

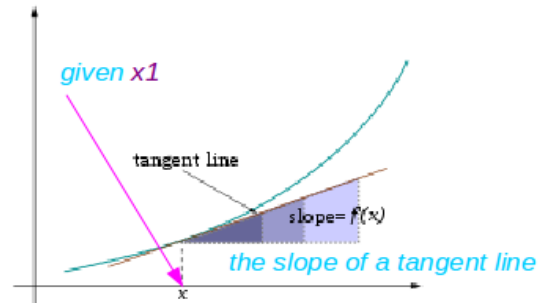
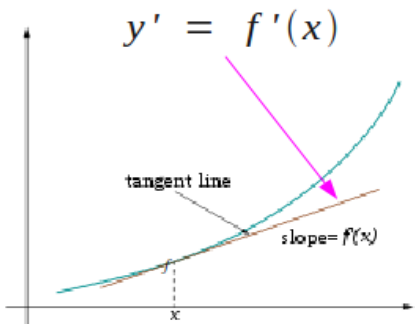
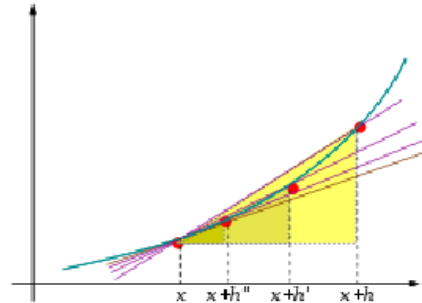
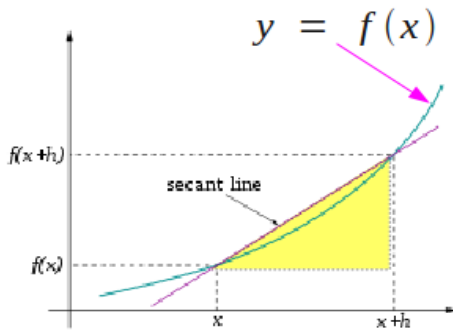
# First Order ODEs (H.1)

20151224

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# Another kind of triangles and their slopes



<http://en.wikipedia.org/wiki/Derivative>

# Differentials and Derivatives (1)

$$dy = f'(x) dx$$

$$dy = \frac{df}{dx} dx$$

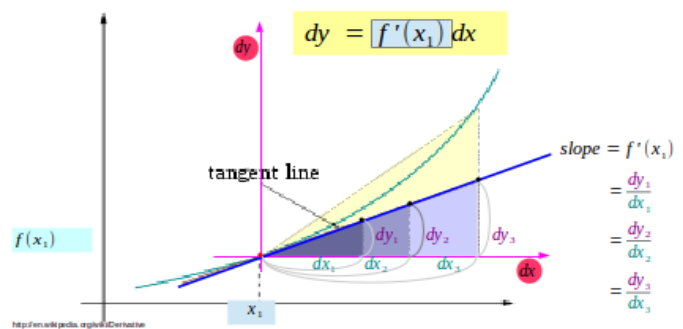
differentials

derivative

$$\frac{dy}{dx} = f'(x)$$

ratio

not a ratio



## Differentials and Derivatives (2)

$$f(x_1 + dx) \approx f(x_1) + dy$$

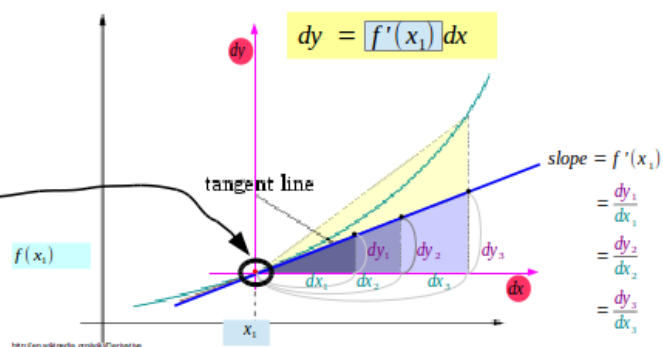
$$= f(x_1) + f'(x_1) dx$$

for small enough  $dx$

$$f(x_1 + dx) \approx f(x_1) + dy$$

$$= f(x_1) + f'(x_1) dx$$

$$\lim_{dx \rightarrow 0} \frac{f(x_1 + dx) - f(x_1)}{dx} \approx f'(x_1)$$



## Differentials and Derivatives (3)

$$dy = f'(x) dx \quad \Rightarrow \quad \int dy = \int f'(x) dx$$

$$dy = \frac{df}{dx} dx \quad \Rightarrow \quad \int dy = \int \frac{df}{dx} dx$$

$$dy = \dot{f} dx \quad y = f(x)$$

$$dy = D_x f dx \quad \int dy = \int 1 dy = y$$

# Integration Constant C

place a constant

$$\int dy = \int f'(x) dx$$

$$\int dy = \int \frac{df}{dx} dx$$

$$y + C_1 = f(x) + C_2$$

$$y = f(x) + C$$

place another constant

differs by a constant

$$\int dy = \int f'(x) dx + C$$

$$\int dy = \int \frac{df}{dx} dx + C$$

place only one constant from the beginning

$$y = f(x) + C$$

# Types of First Order ODEs

---

## A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

### Separable Equations

$$\frac{dy}{dx} = g_1(x)g_2(y)$$

$$y' = g_1(x)g_2(y)$$

$$y = f(x)$$

### Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_1(x)y' + a_0(x)y = g(x)$$

$$y = f(x)$$

### Exact Equations

$$M(x, y)dx + N(x, y)dy = 0$$

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 0$$

$$z = f(x, y)$$

1차 미분 방정식

ch 2.

- (a) Separable eq.      변수 분리      2.2
  - (b) Linear eq      선형 방정식      2.3
  - (c) Exact eq      완전 방정식      2.4
- ↑

$y = f(x)$

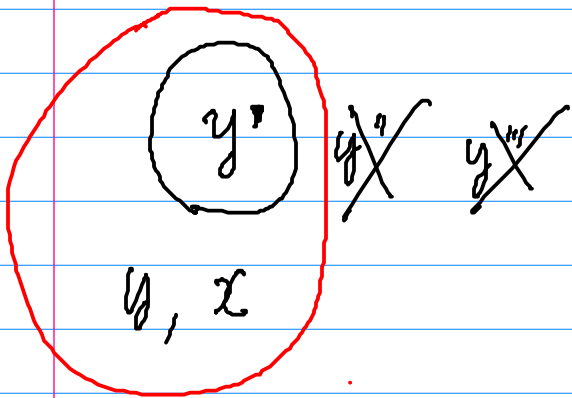
$y = g(x)$  ↕  
 $f(x, y)$

$= f(x, g(x))$



1st order differential eq

Solution:  $y(x)$ :  $y$ 는  $x$ 의 함수



1차 미분 방정식

ODE Ordinary Differential Equation 상미분방정식

PDE Partial Differential Equation 편미분방정식

$$\frac{df}{dx}$$

$f(x)$  ..... 2차원 2점

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$$

$f(x, y)$  ... 3차원 2점

~~~~~  
partial derivative.

# Separable Equation

1차 미분 방정식  $y' = g(x, y)$   $y(x) = ?$

$x$ 에 대해서  $y$ 를 미분  $x, y$ 의 식

$$y' = \boxed{g_1(x)} \times \boxed{g_2(y)}$$

( $x$  식)      ( $y$  식)      separable

$$y' \left( \frac{1}{g_2(y)} \right) = (g_1(x))$$

$$\underline{y'} \text{ (} y \text{ 식)} = (x \text{ 식)}$$

$$\underline{y'} \text{ (} y \text{ 식)} dx = (x \text{ 식)} dx$$

정규화시키기

$$(y \text{ 식}) \boxed{\frac{dy}{dx}} dx = (x \text{ 식}) dx$$

$$(y \text{ 식}) \boxed{dy} = (x \text{ 식}) dx$$

$$\int (y \text{ 식}) dy = \int (x \text{ 식}) dx$$

( $y$  식) 은

$y$ 에 대하여 적분

( $x$  식) 은

$x$ 에 대하여 적분



$$\frac{dy}{dx} = 6y^2 x = (6x)(y^2)$$

$$\textcircled{1} \quad y' = \frac{3x^2 + 4x - 4}{2y - 4} = (3x^2 + 4x - 4) \left( \frac{1}{2y - 4} \right)$$

$$\textcircled{2} \quad y' = \frac{xy^3}{\sqrt{1+x^2}} = \left( \frac{x}{\sqrt{1+x^2}} \right) (y^3)$$

$$\textcircled{3} \quad y' = e^{-y} (2x - 4) = (2x - 4) (e^{-y})$$

$$\textcircled{4} \quad \frac{dr}{d\theta} = \frac{r^2}{\theta} \quad r(\theta) \text{ z\u00e4t f\u00fcr } \theta \text{ auf en\u00f6rken v\u00f6l\u00f6}$$

$$\frac{d}{d\theta} r(\theta) = \frac{r^2(\theta)}{\theta} \Rightarrow \underline{r(\theta)?}$$

$$\frac{dr}{d\theta} = \frac{r^2}{\theta} = \left( \frac{1}{\theta} \right) (r^2)$$

$$\textcircled{5} \quad \frac{dy}{dt} = e^{y-t} \sec(y) (1+t^2) = (e^{-t} (1+t^2)) (e^y \sec(y))$$

$$\frac{dy}{dx} = 6y^2x \approx (6x)(y^2)$$

$$\frac{dy}{dx} \frac{1}{y^2} = 6x$$

$$\frac{1}{y^2} \frac{dy}{dx} dx = 6x dx$$

$$\int \frac{1}{y^2} \frac{dy}{dx} dx = \int 6x dx$$

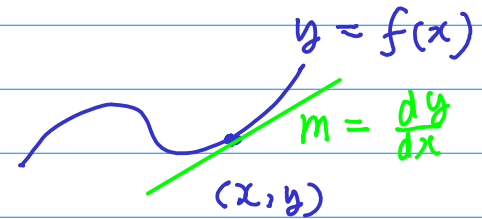
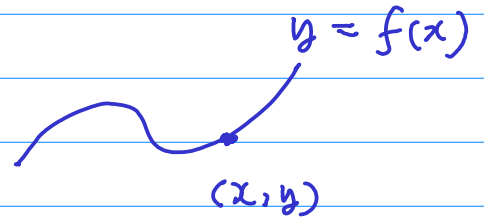
$$\int \frac{1}{y^2} dy = \int 6x dx$$

$$-\frac{1}{3} y^{-3} = 3x^2 + C$$

$$y^{-3} = -9x^2 - 3C$$

$$y^3 = \frac{1}{-9x^2 + C'}$$

$$y = \sqrt[3]{\frac{1}{-9x^2 + C'}}$$





# Linear Equation

1차 미분 방정식

$$y' = g(x, y)$$

$y(x)$ ?

$x$ 에 대해서  $y$ 의  
미분

$x, y$ 의 식

Linear Eq

$$a_1(x) \boxed{y'} + a_2(x) \boxed{y} = g(x)$$

$x$ 의 식

$x$ 의 식

$x$ 의 식

의 상수

의 상수

$\frac{1}{a_1(x)}$

$$1 \cdot \boxed{y'} + p(x) \boxed{y} = f(x)$$

$x$ 의 식

$x$ 의 식

의 상수

$$1. \boxed{y'} + p(x) \boxed{y} = q(x)$$

$$\textcircled{1} \quad \boxed{\frac{dv}{dt}} = 9.8 - 0.196 \boxed{v} \quad v(t) = ?$$

$$| \quad \boxed{\frac{dv}{dt}} + 0.196 \boxed{v} = 9.8$$

$$\textcircled{2} \quad \cos(x) \boxed{y'} + \sin(x) \boxed{y} = 2 \cos^2(x) \sin(x) - 1$$

$$| \quad \boxed{y'} + \frac{\sin(x)}{\cos(x)} \boxed{y} = 2 \cos^2(x) \sin(x) - \frac{1}{\cos(x)}$$

$$\textcircled{3} \quad t \boxed{y'} + 2 \boxed{y} = t^2 - t + 1$$

$$\boxed{y'} + \frac{2}{t} \boxed{y} = t - 1 + \frac{1}{t}$$

$$\textcircled{4} \quad t \boxed{y'} - 2 \boxed{y} = t^5 \sin(2t) - t^3 + 4t^4$$

$$| \quad \boxed{y'} - \frac{2}{t} \boxed{y} = t^4 \sin(2t) - t^2 + 4t^3$$

$$\textcircled{5} \quad 2 \boxed{y'} - \boxed{y} = 4 \sin(3t) \quad y(t) = ?$$

$$| \quad \boxed{y'} - \frac{1}{2} \boxed{y} = 2 \sin(3t)$$

homogeneous eq

$$y' + p(x)y = 0 \quad \text{Sol } \underline{y_h}$$

$$\Leftrightarrow y'_h + p(x)y_h = 0$$

non-homogeneous eq

$$y' + p(x)y = Q(x) \quad \text{Sol } \underline{y_p}$$

$$\Leftrightarrow y'_p + p(x)y_p = Q(x)$$

$$y'_h + p(x)y_h = 0$$

$$y'_p + p(x)y_p = Q(x)$$

$$(y_p + y_h)' + p(x)(y_p + y_h) = Q(x)$$

$$\Leftrightarrow \boxed{y' + p(x)y = Q(x)} \quad \text{Sol } \underline{(y_p + y_h)}$$

homogeneous eq

$$y' + p(x)y = 0 \quad \text{sol } \underline{y_h} = c \cdot y_1 \\ = c \cdot e^{-\int p(x) dx}$$

non-homogeneous eq

$$y' + p(x)y = Q(x) \quad \text{sol } \underline{y_p} = u \cdot y_1 \\ = u(x) e^{-\int p(x) dx}$$

①

$$\frac{dy}{dx} + y = 1$$

$$y = 1 + ce^{-x}$$

②

$$\frac{dy}{dx} + y = x$$

$$y = (x-1) + ce^{-x}$$

③

$$\frac{dy}{dx} + y = x^2$$

$$y = x^2 - 2x + 2 + ce^{-x}$$

particular solutions

$$y = 1$$

$$\frac{d}{dx}(1) + (1) = 0 + 1 = 1$$

$$y = x - 1$$

$$\frac{d}{dx}(x-1) + (x-1) = 1 + x - 1 = x$$

$$y = x^2 - 2x + 2$$

$$\begin{aligned} (x^2 - 2x + 2)' + (x^2 - 2x + 2) \\ = 2x - 2 + x^2 - 2x + 2 \\ = x^2 \end{aligned}$$

 $y_p$ 

homogeneous solutions

$$y = ce^{-x}$$

$$\begin{aligned} (ce^{-x})' + (ce^{-x}) \\ = -ce^{-x} + ce^{-x} = 0 \end{aligned}$$

$$y = ce^{-x}$$

$$\begin{aligned} (ce^{-x})' + (ce^{-x}) \\ = -ce^{-x} + ce^{-x} = 0 \end{aligned}$$

$$y = ce^{-x}$$

$$\begin{aligned} (ce^{-x})' + (ce^{-x}) \\ = -ce^{-x} + ce^{-x} = 0 \end{aligned}$$

 $y_h$ 

general solutions

$$y = 1 + ce^{-x}$$

$$\begin{aligned} (1 + ce^{-x})' + (1 + ce^{-x}) \\ = -ce^{-x} + 1 + ce^{-x} \\ = 1 \end{aligned}$$

$$y = x - 1 + ce^{-x}$$

$$\begin{aligned} (x - 1 + ce^{-x})' + (x - 1 + ce^{-x}) \\ = 1 - ce^{-x} + x - 1 + ce^{-x} \\ = x \end{aligned}$$

$$\begin{aligned} y = x^2 - 2x + 2 + ce^{-x} \\ = x^2 \end{aligned}$$

 $y_p$ 

+

 $y_h$



$$a_1(x) \boxed{y'} + a_2(x) \boxed{y} = g(x)$$

General Solution

$y_p$   
 $y_p + y_h$   
 infinitely many solutions

$y_p$   
 $y_p + C \cdot \boxed{x^2}$   
 $y_h$

$a_1(x) \boxed{y'} + a_2(x) \boxed{y} = \boxed{0}$

$\boxed{1} \boxed{y'} + p(x) \boxed{y}$

$C \cdot e^{-\int p(x) dx}$

# Solving the Homogeneous DE : finding $y_h$

$$\left[ \frac{dy}{dx} + P(x)y = 0 \right]$$

$$y' + P(x)y = 0$$

homogeneous solution

$$y_h = f_h(x)$$

$$\frac{dy}{dx} = -P(x)y$$

$$y' = -P(x)y$$

$$\frac{1}{y} \frac{dy}{dx} = -P(x)$$

$$\frac{1}{y} y' = -P(x)$$

$$\int \frac{1}{y} \frac{dy}{dx} dx = -\int P(x) dx$$

$$\int \frac{1}{y} y' dx = -\int P(x) dx$$

not a ratio

$$dy = \frac{df}{dx} dx$$

$$\int \frac{1}{y} dy = -\int P(x) dx$$

$$\int \frac{1}{y} dy = -\int P(x) dx$$

$$\ln|y| = -\int P(x) dx + C$$

$$\ln|y| = -\int P(x) dx + C$$

$$|y| = e^{-\int P(x) dx + C}$$

$$|y| = e^{-\int P(x) dx + C}$$

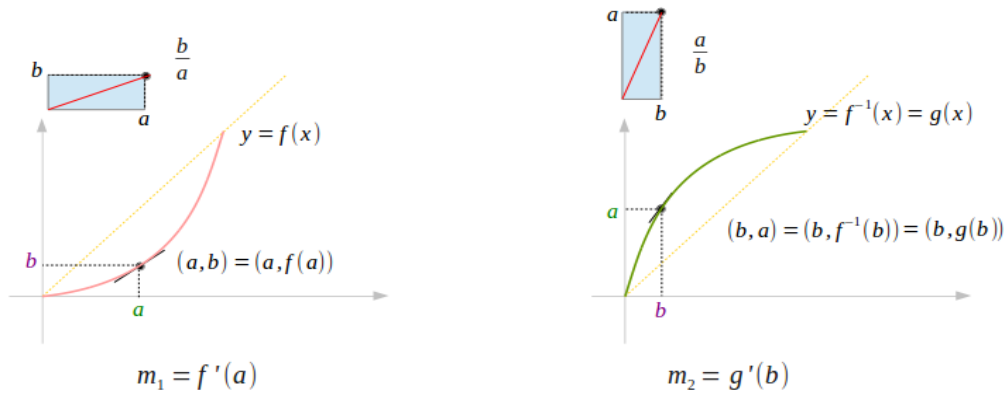
$$y = \pm e^{-\int P(x) dx} \cdot e^C$$

$$= \pm e^C \cdot e^{-\int P(x) dx}$$

$$y = c e^{-\int P(x) dx}$$

$$y = c e^{-\int P(x) dx}$$

## Derivatives of Inverse Functions



$$m_1 m_2 = f'(a) g'(b) = 1 \quad \Rightarrow \quad g'(b) = \frac{1}{f'(a)} \quad \Rightarrow \quad g'(b) = \frac{1}{f'(g(b))}$$

$g(b) = a$

## Derivatives of Inverse Functions

$$m_1 m_2 = f'(a) g'(b) = 1 \quad \Rightarrow \quad g'(b) = \frac{1}{f'(a)} \quad \Rightarrow \quad g'(b) = \frac{1}{f'(g(b))}$$

$g(b) = a$

$$g'(x) = \frac{1}{f'(g(x))}$$

- To find  $g'(x)$
- (1) find  $f'(x)$
  - (2) find  $1 / f'(x)$
  - (3) substitute  $x$  with  $g(x)$

|                   |                                                 |                                    |
|-------------------|-------------------------------------------------|------------------------------------|
|                   | $\frac{d}{dx} \ln x \Rightarrow \frac{1}{x}$    | $\frac{d}{dx} e^x \Rightarrow e^x$ |
| $f'(x)$           | $(e^x)' \Rightarrow e^x$                        | $(\ln x)' = \frac{1}{x}$           |
| $\frac{1}{f'(x)}$ | $\rightarrow \frac{1}{e^x}$                     | $\rightarrow \frac{1}{1/x} = x$    |
| $g'(x)$           | $\rightarrow \frac{1}{e^{\ln x}} = \frac{1}{x}$ | $\rightarrow e^x$                  |

## Derivative of $\ln |x|$

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad x > 0$$

$$\frac{d}{dx} \ln(-x) = \frac{1}{(-x)} \cdot (-1) = \frac{1}{x} \quad x < 0$$

$$\begin{array}{llll} \ln x & x > 0 & \xrightarrow{\frac{d}{dx}} & \frac{1}{x} \\ \ln(-x) & x < 0 & \xrightarrow{\frac{d}{dx}} & \frac{1}{x} \end{array}$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x} \quad x \neq 0$$

# Method of Solving First Order ODEs

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \leftarrow y_p? \quad y' + P(x)y = Q(x) \quad y = f(x)$$

$$\left[ e^{+\int P(x)dx} \right] \left[ \frac{dy}{dx} + P(x)y \right] = \left[ e^{+\int P(x)dx} \right] Q(x) \quad y_1 = e^{-\int P(x)dx} \quad \frac{1}{y_1} = e^{+\int P(x)dx}$$

$$\left[ e^{+\int P(x)dx} \right] \frac{dy}{dx} + \left[ e^{+\int P(x)dx} \right] P(x)y = \left[ e^{+\int P(x)dx} \right] Q(x) \quad \left[ e^{+\int P(x)dx} \right] \cdot P(x) = \frac{d}{dx} \left[ e^{+\int P(x)dx} \right] \quad f'(g(x))g'(x)$$

$$\left[ e^{+\int P(x)dx} \right] \frac{dy}{dx} + \frac{d}{dx} \left[ e^{+\int P(x)dx} \right] y = \left[ e^{+\int P(x)dx} \right] Q(x) \quad \left[ e^{+\int P(x)dx} \right] \frac{dy}{dx} + \frac{d}{dx} \left[ e^{+\int P(x)dx} \right] y = \frac{d}{dx} \left[ e^{+\int P(x)dx} \cdot y \right]$$

$$\frac{d}{dx} \left[ e^{+\int P(x)dx} \cdot y \right] = \left[ e^{+\int P(x)dx} \right] Q(x) \quad f'(x)g(x) + f(x)g'(x)$$

$$\int \frac{d}{dx} \left[ e^{+\int P(x)dx} \cdot y \right] dx = \int \left[ e^{+\int P(x)dx} \right] Q(x) dx + c$$

$$\left[ e^{+\int P(x)dx} \right] \cdot y = \int \left[ e^{+\int P(x)dx} \right] Q(x) dx + c \quad \rightarrow \quad y(x) = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \cdot \left[ \int Q(x) \cdot e^{+\int P(x)dx} dx \right]$$

First Order ODEs (1A)

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3/24/15

$$\left( e^{+\int P(x)dx} \cdot y \right)' = e^{+\int P(x)dx} \cdot y' + e^{+\int P(x)dx} \cdot y$$

$$= e^{+\int P(x)dx} \cdot y' + e^{+\int P(x)dx} P(x) \cdot y$$

$$= e^{+\int P(x)dx} \cdot \left( y' + P(x) \cdot y \right)$$

think the reverse order

the given problem

$$y' + p(x)y = f(x)$$

$$e^{+\int P(x)dx} \cdot (y' + p(x)y) = f(x) \cdot e^{+\int P(x)dx}$$

$$\left( e^{+\int P(x)dx} \cdot y \right)' = f(x) \cdot e^{+\int P(x)dx}$$

$$\left( e^{+\int P(x)dx} \cdot y \right) = \int f(x) \cdot e^{+\int P(x)dx} dx$$

the given problem  $y' + p(x)y = f(x)$

$$\Rightarrow e^{\int p(x) dx} \cdot (y' + p(x)y) = f(x) \cdot e^{\int p(x) dx}$$

$$\Rightarrow \left( e^{\int p(x) dx} \cdot y \right)' = f(x) \cdot e^{\int p(x) dx}$$

$$\left( e^{\int p(x) dx} \cdot y \right) = \int f(x) \cdot e^{\int p(x) dx} dx + C$$

$$\left( e^{\int p(x) dx} \cdot y \right) = \int f(x) \cdot e^{\int p(x) dx} dx + C$$

$$y = \frac{1}{e^{\int p(x) dx}} \int f(x) \cdot e^{\int p(x) dx} dx + \frac{C}{e^{\int p(x) dx}}$$

$$y = \underbrace{e^{-\int p(x) dx} \int f(x) \cdot e^{\int p(x) dx} dx}_{y_p} + \underbrace{C e^{-\int p(x) dx}}_{y_h}$$

# 1st Order Linear Equations

y' + p(x)y = Q(x)

$$y' + p(x)y = Q(x)$$

$$y_h = c \cdot e^{-\int p(x) dx}$$

$$y_p = u(x) e^{-\int p(x) dx}$$

$$e^{+\int p(x) dx}$$

Integrating Factor

- Const. c X
- sign X

$$e^{+\int p(x) dx}$$

$$(y' + p(x)y) = Q(x)$$

$$e^{+\int p(x) dx}$$

$$\left( e^{+\int p(x) dx} \cdot y_p \right)' = Q(x) e^{+\int p(x) dx}$$

$$e^{+\int p(x) dx} \cdot y_p = \int Q(x) e^{+\int p(x) dx} dx$$

$$y_p = e^{-\int p(x) dx} \int Q(x) e^{+\int p(x) dx} dx$$

$$y = y_h + y_p$$

$$y = \underbrace{c \cdot e^{-\int p(x) dx}}_{y_h} + \underbrace{e^{-\int p(x) dx} \int Q(x) e^{+\int p(x) dx} dx}_{y_p}$$

## Calculus 1

Review : Exponential

Review : Logarithmic

Derivatives : Trig Derivatives

## Differential Equation

First Order : Linear Equation

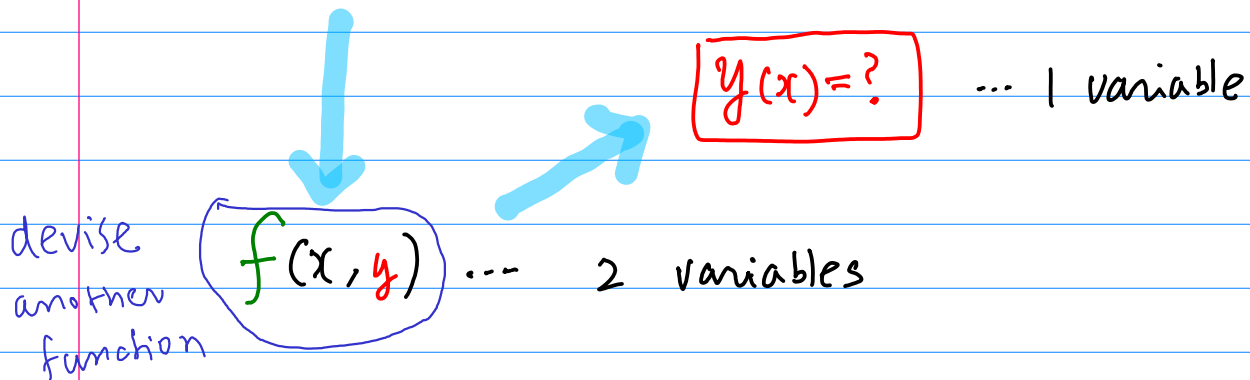


First Order Differential Equations (1P.pdf) .....  
Linear Equation (2A.pdf)

cf) Partial Derivatives (9.4 Zill & Wright)

# Exact Equation

$$M(x, y) dx + N(x, y) dy = 0$$



**Exact** →  $f(x, y)$  exists

total differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$
$$= M(x, y) dx + N(x, y) dy = 0$$

check  $\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$

$$\frac{\partial}{\partial y} (M) = \frac{\partial}{\partial x} (N)$$

$$df = 0$$

$$f(x, y) = C$$

$$y(x)$$

← Curve ← Surface ∩  $\mathbb{R}^2$

$$M(x, y) dx + N(x, y) dy = 0$$

$$\text{check } \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \stackrel{?}{=} \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$$

if yes, there exists  $f(x, y)$  and

$$M(x, y) dx + N(x, y) dy = 0 \rightarrow df = 0$$

$$\rightarrow f = c$$

$$f(x, y) = \text{const.}$$

||

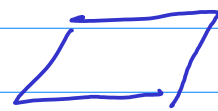
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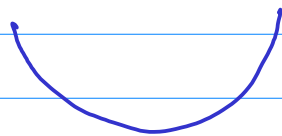
z

a surface  
in  $\mathbb{R}^3$

a plane



parallel to x-y plane



intersection

" $\cap$ "



curve ...



$y = f(x)$  ... solution.

$$(2xy - 9x^2) + (2y + x^2 + 1) \frac{dy}{dx} = 0$$

$$(2xy - 9x^2) dx + (2y + x^2 + 1) dy = 0$$

$$\begin{array}{l} \boxed{\begin{array}{c} \parallel \\ M \\ \parallel \\ \frac{\partial f}{\partial x} \end{array}} dx + \boxed{\begin{array}{c} \parallel \\ N \\ \parallel \\ \frac{\partial f}{\partial y} \end{array}} dy = 0 \\ \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \end{array}$$

\* Exact?

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$2x = 2x \quad \therefore \text{exact}$$

$$f(x, y) = \int \frac{\partial f}{\partial x} dx + C$$

variable  $y$  may be included

$$= \int \frac{\partial f}{\partial x} dx + g(y)$$

$$= \int (2xy - 9x^2) dx + g(y)$$

$$f(x, y) = x^2y - 3x^3 + g(y) ?$$

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = (2y + x^2 + 1)$$

$$g'(y) = 2y + 1$$

$$g(y) = y^2 + y$$

$$f(x, y) = x^2y - 3x^3 + y^2 + y$$

$$df = 0$$

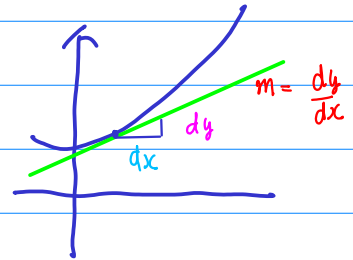
$$f(x, y) = x^2y - 3x^3 + y^2 + y = C$$

$$a(x, y) + b(x, y) \frac{dy}{dx} = 0$$

$$\underline{y(x) = ?}$$

$$\Downarrow$$
$$a(x, y) dx + b(x, y) \frac{dy}{dx} dx = 0$$

$$a(x, y) dx + b(x, y) dy = 0$$



$$\boxed{M(x, y) dx + N(x, y) dy = 0} \implies \boxed{df = 0}$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$\Downarrow$$
$$\boxed{f(x, y) = C}$$

$f(x, y)$

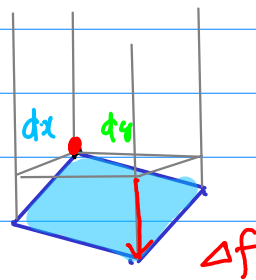
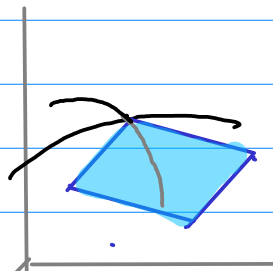
When

$x : dx$   
 $y : dy$   
 $f : df$

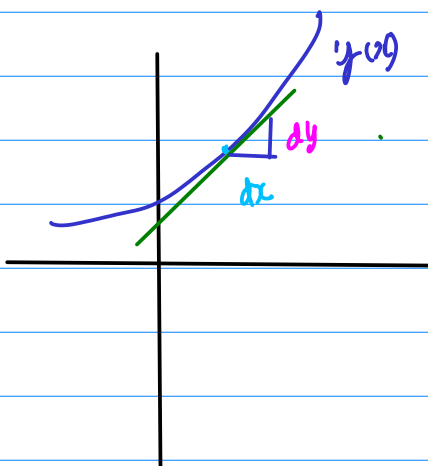
$x \rightarrow x + dx$   
 $y \rightarrow y + dy$

change in the value of  $f(x, y)$

$$df = 0 \Rightarrow \text{no change} \Rightarrow \text{constant}$$



total differential  $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \dots \dots \mathbb{R}^3, z = f(x, y)$



$$dy = \frac{dy}{dx} dx \dots \dots \mathbb{R}^2, y = f(x)$$

$$\left( df = \frac{df}{dx} dx \right)$$

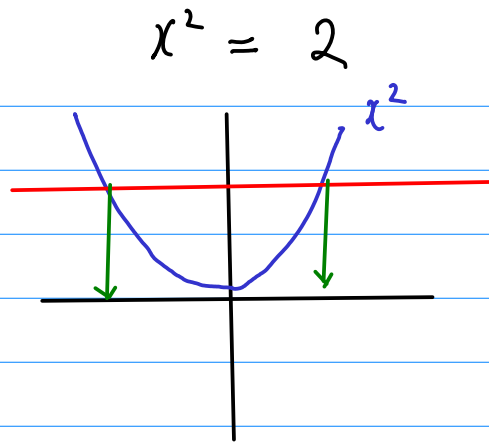
\* exactness condition

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$$

$$\frac{\partial}{\partial y} (M) = \frac{\partial}{\partial x} (N)$$

if this condition is not met,

there is no such  $f(x, y)$



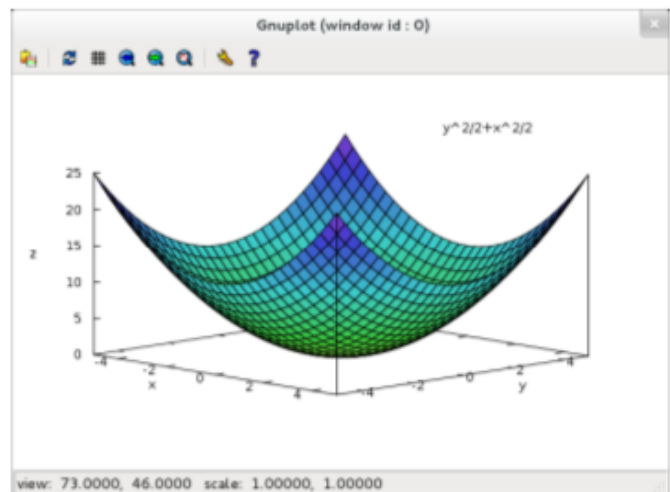
## Exact Equation Method (2)

### A differential form

$$P(x, y)dx + Q(x, y)dy$$

this differential form is **exact** in a region **R** if there is a function  $f(x, y)$  such that

$$\begin{aligned} P(x, y)dx + Q(x, y)dy \\ = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df \end{aligned}$$



### differential form

$$x dx + y dy \quad \Rightarrow$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$$

### exact differential form

Since there exists such

$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + c_1$$

$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + C$$



$$df = 0 \Leftrightarrow f = c$$

$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} = c'$$

## Exact Equation Method (4)

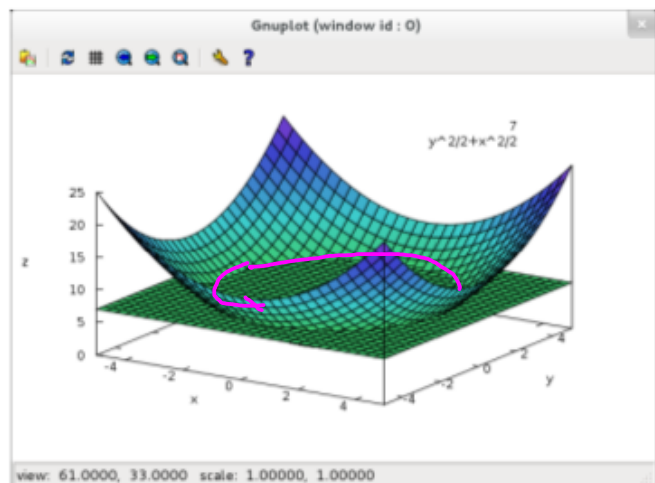
differential equation

$$x dx + y dy = 0 \quad \Rightarrow$$

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df = 0$$

The implicit solution is

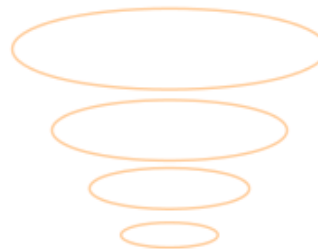
$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + c_1 = c$$



$$f(x, y) = \frac{x^2}{2} + \frac{y^2}{2} = 7$$

The solution is all the points  $(x, y)$  that satisfies  $f(x, y) = 7$

different  $c$   
In 3-d space



different  $c$   
In 2-d space



$$f(x, y)$$

$$df = x : dx \rightarrow \\ y : dy \uparrow$$

$$\boxed{M(x, y) dx + N(x, y) dy = 0} \implies \boxed{df = 0}$$

$f(x, y) \dots 2 \text{ variables}$

$y(x) = ? \dots 1 \text{ variable}$

$f(x, y) = \underline{c}$

$f(x, y(x))$

$$\underbrace{M(x, y)}_{\frac{\partial f}{\partial x}} dx + \underbrace{N(x, y)}_{\frac{\partial f}{\partial y}} dy = 0$$

① Verify "exactness"

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

②

$$\underbrace{M(x, y)}_{\frac{\partial f}{\partial x}}$$

$$f(x, y) = \int M(x, y) dx + g(y)$$

③

$$\underbrace{N(x, y)}_{\frac{\partial f}{\partial y}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left[ \int M(x, y) dx \right] + g'(y) \\ = N(x, y)$$

④

$$\underbrace{f(x, y)}_{=} = \underbrace{c}_{}$$

level surface  $\cap \mathbb{R}^2 \Rightarrow y(x)$

## Non-Exact Case

Zill (4.4)

$$xy \, dx + (2x^2 + 3y^4 - 20) \, dy = 0$$

$$\frac{\partial}{\partial y} (xy) = x \neq \frac{\partial}{\partial x} (2x^2 + 3y^4 - 20) = 4x$$

non-exact eq

$$\boxed{y^3} \leftarrow$$

Integrating Factor  $\mu(x)$   $\mu(y)$

$$\boxed{xy \cdot y^3} \, dx + \boxed{(2x^2 + 3y^4 - 20) y^3} \, dy = 0$$

$$xy^4 \, dx + (2x^2y^3 + 3y^5 - 20y^3) \, dy = 0$$

$$\frac{\partial}{\partial y} (xy^4) = 4xy^3 = \frac{\partial}{\partial x} (2x^2y^3 + 3y^5 - 20y^3) = 4xy^3$$

Now exact!

## Non-Exact Case : Finding Integrating Factor

$$\text{Exact} \rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad M_y = N_x$$

$$M_y - N_x = 0$$

$$\text{Non-Exact} \rightarrow M_y - N_x \neq 0$$

$$\frac{M_y - N_x}{N} : x \text{의 함수} \checkmark \quad \text{I.F.} = e^{\int \frac{M_y - N_x}{N} dx}$$

$$- \frac{M_y - N_x}{M} : y \text{의 함수} \checkmark \quad \text{I.F.} = e^{\int \frac{M_y - N_x}{M} dy}$$

# Integrating Factor

- sign x
- const c x

$$\mu(x) = e^{\int \frac{My - Nx}{N} dx} \Leftrightarrow \frac{My - Nx}{N} \dots x \text{의 함수}$$

$$\mu(y) = e^{\int \frac{-My + Nx}{M} dy} \Leftrightarrow -\frac{My - Nx}{M} \dots y \text{의 함수}$$

$$M(x, y) dx + N(x, y) dy = 0$$

$$\frac{\partial M}{\partial y} = M_y$$

$$\frac{\partial N}{\partial x} = N_x$$

difference

x 또는  
y 함수

$$xy \, dx + (2x^2 + 3y^2 - 20) \, dy = 0$$

$$\frac{\partial}{\partial y} (xy) = x \neq \frac{\partial}{\partial x} (2x^2 + 3y^2 - 20) = 4x$$

non-exact eq

$$\frac{\partial M}{\partial y} = M_y = x \quad \frac{\partial N}{\partial x} = N_x = 4x$$

$x$  only expression

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

$$\mu(x) = e^{\int \frac{-3x}{(2x^2 + 3y^2 - 20)} dx}$$

$$xy \, dx + (2x^2 + 3y^2 - 20) \, dy = 0$$

$$\frac{\partial}{\partial y} (xy) = x \neq \frac{\partial}{\partial x} (2x^2 + 3y^2 - 20) = 4x$$

non-exact eq

$$\frac{\partial M}{\partial y} = M_y = x \quad \frac{\partial N}{\partial x} = N_x = 4x$$

$x$  only expression

$$\mu(y) = e^{\int \frac{-M_y + N_x}{N} dy} \Leftrightarrow -M + N \dots y \text{ only}$$

$$\begin{aligned} \mu(y) &= e^{\int \frac{3x}{xy} dy} = e^{\int \frac{3}{y} dy} = e^{\ln|y|^3} \\ &= |y|^3 \end{aligned}$$

$$\mu(y) = y^3, \quad -y^{-3} \Rightarrow y^3$$

$$x y \cdot y^3 dx + (2x^2 + 3y^2 - 20) y^3 dy = 0$$

# integrating factor 공식 유도 과정

$$(f \cdot g)' = f'g + fg'$$

$$\textcircled{2} \quad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

$$\begin{aligned} \frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} &= \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x} \\ \mu \frac{\partial M}{\partial y} &= \frac{d\mu}{dx} N + \mu \frac{\partial N}{\partial x} \\ \frac{d\mu}{dx} N &= \mu \frac{\partial M}{\partial y} - \mu \frac{\partial N}{\partial x} \end{aligned}$$

$\mu(x)$   
y가 없는 수식

$$\textcircled{1} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{d\mu}{dx} N = \mu M_y - \mu N_x$$

$$\frac{d\mu}{dx} = \left( \frac{M_y - N_x}{N} \right) \mu$$

$$\frac{d\mu}{dx} = P(x)\mu(x)$$

한번  $\square$ 에 y가 없는 수식이

이

$$\frac{d\mu}{dx} - P(x)\mu = 0$$

$$\mu(x) = c e^{\int P(x) dx}$$

$$\mu(x) = e^{\int \left( \frac{M_y - N_x}{N} \right) dx}$$



# Substitution Method

$$\frac{dy}{dx} = g(x, y) \implies \left(\frac{y}{x}\right) \text{ 항상 미분 가능한 일.}$$

$$= u \quad y = ux$$

Substitute

ex)  $y = ux$

$$y = h(x, u)$$

$$u = \Phi(x)$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

$$\frac{d}{dx} y = \frac{d}{dx} h(x, u)$$

$$\frac{\partial h}{\partial x} = u \quad \frac{\partial h}{\partial u} = x \quad \frac{du}{dx} = u' \implies u + xu'$$

$$g(x, h(x, u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

$$g(x, h(x, u)) = h_x(x, y) + h_u(x, u) \frac{du}{dx}$$

$$\frac{dy}{dx} = g(x, y)$$

\* Substitute  $y$   
and  $y'$

$$\left(\frac{y}{x}\right) \rightarrow u$$

$$y \rightarrow ux$$

$$y' \rightarrow u'x + u$$

both  $y$  &  $y'$

$$x y y' + 4x^2 + y^2 = 0$$

divide by  $x^2$

$$\left(\frac{y}{x}\right)y' + 4 + \left(\frac{y}{x}\right)^2 = 0$$

$\uparrow$   
 $u$

$\uparrow$   
 $u$

$$u = \frac{y}{x} = \frac{y(x)}{x}$$

$$\frac{x \frac{dy}{dx}}{x^2 \frac{dx}{dx}}$$

Substitute both  $y$  and  $y'$

$$y = xu$$

$$y' = u + xu'$$

$$\left(\frac{y}{x}\right)y' + 4 + \left(\frac{y}{x}\right)^2 = 0$$

$$\left(\frac{xu}{x}\right) \cdot (u + xu') + 4 + \left(\frac{xu}{x}\right)^2 = 0$$

$$u(u + xu') + 4 + u^2 = 0$$

$$2u^2 + xu'u' + 4 = 0$$

$$2u^2 + xu' + 4 = 0$$

$$\frac{xu' + 4}{u} = -2u$$

$$xu' = -\frac{4 + 2u^2}{u}$$

$$\left(\frac{u}{4 + 2u^2}\right) u' = -\frac{1}{x}$$

$$u' = \frac{du}{dx}$$

$$\frac{u}{4 + 2u^2} \cdot \frac{du}{dx} dx = -\frac{1}{x} dx$$

$$\frac{u}{4 + 2u^2} du = -\frac{1}{x} dx$$

$$\frac{1}{4} \int \frac{4u}{4 + 2u^2} du = \int -\frac{1}{x} dx$$

$$\ln(4 + 2v^2)^{\frac{1}{4}} = \ln(x)^{-1} + c$$

sides and do a little rewriting

$$(4 + 2v^2)^{\frac{1}{4}} = e^{\ln(x)^{-1} + c} = e^c e^{\ln(x)^{-1}} = \frac{c}{x}$$

$$\frac{1}{4} \ln|4 + 2u^2| = -\ln|x| + C$$

$$\ln|4 + 2u^2|^{\frac{1}{4}} = \ln|x|^{-1} + C$$

$$|4 + 2u^2|^{\frac{1}{4}} = e^{\ln|x|^{-1} + C} = e^{\ln|x|^{-1}} e^C$$

$$(4 + 2u^2)^{\frac{1}{4}} = \frac{1}{x} C$$

$$(4+2u^2)^{\frac{1}{4}} = \frac{1}{x} \left( \frac{C}{x} \right)$$

$$(4+2u^2) = \left( \frac{C}{x} \right)^4$$

$$u = \frac{y}{x}$$

$$4 + 2 \left( \frac{y}{x} \right)^2 = \left( \frac{C}{x} \right)^4$$

$$2 \left( \frac{y}{x} \right)^2 = \left( \frac{C_1}{x^4} - 4 \right)$$

$$\frac{y^2}{x^2} = \frac{1}{2} \left( \frac{C_1 - 4x^4}{x^4} \right)$$

$$y^2 = \frac{1}{2} x^2 \left( \frac{C_1 - 4x^4}{x^4} \right)$$

$$= \frac{C_1 - 4x^4}{2x^2}$$

$$x y' = y (\ln x - \ln y)$$

$$\boxed{x > 0}$$

$$x y' = y \ln\left(\frac{x}{y}\right)$$

$$y' = \left(\frac{y}{x}\right) \ln\left(\frac{x}{y}\right)$$

$$\frac{y}{x} = u$$

$$y = x u$$

$$y' = u + x u'$$

$$(u + x u') = u \ln\left(\frac{1}{u}\right)$$

$$x u' = u \ln\left(\frac{1}{u}\right) - u$$

$$\int \frac{1}{u (\ln(\frac{1}{u}) - 1)} \frac{du}{dx} dx = \int \frac{1}{x} dx$$

$$\int \frac{1}{u (\ln(\frac{1}{u}) - 1)} du = \int \frac{1}{x} dx$$

$$\int \frac{-1}{(\ln(\frac{1}{u}) - 1)} \left(\frac{-1}{u} du\right)$$

$$- \int \frac{1}{(\ln(\frac{1}{u}) - 1)} dt = \int \frac{1}{x} dx$$

$$\ln\left(\ln\left(\frac{1}{u}\right) - 1\right) = -(\ln x + c)$$

$$\ln\left(\ln\left(\frac{1}{u}\right) - 1\right) = c - \ln x$$

$$\begin{aligned} t &= \ln\left(\frac{1}{u}\right) - 1 \\ dt &= \left(\frac{1}{u}\right)' du \\ dt &= u \cdot (-u^{-2}) du \\ dt &= -\frac{1}{u} du \end{aligned}$$

$$dt = u du$$

$$\ln(\ln\left(\frac{1}{u}\right) - i) = C - \ln x$$

$$\begin{aligned}\ln\left(\frac{1}{u}\right) - i &= e^{C - \ln x} \\ &= c_1 e^{\ln \frac{1}{x}} \\ &= \frac{c_1}{x}\end{aligned}$$

$$\ln\left(\frac{1}{u}\right) - i = \frac{c_1}{x}$$

$$\ln\left(\frac{1}{u}\right) = \frac{c_1}{x} + 1$$

$$\frac{1}{u} = e^{\frac{c_1}{x} + 1}$$

$$\underline{u} = e^{-\frac{c_1}{x} - 1}$$

$$\frac{y}{x} = e^{-\frac{c_1}{x} - 1}$$

$$y = x \cdot e^{-\frac{c_1}{x} - 1}$$

\*  $\int \tan x \, dx \rightarrow$  substitution  $u = \cos x$

$$\int t^2 \sin(2t) \, dt \rightarrow$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

$$u = \cos x$$

$$du = -\sin x \, dx$$

$$= \int \frac{-1}{u} \, du = -\ln|u| = \ln|u|^{-1}$$

$$= \ln\left|\frac{1}{\cos x}\right| = \ln|\sec x|$$

integrating factor  $\mu(x)$

$\rightarrow$

Multiplied to the original differential eq

$$y' + p(x)y = Q(x)$$

$$\pm \mu(x) (y' + p(x)y) = (Q(x)) \pm \mu(x)$$

40p Ex 6)

$$\frac{2x}{39}$$

$$\int f g' dt = f g - \int f' g dt$$

$$\int t^2 \sin(2t) dt$$

$$= -\frac{1}{2} t^2 \cos(2t) - \int (t^2)' \left[ \frac{1}{2} \cos(2t) \right] dt$$

$$= -\frac{1}{2} t^2 \cos(2t) + \left[ \int t \cos(2t) dt \right]$$

$$\left[ t \left[ \frac{1}{2} \sin(2t) \right] - \int t' \left[ \frac{1}{2} \sin(2t) \right] dt \right]$$

$$\frac{1}{2} t \sin(2t) - \frac{1}{2} \int \sin(2t) dt$$

$$- \frac{1}{2} \left[ -\frac{1}{2} \cos(2t) \right]$$

$$-\frac{1}{2} t^2 \cos(2t) + \frac{1}{2} t \sin(2t) + \frac{1}{4} \cos(2t)$$

$$-\frac{1}{2} \left( t^2 - \frac{1}{2} \right) \cos(2t) + \frac{1}{2} t \sin(2t)$$

(%i2) integrate(%, t);

(%o2)  $\frac{4 t \sin(2 t) + (2 - 4 t^2) \cos(2 t)}{8}$



$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + c_1 = (x-1)e^x + c_1$$

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx = x^2 e^x - 2x e^x + 2e^x + c_2 = (x^2 - 2x + 2)e^x + c_2$$

$$\int x^3 e^x dx = x^3 e^x - 3 \int x^2 e^x dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c_3 = (x^3 - 3x^2 + 6x - 6)e^x + c_3$$

$$\int x e^x dx = (x-1)e^x + c_1$$

$$\int x^2 e^x dx = (x^2 - 2x + 2)e^x + c_2$$

$$\int x^3 e^x dx = (x^3 - 3x^2 + 6x - 6)e^x + c_3$$

$$\int e^x dx = \int \left\{ \frac{d}{dx} e^x \right\} dx = e^x + c$$

$$\int x e^{x^2/2} dx = \int \left\{ \frac{d}{dx} e^{x^2/2} \right\} dx = e^{x^2/2} + c$$

$$\int x^2 e^{x^3/3} dx = \int \left\{ \frac{d}{dx} e^{x^3/3} \right\} dx = e^{x^3/3} + c$$

Chain Rule

$$f(g(x)) \xrightarrow{\frac{d}{dx}} f'(g(x)) g'(x)$$

:  $\int \cdot dx$

Substitution rule

Product Rule

$$(fg)' = f'g + fg'$$

$$fg' = (fg)' - f'g$$

$$\int fg' dx = \int (fg)' dx - \int f'g dx$$

$$\int fg' dx = fg - \int f'g dx$$

$$(x^n)' = n x^{n-1}$$

$$\int x^n dx = \frac{1}{n+1} x^{n+1}$$

$$\int \frac{1}{t^2} dt = \int t^{-2} dt = \frac{1}{-2+1} t^{-2+1} = -1 \cdot t^{-1} = \frac{-1}{t}$$

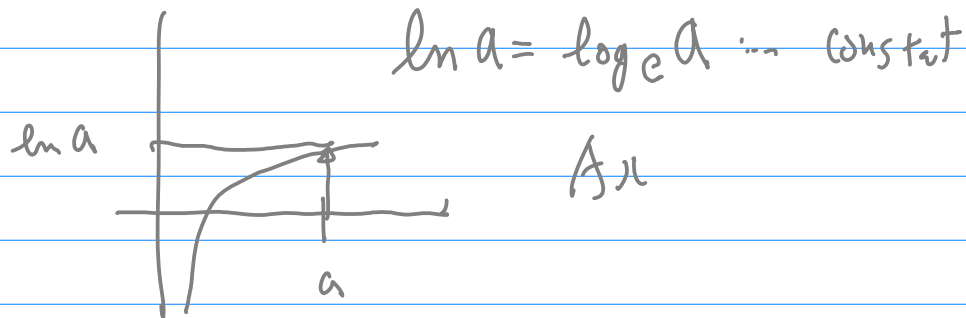
$$\left(-\frac{1}{t}\right)' = (-t^{-1})' = -(-1) t^{-1-1} = t^{-2} = \frac{1}{t^2}$$

$$(e^x)' = e^x$$

$$(e^{2x+3})' = e^{2x+3} \cdot (2x+3)' = e^{2x+3} \cdot 2$$

$$(e^{x^2+1})' = e^{x^2+1} \cdot (x^2+1)' = e^{x^2+1} \cdot 2x$$

$$(e^{x \ln a})' = e^{x \ln a} \cdot (x \ln a)' = e^{x \ln a} \cdot \ln a$$



$$a = e^{\log_e a} = 10^{\log_{10} a} = 99^{\log_{99} a}$$
$$= e^{\ln a}$$

$$e^{x \ln a} = (e^x)^{\ln a} = (e^{\ln a})^x$$

$$(2^2)^3 = 4^3 = 64$$

$$(2^3)^2 = 8^2 = 64$$

$\Downarrow$   
 $a^x$

$$(e^{x \ln a})' = e^{x \ln a} \cdot \ln a$$

$$(a^x)' = (a^x) \cdot \ln a$$

$$\begin{aligned}\frac{d}{dx}(\log_a x) &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\ &= \frac{1}{\ln a} \frac{d}{dx}(\ln x) \\ &= \frac{1}{x \ln a}\end{aligned}$$

that  $a$  was a constant and so  $\ln a$  is also  
Putting all this together gives,

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Here is a summary of the derivatives in this section:

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Pauls online math note  
Calculus 1

Integral :  
Computing Indefinite Integrals  
Substitution Rule  
More about substitution rule

a.     a  
c.     c  
e.     e

b  
d  
f  
h

Differential Equation  
1st Order

Exact Equation  
Substitution

다 풀고

range

Graph x

성질 정리

initial     condi

IVP

$$f(x,y) = c$$

## 2.2 separable

Sop Ex1)  $(1+x) dy - y dx = 0$

Ex2)  $\frac{dy}{dx} = -\frac{x}{y}$

$$y(4) = -3$$

$$\begin{pmatrix} x=4 \\ y=-3 \end{pmatrix} \quad (4, -3)$$

Ex3)  $\frac{dy}{dx} = y^2 - 4$

## 2.3 Linear Eq

60p Ex 1)  $\frac{dy}{dx} - 3y = 6$

Ex 2)  $x \frac{dy}{dx} - 4y = x^4 e^x$

Ex 3)  $(x^2 - 9) \frac{dy}{dx} + xy = 0$

Exact Equation

P 90 Ex 1)  $2xy dx + (x^2 - 1) dy = 0$

Ex 2)  $(e^{2y} - y \cos xy) dx + (2xe^{2y} - x \cos xy + 2y) dy = 0$

Ex 3)  $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}$ ,  $y(0) = 2$

$$\int x e^x dx =$$

$$\int x^2 e^x dx =$$

$$\int x^3 e^x dx =$$

$$\int x \cos x dx =$$

$$\int x^2 \cos x dx =$$

$$\int x \sin x dx =$$

$$\int x^2 \sin x dx =$$

```
(%i1) integrate( x*e^x, x );
```

```
(%o1) (x-1)%e^x
```

```
(%i2) integrate( x^2*e^x, x );
```

```
(%o2) (x^2-2x+2)%e^x
```

```
(%i3) integrate( x^3*e^x, x );
```

```
(%o3) (x^3-3x^2+6x-6)%e^x
```

```
(%i4) integrate( x*cos(x), x );
```

```
(%o4) x sin(x)+cos(x)
```

```
(%i5) integrate( x^2*cos(x), x );
```

```
(%o5) (x^2-2) sin(x)+2x cos(x)
```

```
(%i6) integrate( x*sin(x), x );
```

```
(%o6) sin(x)-x cos(x)
```

```
(%i7) integrate( x^2*sin(x), x );
```

```
(%o7) 2x sin(x)+(2-x^2) cos(x)
```



Substitution Rule

$$\int \underset{\substack{\parallel \\ u}}{f(g(x))} g'(x) dx = \int f(u) du$$

Chain rule

$$\frac{d}{dx} f(g(x)) = \frac{d}{dg} f(g) \frac{d}{dx} g(x)$$

$$\begin{aligned}
 \textcircled{1} \quad \int \cos x \sin x \, dx &= -\int (\cos x)(\cos x)' \, dx \\
 &= -\int u \, du \\
 &= -\frac{1}{2} u^2 + C \\
 &= \underline{\underline{-\frac{1}{2} (\cos x)^2 + C}}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{2} \quad \int \frac{1}{2} \sin 2x \, dx &= -\frac{1}{4} \cos 2x + C \\
 &= -\frac{1}{4} ((\cos x)^2 - (\sin x)^2) + C \\
 &= -\frac{1}{4} ((\cos x)^2 - (1 - (\cos x)^2)) + C \\
 &= -\frac{1}{2} (\cos x)^2 + \frac{1}{4} + C \\
 &= \underline{\underline{-\frac{1}{2} (\cos x)^2 + C'}}
 \end{aligned}$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$





