \overline{X}^2$$

$$\sigma_X^2 = E\left[\left\{X(t) - E\left[X(t)\right]\right\}^2\right] = C_{XX}(0)$$

it the two random processes uncorrelated

$$C_{XY}(t,t+\tau) = R_{XY}(t,t+\tau) - m_X(t)m_Y(t+\tau) = 0$$

$$R_{XY}(t, t+\tau) = m_X(t)m_Y(t+\tau)$$

## Discrete-Time Processes and Sequences (1)

N Gaussian random variables

$$m_{X}[n] = \overline{X}, m_{Y}[n] = \overline{Y}$$

$$R_{XX}[n, n+k] = R_{XX}[k]$$

$$R_{YY}[n, n+k] = R_{YY}[k]$$

$$C_{XX}[n, n+k] = R_{XX}[k] - \overline{X}^{2}$$

$$C_{YY}[n, n+k] = R_{YY}[k] - \overline{Y}^{2}$$

## Discrete-Time Processes and Sequences (2)

N Gaussian random variables

$$m_{X}[n] = \overline{X}, m_{Y}[n] = \overline{Y}$$

$$R_{XY}[n, n+k] = R_{XY}[k]$$

$$R_{YX}[n, n+k] = R_{YX}[k]$$

$$C_{XY}[n, n+k] = R_{XY}[k] - \overline{XY}$$

$$C_{YX}[n, n+k] = R_{YX}[k] - \overline{YX}$$