ASP Background (1A)

- Linear Regression
- Polynomial Regression
- Multiple Regression
- General Multiple Regression
- Least Squares
- Linear Least Squares

Young Won Lim
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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Regression

Linear Regression

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - (a_0 + a_1 x_i)\right)^2$$

Polynomial Regression

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - (a_0 + a_1 x_i + a_2 x_i^2)\right)^2$$

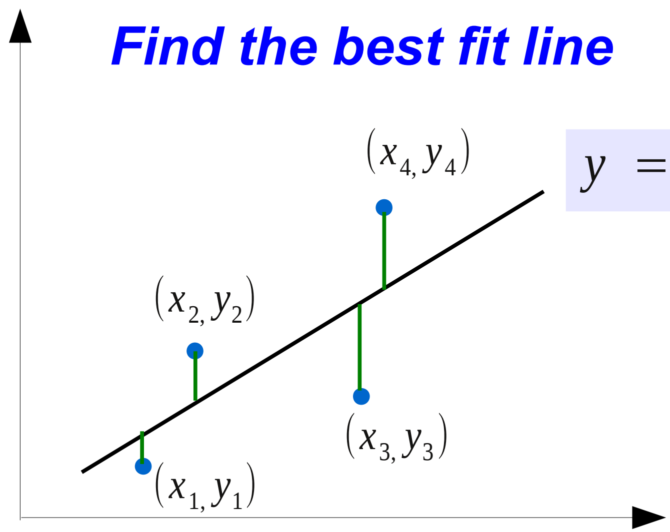
Multiple Linear Regression

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - (a_0 + a_1 x_{i,\mathbf{1}} + a_2 x_{i,\mathbf{2}})\right)^2$$

General Multiple Linear Regression

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(\beta_{\mathbf{0}} + \sum_{\mathbf{j}=1}^{m} x_{i\mathbf{j}} \beta_{\mathbf{j}}\right)\right)^2$$

Linear Regression (1)



$$y = a_0 + a_1 x$$

$a_0, \ a_1$ *unknowns*

$(x_i, \ y_i)$ *measured data*

random

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(a_0 + a_1 x_i\right)\right)^2$$

Linear Regression (2)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - (a_0 + a_1 x_i)\right)^2$$

$a_0, \; a_1$ *unknowns*

$(x_i, \; y_i)$ *measured data*

random

Minimum Condition

$$\frac{\partial S_r}{\partial a_0} = 2 \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)(-1) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} a_0 + \sum_{i=1}^{n} a_1 x_i = \sum_{i=1}^{n} y_i$$

$$\frac{\partial S_r}{\partial a_1} = 2 \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)(-x_i) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} a_0 x_i + \sum_{i=1}^{n} a_1 x_i^2 = \sum_{i=1}^{n} y_i x_i$$

$$\begin{bmatrix} \sum_{i=1}^{n} 1 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

Linear Regression (3)

$$\sum_{i=1}^{n} a_0 + \sum_{i=1}^{n} a_1 x_i = \sum_{i=1}^{n} y_i$$

$$n \cdot a_0 + \sum_{i=1}^{n} a_1 x_i = \sum_{i=1}^{n} y_i$$

$$a_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{n}\sum_{i=1}^{n} a_1 x_i$$

$$\sum_{i=1}^{n} a_0 x_i + \sum_{i=1}^{n} a_1 x_i^2 = \sum_{i=1}^{n} y_i x_i$$

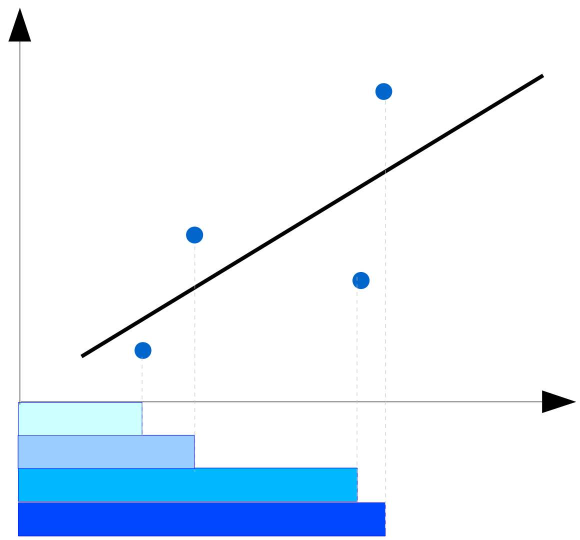
$$\left(\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{n}\sum_{i=1}^{n} a_1 x_i\right) + \sum_{i=1}^{n} a_1 x_i^2 = \sum_{i=1}^{n} y_i x_i$$

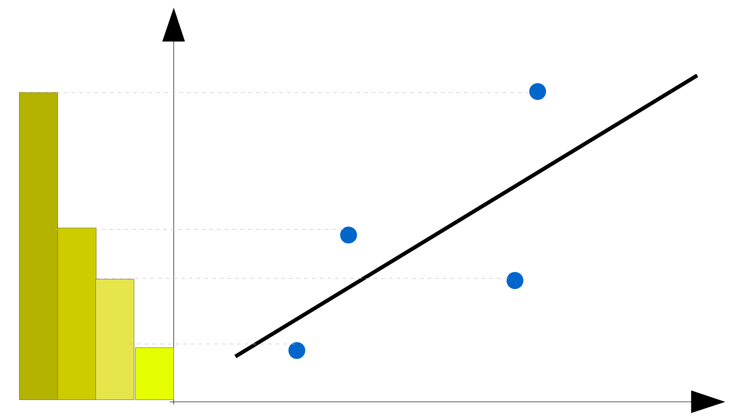
$$\frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2 a_1 + \left(\sum_{i=1}^{n} x_i^2\right) a_1 = \left(\sum_{i=1}^{n} y_i x_i\right)$$

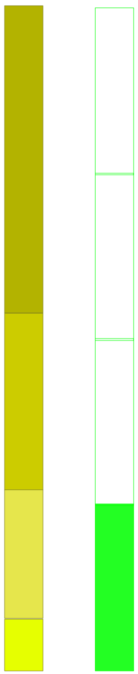
$$n\left(\sum_{i=1}^{n} x_i^2\right) a_1 - \left(\sum_{i=1}^{n} x_i\right)^2 a_1 = n\left(\sum_{i=1}^{n} y_i x_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$$

$$a_1 = \frac{n\left(\sum_{i=1}^{n} y_i x_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2}$$

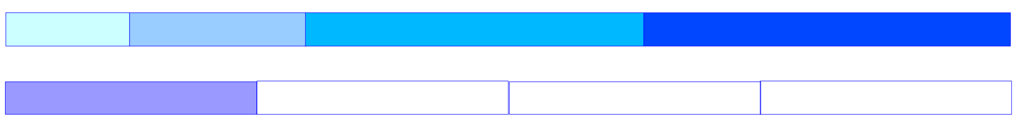
Mean Values of x_i, y_i



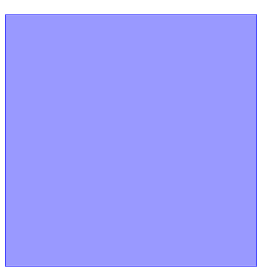




$$\frac{1}{n}\sum_{i=1}^{n} x_i$$

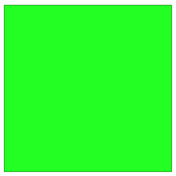


$$\frac{1}{n}\sum_{i=1}^{n} y_i$$



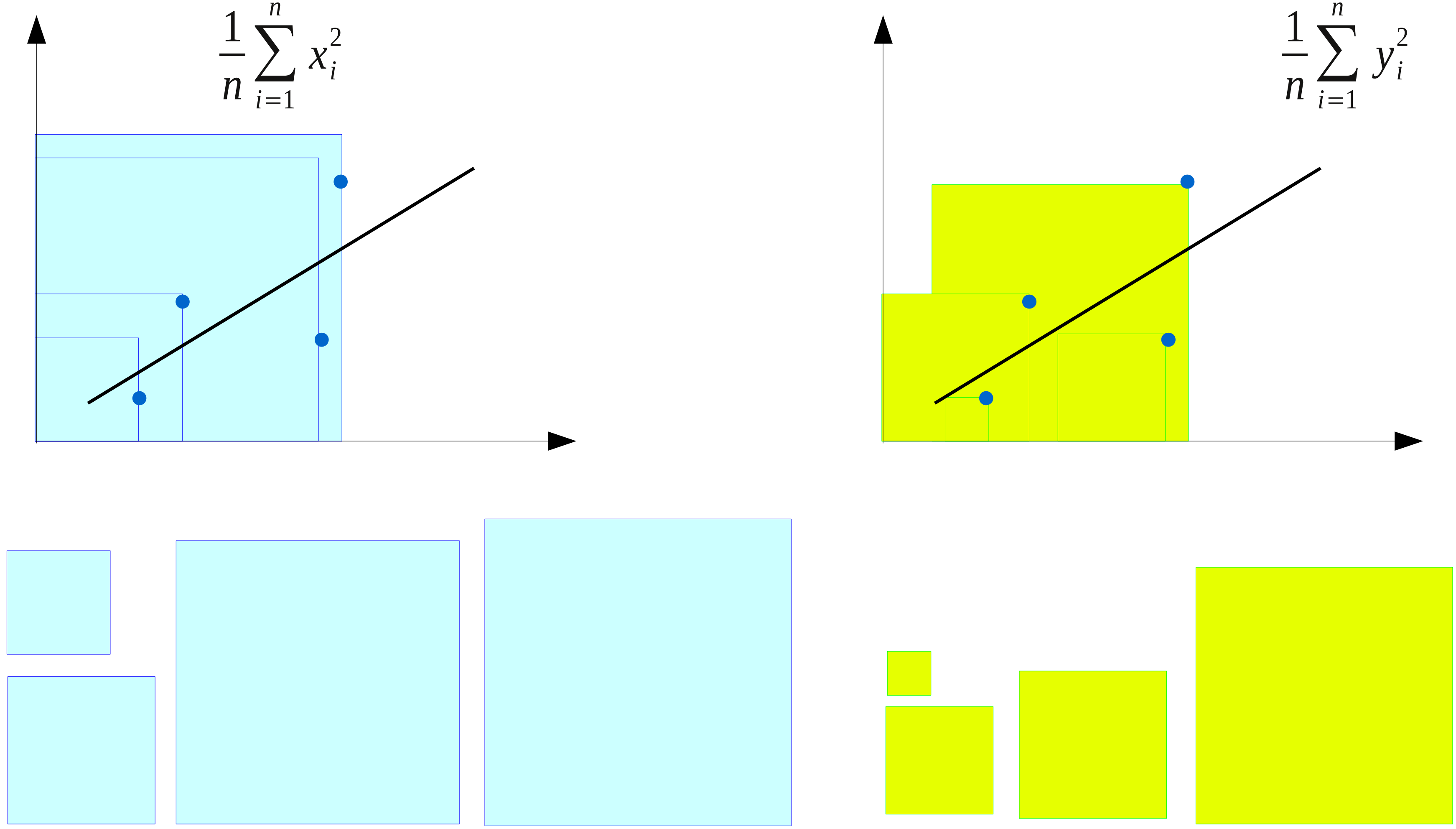
$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$

$$\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)^2$$

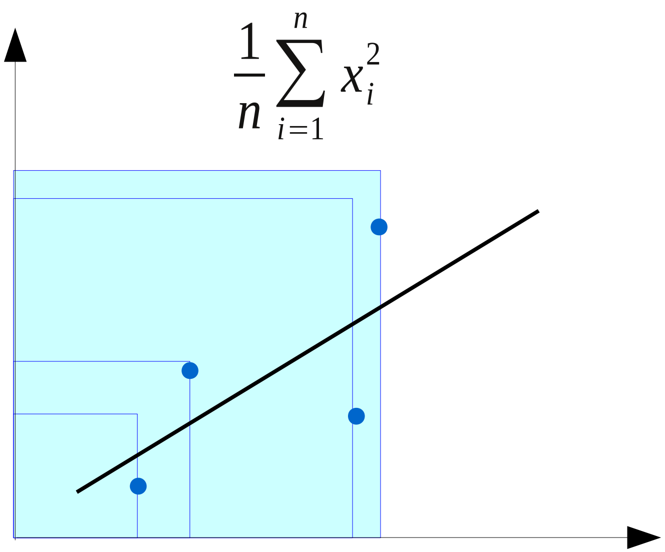


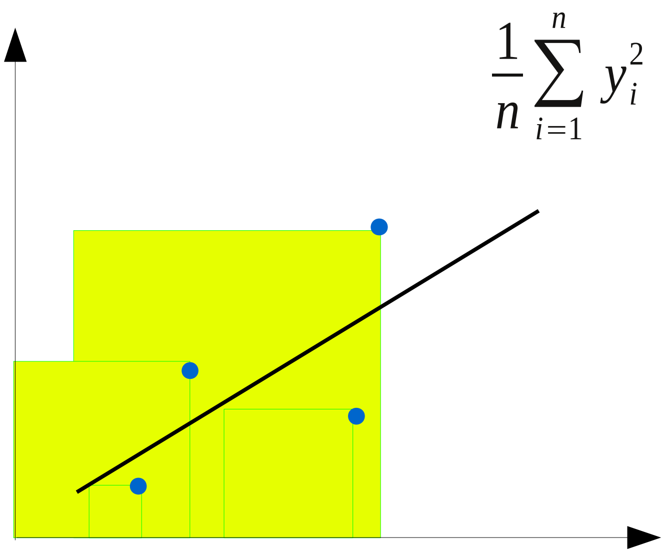
Mean Values of x_i^2, y_i^2, $x_i y_i$ (1)

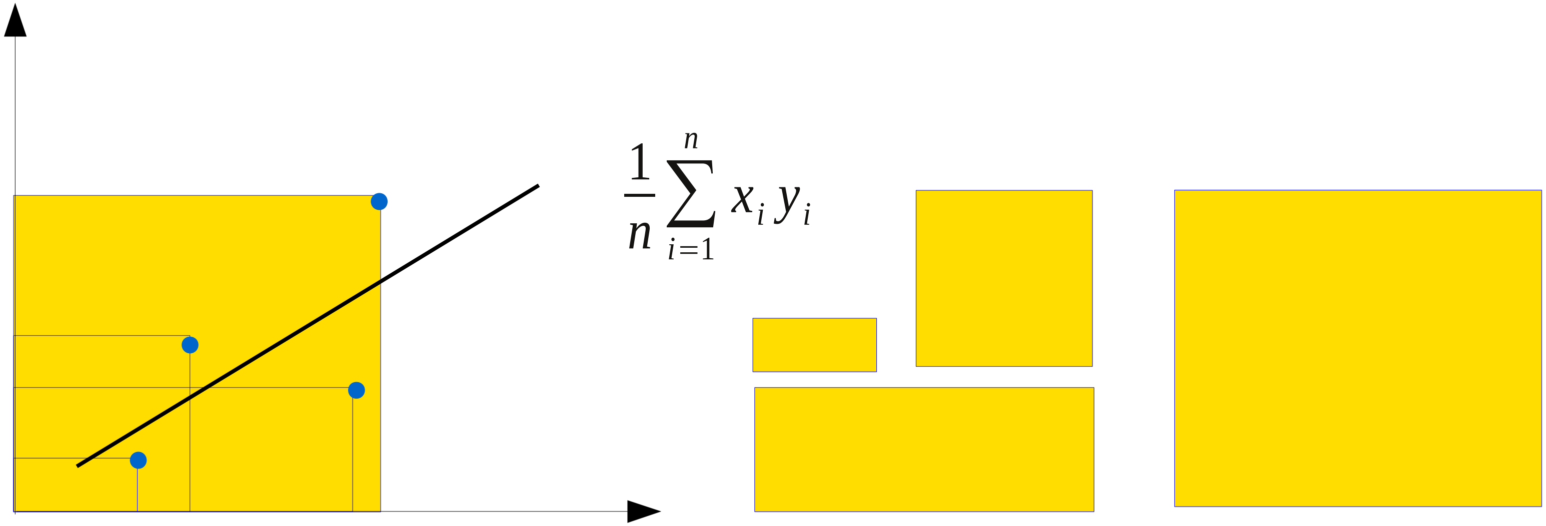
$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \qquad\qquad \frac{1}{n}\sum_{i=1}^{n} y_i^2$$



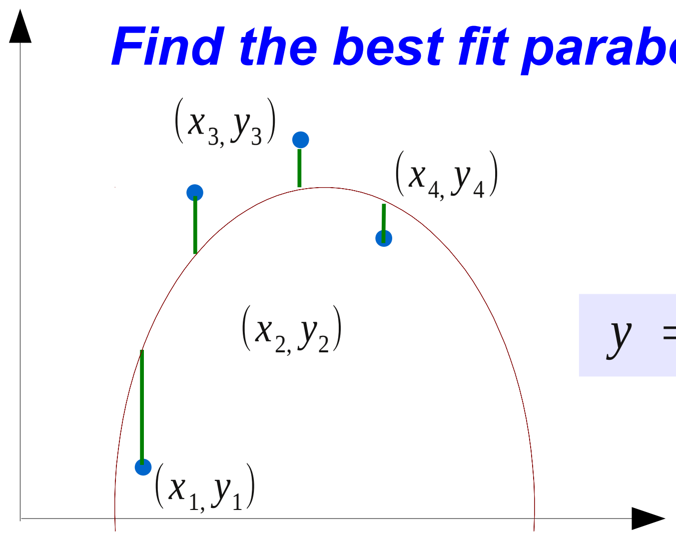
Mean Values of x_i^2, y_i^2, $x_i y_i$ (2)







Polynomial Regression (1)



$$y = a_0 + a_1 x + a_2 x$$

$a_{0,}$ $a_{1,}$ a_2 unknowns

(x_i, y_i) measured data

random

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(a_0 + a_1 x_i + a_2 x_i^2\right)\right)^2$$

Polynomial Regression (2)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (y_i - (a_0 + a_1 x_i + a_2 x_i^2))^2$$

$a_{0,}\ \ a_{1,}\ \ a_2$ *unknowns*

$(x_i,\ y_i)$ *measured data*

random

Minimum Condition

Find the best fit parabola

$$\frac{\partial S_r}{\partial a_0} = 2 \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)(-1) = 0$$

$$\frac{\partial S_r}{\partial a_1} = 2 \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)(-x_i) = 0$$

$$\frac{\partial S_r}{\partial a_2} = 2 \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)(-x_i^2) = 0$$

Polynomial Regression (3)

$$\left(\sum_{i=1}^{n} 1\right) \cdot a_0 + \left(\sum_{i=1}^{n} x_i\right) \cdot a_1 + \left(\sum_{i=1}^{n} x_i^2\right) \cdot a_2 = \left(\sum_{i=1}^{n} y_i\right)$$

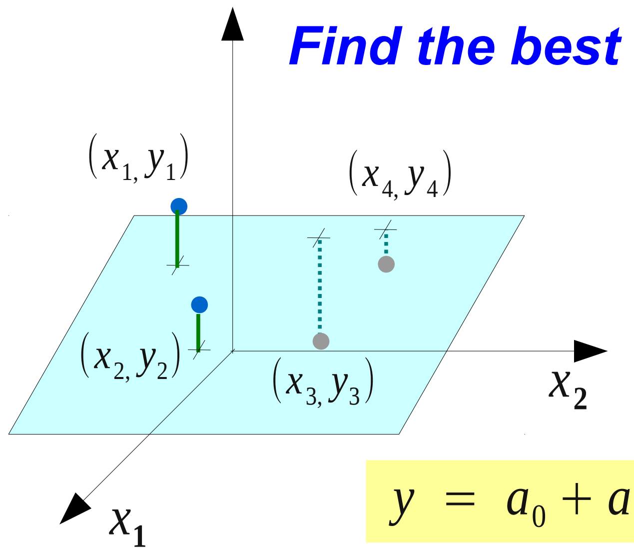
$$\left(\sum_{i=1}^{n} x_i\right) \cdot a_0 + \left(\sum_{i=1}^{n} x_i^2\right) \cdot a_1 + \left(\sum_{i=1}^{n} x_i^3\right) \cdot a_2 = \left(\sum_{i=1}^{n} x_i y_i\right)$$

$$\left(\sum_{i=1}^{n} x_i^2\right) \cdot a_0 + \left(\sum_{i=1}^{n} x_i^3\right) \cdot a_1 + \left(\sum_{i=1}^{n} x_i^4\right) \cdot a_2 = \left(\sum_{i=1}^{n} x_i^2 y_i\right)$$

$$\begin{pmatrix} \left(\sum_{i=1}^{n} 1\right) & \left(\sum_{i=1}^{n} x_i\right) & \left(\sum_{i=1}^{n} x_i^2\right) \\ \left(\sum_{i=1}^{n} x_i\right) & \left(\sum_{i=1}^{n} x_i^2\right) & \left(\sum_{i=1}^{n} x_i^3\right) \\ \left(\sum_{i=1}^{n} x_i^2\right) & \left(\sum_{i=1}^{n} x_i^3\right) & \left(\sum_{i=1}^{n} x_i^4\right) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \left(\sum_{i=1}^{n} y_i\right) \\ \left(\sum_{i=1}^{n} x_i y_i\right) \\ \left(\sum_{i=1}^{n} x_i^2 y_i\right) \end{pmatrix}$$

Multiple Linear Regression (1)

Find the best fit plane



$$y = a_0 + a_1 x_1 + a_2 x_2$$

$a_0, \; a_1, \; a_2$	*unknowns*
$(x_{i,1}, \; x_{i,2}, \; y_i)$	*measured data*

random

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(a_0 + a_1 x_{i,1} + a_2 x_{i,2} \right) \right)^2$$

Multiple Linear Regression (2)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(a_0 + a_1 x_{i,\mathbf{1}} + a_2 x_{i,\mathbf{2}}\right)\right)^2$$

$a_{0,}\ \ a_{1,}\ \ a_2$ *unknowns*

$(x_{i,\mathbf{1}},\ \ x_{i,\mathbf{2}},\ \ y_i)$ *measured data*

random

Minimum Condition

$$\frac{\partial S_r}{\partial a_0} = 2\sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_{i,\mathbf{1}} - a_2 x_{i,\mathbf{2}}\right)(-1) = 0$$

$$\frac{\partial S_r}{\partial a_1} = 2\sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_{i,\mathbf{1}} - a_2 x_{i,\mathbf{2}}\right)(-x_{i,\mathbf{1}}) = 0$$

$$\frac{\partial S_r}{\partial a_2} = 2\sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_{i,\mathbf{1}} - a_2 x_{i,\mathbf{2}}\right)(-x_{i,\mathbf{2}}) = 0$$

Multiple Linear Regression (3)

$$\left(\sum_{i=1}^{n} 1\right) \cdot a_0 + \left(\sum_{i=1}^{n} x_{i,\mathbf{1}}\right) \cdot a_1 + \left(\sum_{i=1}^{n} x_{i,\mathbf{2}}\right) \cdot a_2 = \left(\sum_{i=1}^{n} y_i\right)$$

$$\left(\sum_{i=1}^{n} x_{i,\mathbf{1}}\right) \cdot a_0 + \left(\sum_{i=1}^{n} x_{i,\mathbf{1}}^2\right) \cdot a_1 + \left(\sum_{i=1}^{n} x_{i,\mathbf{1}} x_{i,\mathbf{2}}\right) \cdot a_2 = \left(\sum_{i=1}^{n} x_{i,\mathbf{1}} y_i\right)$$

$$\left(\sum_{i=1}^{n} x_{i,\mathbf{2}}\right) \cdot a_0 + \left(\sum_{i=1}^{n} x_{i,\mathbf{1}} x_{i,\mathbf{2}}\right) \cdot a_1 + \left(\sum_{i=1}^{n} x_{i,\mathbf{2}}^2\right) \cdot a_2 = \left(\sum_{i=1}^{n} x_{i,\mathbf{2}} y_i\right)$$

$$\begin{bmatrix} \left(\sum_{i=1}^{n} 1\right) & \left(\sum_{i=1}^{n} x_{i\mathbf{1}}\right) & \left(\sum_{i=1}^{n} x_{i\mathbf{2}}\right) \\ \left(\sum_{i=1}^{n} x_{i\mathbf{1}}\right) & \left(\sum_{i=1}^{n} x_{i\mathbf{1}}^2\right) & \left(\sum_{i=1}^{n} x_{i\mathbf{1}} x_{i\mathbf{2}}\right) \\ \left(\sum_{i=1}^{n} x_{i\mathbf{2}}\right) & \left(\sum_{i=1}^{n} x_{i\mathbf{1}} x_{i\mathbf{2}}\right) & \left(\sum_{i=1}^{n} x_{i\mathbf{2}}^2\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \left(\sum_{i=1}^{n} y_i\right) \\ \left(\sum_{i=1}^{n} x_{i\mathbf{1}} y_i\right) \\ \left(\sum_{i=1}^{n} x_{i\mathbf{2}} y_i\right) \end{bmatrix}$$

Multiple Linear Regression – General (1)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \left(\beta_0 + \sum_{j=1}^{m} x_{ij} \beta_j \right) \right)^2$$

$\beta_0,\ \beta_1, \cdots ,\ \beta_m$ *unknowns*

$(x_{i1},\ x_{i2}, \cdots , x_{im},\ \ y_i)$ *measured data*

random

$$y = \beta_0 + \sum_{j=1}^{m} x_j \beta_j = \beta_0 + x_1 \beta_1 + x_2 \beta_2 + \cdots + x_m \beta_m$$

Minimum Condition

$$\frac{\partial S_r}{\partial \beta_0} = 2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_m x_{im})(-1) = 0$$

$$\frac{\partial S_r}{\partial \beta_1} = 2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_m x_{im})(-x_{i1}) = 0$$

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$$\frac{\partial S_r}{\partial \beta_m} = 2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_m x_{im})(-x_{im}) = 0$$

Multiple Linear Regression – General (2)

$$\begin{pmatrix} \left(\sum_{i=1}^{n} 1\right) & \left(\sum_{i=1}^{n} x_{i1}\right) & \left(\sum_{i=1}^{n} x_{i2}\right) & \cdots & \left(\sum_{i=1}^{n} x_{im}\right) \\ \left(\sum_{i=1}^{n} x_{i1}\right) & \left(\sum_{i=1}^{n} x_{i1}^2\right) & \left(\sum_{i=1}^{n} x_{i1}x_{i2}\right) & \cdots & \left(\sum_{i=1}^{n} x_{i1}x_{im}\right) \\ \left(\sum_{i=1}^{n} x_{i2}\right) & \left(\sum_{i=1}^{n} x_{i2}x_{i1}\right) & \left(\sum_{i=1}^{n} x_{i2}^2\right) & \cdots & \left(\sum_{i=1}^{n} x_{i1}x_{im}\right) \\ \vdots & \vdots & \vdots & & \vdots \\ \left(\sum_{i=1}^{n} x_{im}\right) & \left(\sum_{i=1}^{n} x_{im}x_{i1}\right) & \left(\sum_{i=1}^{n} x_{im}x_{i2}\right) & \cdots & \left(\sum_{i=1}^{n} x_{im}^2\right) \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} \left(\sum_{i=1}^{n} y_i\right) \\ \left(\sum_{i=1}^{n} x_{i1}y_i\right) \\ \left(\sum_{i=1}^{n} x_{i2}y_i\right) \\ \vdots \\ \left(\sum_{i=1}^{n} x_{im}y_i\right) \end{pmatrix}$$

Multiple Linear Regression – General (3)

$i = 1$ *measured data*

$$\begin{matrix} 1 & x_{11} & x_{12} & \cdots & x_{1m} \\ x_{11} & x_{11}^2 & x_{11}x_{12} & \cdots & x_{11}x_{1m} \\ x_{12} & x_{12}x_{11} & x_{12}^2 & \cdots & x_{11}x_{1m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{1m} & x_{1m}x_{11} & x_{1m}x_{12} & \cdots & x_{1m}^2 \end{matrix}$$

$i = 2$ *measured data*

$$\begin{matrix} 1 & x_{21} & x_{22} & \cdots & x_{2m} \\ x_{21} & x_{21}^2 & x_{21}x_{22} & \cdots & x_{21}x_{2m} \\ x_{22} & x_{22}x_{21} & x_{22}^2 & \cdots & x_{21}x_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{2m} & x_{2m}x_{21} & x_{2m}x_{22} & \cdots & x_{2m}^2 \end{matrix}$$

$i = 3$ *measured data*

$$\begin{matrix} 1 & x_{31} & x_{32} & \cdots & x_{3m} \\ x_{31} & x_{31}^2 & x_{31}x_{32} & \cdots & x_{31}x_{3m} \\ x_{32} & x_{32}x_{31} & x_{32}^2 & \cdots & x_{31}x_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{3m} & x_{3m}x_{31} & x_{3m}x_{32} & \cdots & x_{3m}^2 \end{matrix}$$

$i = 4$ *measured data*

$$\begin{matrix} 1 & x_{41} & x_{42} & \cdots & x_{4m} \\ x_{41} & x_{41}^2 & x_{41}x_{42} & \cdots & x_{41}x_{4m} \\ x_{42} & x_{42}x_{41} & x_{42}^2 & \cdots & x_{41}x_{4m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{4m} & x_{4m}x_{41} & x_{4m}x_{42} & \cdots & x_{4m}^2 \end{matrix}$$

Multiple Linear Regression – General (4)

$$E\{1\} \quad E\{x_1\} \quad E\{x_2\} \quad E\{x_m\}$$

$$1 \quad \overline{x_1} \quad \overline{x_2} \quad \overline{x_m}$$

$$\frac{1}{n}\sum_{i=1}^{n} 1 \quad \frac{1}{n}\sum_{i=1}^{n} x_{i1} \quad \frac{1}{n}\sum_{i=1}^{n} x_{i2} \quad \frac{1}{n}\sum_{i=1}^{n} x_{im}$$

$$\begin{matrix} 1 & \overline{x_1} & \overline{x_2} & \cdots & \overline{x_m} \\ \overline{x_1} & \overline{x_1^2} & \overline{x_1 x_2} & \cdots & \overline{x_1 x_m} \\ \overline{x_2} & \overline{x_2 x_1} & \overline{x_2^2} & \cdots & \overline{x_1 x_m} \\ \vdots & \vdots & \vdots & & \vdots \\ \overline{x_m} & \overline{x_m x_1} & \overline{x_m x_2} & \cdots & \overline{x_m^2} \end{matrix}$$

$$\begin{matrix} 1 & x_{11} & x_{12} & \cdots & x_{1m} \\ x_{11} & x_{11}^2 & x_{11} x_{12} & \cdots & x_{11} x_{1m} \\ x_{12} & x_{12} x_{11} & x_{12}^2 & \cdots & x_{11} x_{1m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{1m} & x_{1m} x_{11} & x_{1m} x_{12} & \cdots & x_{1m}^2 \end{matrix}$$

$i = 4$ *measured data*

$i = 3$ *measured data*

$i = 2$ *measured data*

$i = 1$ *measured data*

Least Square (1)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - f(x_i, \boldsymbol{\beta}) \right)^2$$

$$\epsilon_i = \left(y_i - f(x_i, \boldsymbol{\beta}) \right)$$

$\beta_1, \beta_2, \cdots, \beta_m$ *unknowns*

$(x_{i1}, x_{i2}, \cdots, x_{im}, y_i)$ *measured data*

random

Minimum Condition

$$\frac{\partial S_r}{\partial \beta_j} = 2 \sum_{i=1}^{n} \epsilon_i \frac{\partial \epsilon_i}{\partial \beta_j} = 0 \qquad j = 1, \ldots, m$$

$$\frac{\partial S_r}{\partial \beta_j} = -2 \sum_{i=1}^{n} \epsilon_i \frac{\partial f(x_i, \boldsymbol{\beta})}{\partial \beta_j} = 0 \qquad j = 1, \ldots, m$$

$$\frac{\partial \epsilon_i}{\partial \beta_j} = -\frac{\partial f(x_i, \boldsymbol{\beta})}{\partial \beta_j}$$

$$\epsilon_i = \left(y_i - f(x_i, \boldsymbol{\beta}) \right)$$

Least Square (2)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - f(x_i, \boldsymbol{\beta}) \right)^2$$

$\beta_1, \beta_2, \cdots, \beta_m$ *unknowns*

$(x_{i1}, x_{i2}, \cdots, x_{im}, y_i)$ *measured data*

random

Minimum Condition

$$\frac{\partial S_r}{\partial \beta_j} = -2 \sum_{i=1}^{n} \epsilon_i \frac{\partial f(x_i, \boldsymbol{\beta})}{\partial \beta_j} = 0 \qquad j = 1, \ldots, m$$

Linear Least Square

$$y = \sum_{j=1}^{m} x_j \beta_j = x_1\beta_1 + x_2\beta_2 + \cdots + x_m\beta_m$$

$$f(x_i, \boldsymbol{\beta}) = \sum_{j=1}^{m} x_{ij}\beta_j = x_{i1}\beta_{i1} + x_{i2}\beta_{i2} + \cdots + x_{im}\beta_{im}$$

i: *measuring index*

Linear Least Square (1)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \sum_{j=1}^{m} x_{ij} \beta_j \right)^2$$

$$y = \sum_{j=1}^{m} x_j \beta_j = x_1 \beta_1 + x_2 \beta_2 + \cdots + x_m \beta_m$$

$\beta_1, \ \beta_2, \cdots, \ \beta_m$ *unknowns*

$(x_{i1}, \ x_{i2}, \cdots, x_{im}, \ y_i)$ *measured data*

random

i: *measuring index*

Minimum Condition

$$\frac{\partial S_r}{\partial \beta_1} = 2 \sum_{i=1}^{n} (y_i - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_m x_{im})(-x_{i1}) = 0$$

$$\frac{\partial S_r}{\partial \beta_2} = 2 \sum_{i=1}^{n} (y_i - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_m x_{im})(-x_{i2}) = 0$$

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$$\frac{\partial S_r}{\partial \beta_m} = 2 \sum_{i=1}^{n} (y_i - \beta_1 x_{i1} - \beta_2 x_{i2} - \cdots - \beta_m x_{im})(-x_{im}) = 0$$

Linear Least Square (2)

$$
\begin{pmatrix}
\left(\sum_{i=1}^{n} x_{i1}^2\right) & \left(\sum_{i=1}^{n} x_{i1} x_{i2}\right) & \left(\sum_{i=1}^{n} x_{i1} x_{i3}\right) & \cdots & \left(\sum_{i=1}^{n} x_{i1} x_{im}\right) \\
\left(\sum_{i=1}^{n} x_{i2} x_{i1}\right) & \left(\sum_{i=1}^{n} x_{i2}^2\right) & \left(\sum_{i=1}^{n} x_{i2} x_{i3}\right) & \cdots & \left(\sum_{i=1}^{n} x_{i2} x_{im}\right) \\
\left(\sum_{i=1}^{n} x_{i3} x_{i1}\right) & \left(\sum_{i=1}^{n} x_{i3} x_{i2}\right) & \left(\sum_{i=1}^{n} x_{i3}^2\right) & \cdots & \left(\sum_{i=1}^{n} x_{i3} x_{im}\right) \\
\vdots & \vdots & \vdots & & \vdots \\
\left(\sum_{i=1}^{n} x_{im} x_{i1}\right) & \left(\sum_{i=1}^{n} x_{im} x_{i2}\right) & \left(\sum_{i=1}^{n} x_{im} x_{i3}\right) & \cdots & \left(\sum_{i=1}^{n} x_{im}^2\right)
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m
\end{pmatrix}
=
\begin{pmatrix}
\left(\sum_{i=1}^{n} x_{i1} y_i\right) \\
\left(\sum_{i=1}^{n} x_{i2} y_i\right) \\
\left(\sum_{i=1}^{n} x_{i3} y_i\right) \\
\vdots \\
\left(\sum_{i=1}^{n} x_{im} y_i\right)
\end{pmatrix}
$$

i: *measuring index*

Linear Least Square (3)

$i = 1$ *measured data*

$$\begin{matrix} x_{11}^2 & x_{11}x_{12} & x_{11}x_{13} & \cdots & x_{11}x_{1m} \\ x_{12}x_{11} & x_{12}^2 & x_{12}x_{13} & \cdots & x_{12}x_{1m} \\ x_{13}x_{11} & x_{13}x_{12} & x_{13}^2 & \cdots & x_{13}x_{1m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{1m}x_{11} & x_{1m}x_{12} & x_{1m}x_{13} & \cdots & x_{1m}^2 \end{matrix}$$

$i = 2$ *measured data*

$$\begin{matrix} x_{21}^2 & x_{21}x_{22} & x_{21}x_{23} & \cdots & x_{21}x_{2m} \\ x_{22}x_{21} & x_{22}^2 & x_{22}x_{23} & \cdots & x_{22}x_{2m} \\ x_{23}x_{21} & x_{23}x_{22} & x_{23}^2 & \cdots & x_{23}x_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{2m}x_{21} & x_{2m}x_{22} & x_{2m}x_{23} & \cdots & x_{2m}^2 \end{matrix}$$

$i = 3$ *measured data*

$$\begin{matrix} x_{31}^2 & x_{31}x_{32} & x_{31}x_{33} & \cdots & x_{31}x_{3m} \\ x_{32}x_{31} & x_{32}^2 & x_{32}x_{33} & \cdots & x_{32}x_{3m} \\ x_{33}x_{31} & x_{33}x_{32} & x_{33}^2 & \cdots & x_{33}x_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{3m}x_{31} & x_{3m}x_{32} & x_{3m}x_{33} & \cdots & x_{3m}^2 \end{matrix}$$

$i = 4$ *measured data*

$$\begin{matrix} x_{41}^2 & x_{41}x_{42} & x_{41}x_{43} & \cdots & x_{41}x_{4m} \\ x_{42}x_{41} & x_{42}^2 & x_{42}x_{43} & \cdots & x_{42}x_{4m} \\ x_{43}x_{41} & x_{43}x_{42} & x_{43}^2 & \cdots & x_{43}x_{4m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{4m}x_{41} & x_{4m}x_{42} & x_{4m}x_{43} & \cdots & x_{4m}^2 \end{matrix}$$

Linear Least Square (4)

$$E\{x_1\} = \overline{x_1^2} = \frac{1}{n}\sum_{i=1}^{n} x_{i1}^2 \qquad E\{x_1 x_2\} = \overline{x_1 x_2} = \frac{1}{n}\sum_{i=1}^{n} x_{i1} x_{i2} \qquad E\{x_1 x_3\} = \overline{x_1 x_3} = \frac{1}{n}\sum_{i=1}^{n} x_{i1} x_{i3} \qquad E\{x_1 x_m\} = \overline{x_1 x_m} = \frac{1}{n}\sum_{i=1}^{n} x_{i1} x_{im}$$

$$\begin{matrix} \overline{x_1^2} & \overline{x_1 x_2} & \overline{x_1 x_3} & \cdots & \overline{x_1 x_m} \\ \overline{x_2 x_1} & \overline{x_2^2} & \overline{x_2 x_3} & \cdots & \overline{x_2 x_m} \\ \overline{x_3 x_1} & \overline{x_3 x_2} & \overline{x_3^2} & \cdots & \overline{x_3 x_m} \\ \vdots & \vdots & \vdots & & \vdots \\ \overline{x_m x_1} & \overline{x_m x_2} & \overline{x_m x_3} & \cdots & \overline{x_m^2} \end{matrix}$$

$$\begin{matrix} x_{11}^2 & x_{11} x_{12} & x_{11} x_{13} & \cdots & x_{11} x_{1m} \\ x_{12} x_{11} & x_{12}^2 & x_{12} x_{13} & \cdots & x_{12} x_{1m} \\ x_{13} x_{11} & x_{13} x_{12} & x_{13}^2 & \cdots & x_{13} x_{1m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{1m} x_{11} & x_{1m} x_{12} & x_{1m} x_{13} & \cdots & x_{1m}^2 \end{matrix}$$

$i = 4$ *measured data*

$i = 3$ *measured data*

$i = 2$ *measured data*

$i = 1$ *measured data*

Linear Least Square (5)

Sum of the square of the residuals

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \sum_{j=1}^{m} x_{ij}\beta_j \right)^2$$

$\beta_1, \cdots, \beta_m$ *unknowns*

$(x_{i1}, x_{i2}, \cdots, x_{im}, y_i)$ *measured data*

random

$$y = \sum_{j=1}^{m} x_j \beta_j \quad \Rightarrow \quad y_i = \sum_{j=1}^{m} x_{ij}\beta_j$$

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta}$$

i: *measuring index* $\qquad$ i: *measuring index*

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} \qquad \boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1m} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2m} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nm} \end{pmatrix} \qquad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_m \end{pmatrix}$$

Linear Least Square (6)

Normal Equations

$$y_i = \sum_{j=1}^{m} x_{ij}\beta_j$$

$$\boldsymbol{X\beta} = \boldsymbol{y}$$

$\beta_1, \cdots, \beta_m$ unknowns

$(x_{i1}, x_{i2}, \cdots, x_{im}, y_i)$ measured data

random

$$\frac{\partial S_r}{\partial \beta_j} = 2\sum_{i=1}^{n} \epsilon_i \frac{\partial \epsilon_i}{\partial \beta_j} \quad (j = 1,2,\dots,m) \qquad \frac{\partial \epsilon_i}{\partial \beta_j} = -x_{ij}$$

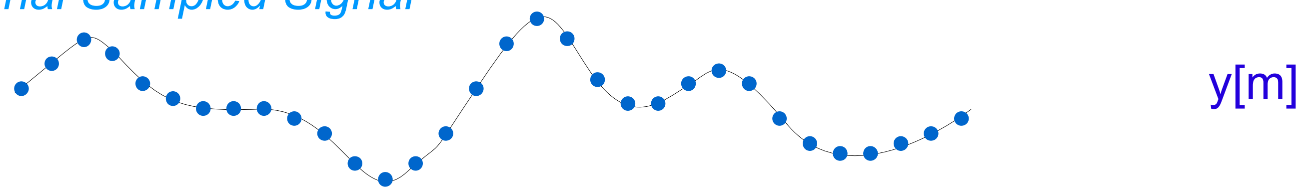
$$\frac{\partial S_r}{\partial \beta_j} = 2\sum_{i=1}^{n} \left(y_i - \sum_{j=1}^{m} x_{ij}\beta_j \right)(-x_{ij}) = 0 \quad (j = 1,2,\dots,m)$$

$$\sum_{i=1}^{n} \left(x_{ij} y_i - \sum_{k=1}^{m} x_{ij} x_{ik} \hat{\beta}_k \right) = 0 \quad (j = 1,2,\dots,m)$$

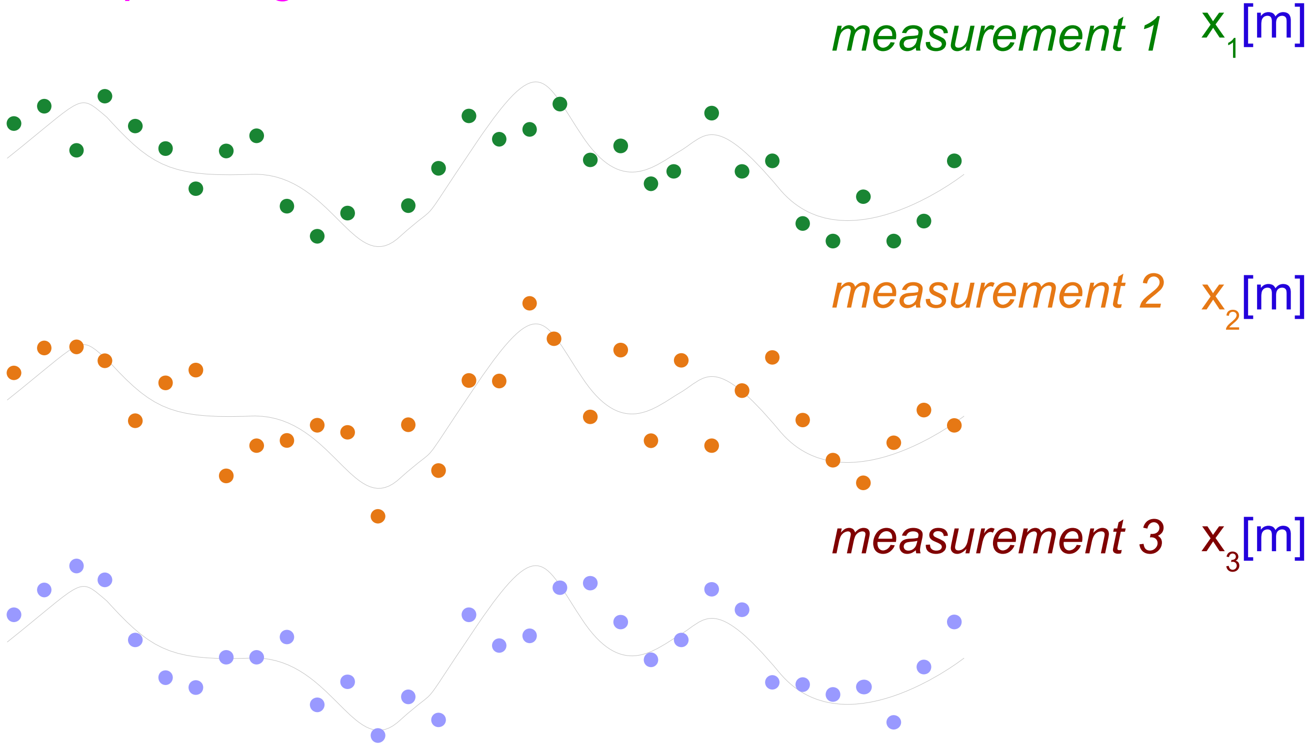
$$\sum_{i=1}^{n} \sum_{k=1}^{n} x_{ij} x_{ik} \hat{\beta}_k = \sum_{i=1}^{n} x_{ij} y_i$$

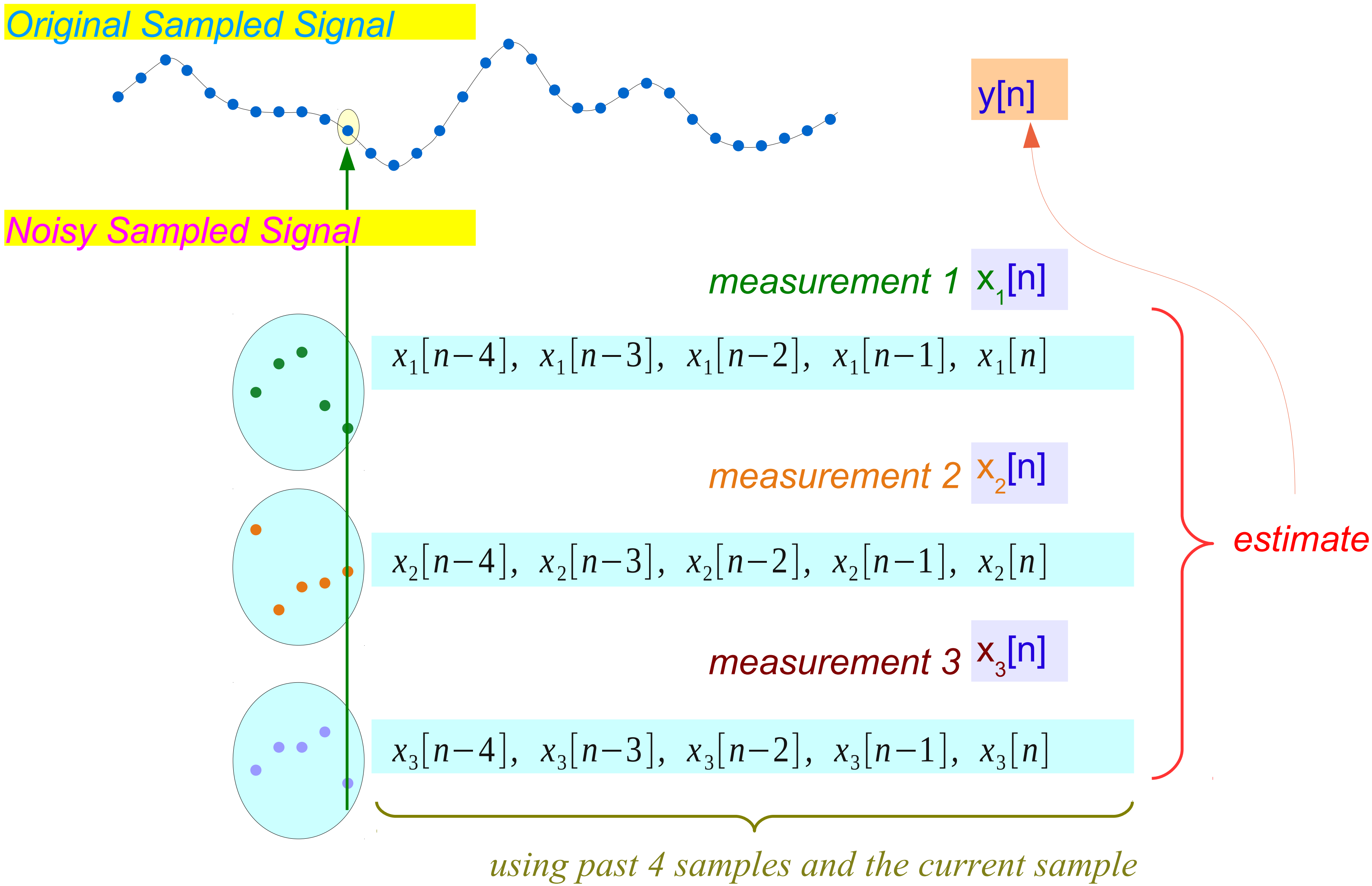
$$\boldsymbol{X}^t \boldsymbol{X} \hat{\boldsymbol{\beta}} = \boldsymbol{X}^t \boldsymbol{y}$$

Original Sampled Signal



Noisy Sampled Signal





Original Sampled Signal
y[n]
Noisy Sampled Signal
measurement 1 $x_1[n]$
$x_1[n-4],\ x_1[n-3],\ x_1[n-2],\ x_1[n-1],\ x_1[n]$
measurement 2 $x_2[n]$
$x_2[n-4],\ x_2[n-3],\ x_2[n-2],\ x_2[n-1],\ x_2[n]$
measurement 3 $x_3[n]$
$x_3[n-4],\ x_3[n-3],\ x_3[n-2],\ x_3[n-1],\ x_3[n]$
estimate
using past 4 samples and the current sample

measurement 1

$x_1[m-4],\ x_1[m-3],\ x_1[m-2],\ x_1[m-1],\ x_1[m]$ ⇒ $x_{1,m-4},\ x_{1,m-3},\ x_{1,m-2},\ x_{1,m-1},\ x_{1,m}$

measurement 2

$x_2[m-4],\ x_2[m-3],\ x_2[m-2],\ x_2[m-1],\ x_2[m]$ ⇒ $x_{2,m-4},\ x_{2,m-3},\ x_{2,m-2},\ x_{2,m-1},\ x_{2,m}$

measurement 3

$x_3[m-4],\ x_3[m-3],\ x_3[m-2],\ x_3[m-1],\ x_3[m]$ ⇒ $x_{3,m-4},\ x_{3,m-3},\ x_{3,m-2},\ x_{3,m-1},\ x_{3,m}$

$$S_r = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} \left(y_i - \sum_{j=1}^{m} x_{ij} \beta_j \right)^2 \qquad y_i = \sum_{j=1}^{m} x_{ij} \beta_j \qquad \mathbf{y} = \mathbf{X} \boldsymbol{\beta}$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} x_{1m} & x_{1,m-1} & x_{1,m-2} & x_{1,m-3} & x_{1,m-4} \\ x_{2m} & x_{2,m-1} & x_{2,m-2} & x_{2,m-3} & x_{2,m-4} \\ x_{3m} & x_{3,m-1} & x_{3,m-2} & x_{3,m-3} & x_{3,m-4} \\ x_{4m} & x_{4,m-1} & x_{4,m-2} & x_{4,m-3} & x_{4,m-4} \\ x_{5m} & x_{5,m-1} & x_{5,m-2} & x_{5,m-3} & x_{5,m-4} \end{pmatrix} \qquad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix}$$

Linear Least Square (4)

$E\{x_m x_m\}$	$E\{x_m x_{m-1}\}$	$E\{x_m x_{m-2}\}$	$E\{x_m x_{m-3}\}$	$E\{x_m x_{m-4}\}$
$\overline{x_m x_m}$	$\overline{x_m x_{m-1}}$	$\overline{x_m x_{m-2}}$	$\overline{x_m x_{m-3}}$	$\overline{x_m x_{m-4}}$
$\frac{1}{n}\sum_{i=1}^{n} x_{im} x_{im}$	$\frac{1}{n}\sum_{i=1}^{n} x_{im} x_{im-1}$	$\frac{1}{n}\sum_{i=1}^{n} x_{im} x_{im-2}$	$\frac{1}{n}\sum_{i=1}^{n} x_{im} x_{im-3}$	$\frac{1}{n}\sum_{i=1}^{n} x_{im} x_{im-4}$

Ensemble Average

x_{1m}^2	$x_{1m} x_{1m-1}$	$x_{1m} x_{1m-2}$	$x_{1m} x_{1m-3}$	$x_{1m} x_{1m-4}$
$x_{1m-1} x_{1m}$	x_{1m-1}^2	$x_{1m-1} x_{1m-2}$	$x_{1m-1} x_{1m-3}$	$x_{1m-1} x_{1m-4}$
$x_{1m-2} x_{1m}$	$x_{1m-2} x_{1m-1}$	x_{1m-2}^2	$x_{1m-2} x_{1m-3}$	$x_{1m-2} x_{1m-4}$
$x_{1m-3} x_{1m}$	$x_{1m-3} x_{1m-1}$	$x_{1m-3} x_{1m-2}$	x_{1m-3}^2	$x_{1m-3} x_{1m-4}$
$x_{1m-4} x_{1m}$	$x_{1m-4} x_{1m-1}$	$x_{1m-4} x_{1m-2}$	$x_{1m-4} x_{1m-3}$	x_{1m-4}^2

$i = 4$ *measured data*

$i = 3$ *measured data*

$i = 2$ *measured data*

$i = 1$ *measured data*

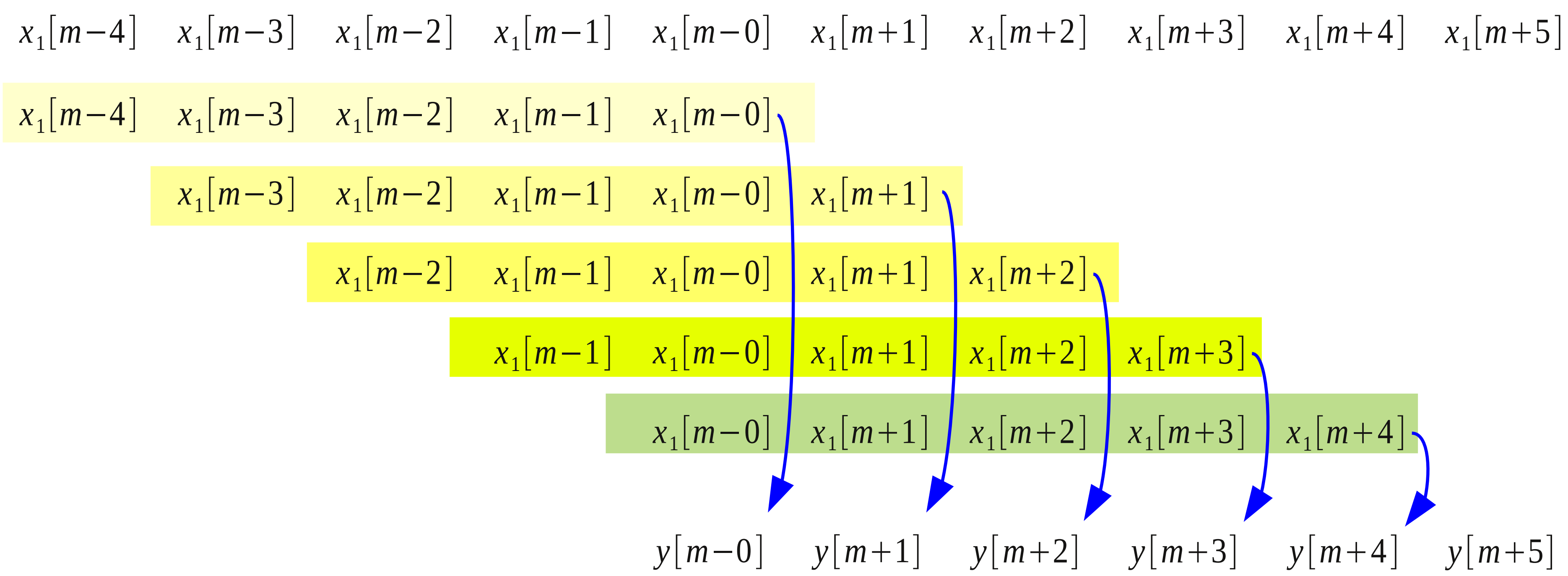
$\overline{x_m^2}$	$\overline{x_m x_{m-1}}$	$\overline{x_m x_{m-2}}$	$\overline{x_m x_{m-3}}$	$\overline{x_m x_{m-4}}$
$\overline{x_{m-1} x_m}$	$\overline{x_{m-1}^2}$	$\overline{x_{m-1} x_{m-2}}$	$\overline{x_{m-1} x_{m-3}}$	$\overline{x_{m-1} x_{1m-4}}$
$\overline{x_{m-2} x_m}$	$\overline{x_{m-2} x_{m-1}}$	$\overline{x_{m-2}^2}$	$\overline{x_{m-2} x_{m-3}}$	$\overline{x_{m-2} x_{m-4}}$
$\overline{x_{m-3} x_m}$	$\overline{x_{m-3} x_{m-1}}$	$\overline{x_{m-3} x_{m-2}}$	$\overline{x_{m-3}^2}$	$\overline{x_{m-3} x_{m-4}}$
$\overline{x_{m-4} x_m}$	$\overline{x_{m-4} x_{m-1}}$	$\overline{x_{m-4} x_{m-2}}$	$\overline{x_{m-4} x_{m-3}}$	$\overline{x_{m-4}^2}$

Auto-correlation Matrix

$r_{xx}[0]$	$r_{xx}[1]$	$r_{xx}[2]$	$r_{xx}[3]$	$r_{xx}[4]$
$r_{xx}[1]$	$r_{xx}[0]$	$r_{xx}[1]$	$r_{xx}[2]$	$r_{xx}[3]$
$r_{xx}[2]$	$r_{xx}[1]$	$r_{xx}[0]$	$r_{xx}[1]$	$r_{xx}[2]$
$r_{xx}[3]$	$r_{xx}[2]$	$r_{xx}[1]$	$r_{xx}[0]$	$r_{xx}[1]$
$r_{xx}[4]$	$r_{xx}[3]$	$r_{xx}[2]$	$r_{xx}[1]$	$r_{xx}[0]$

Auto-correlation Matrix *Cross-correlation*

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} = \begin{pmatrix} r_{xx}[0] & r_{xx}[1] & r_{xx}[2] & r_{xx}[3] & r_{xx}[4] \\ r_{xx}[1] & r_{xx}[0] & r_{xx}[1] & r_{xx}[2] & r_{xx}[3] \\ r_{xx}[2] & r_{xx}[1] & r_{xx}[0] & r_{xx}[1] & r_{xx}[2] \\ r_{xx}[3] & r_{xx}[2] & r_{xx}[1] & r_{xx}[0] & r_{xx}[1] \\ r_{xx}[4] & r_{xx}[3] & r_{xx}[2] & r_{xx}[1] & r_{xx}[0] \end{pmatrix}^{-1} \begin{pmatrix} r_{xy}[0] \\ r_{xy}[1] \\ r_{xy}[2] \\ r_{xy}[3] \\ r_{xy}[4] \end{pmatrix}$$


$x_1[m-4]$ $x_1[m-3]$ $x_1[m-2]$ $x_1[m-1]$ $x_1[m-0]$ $x_1[m+1]$ $x_1[m+2]$ $x_1[m+3]$ $x_1[m+4]$ $x_1[m+5]$
$x_1[m-4]$ $x_1[m-3]$ $x_1[m-2]$ $x_1[m-1]$ $x_1[m-0]$
$x_1[m-3]$ $x_1[m-2]$ $x_1[m-1]$ $x_1[m-0]$ $x_1[m+1]$
$x_1[m-2]$ $x_1[m-1]$ $x_1[m-0]$ $x_1[m+1]$ $x_1[m+2]$
$x_1[m-1]$ $x_1[m-0]$ $x_1[m+1]$ $x_1[m+2]$ $x_1[m+3]$
$x_1[m-0]$ $x_1[m+1]$ $x_1[m+2]$ $x_1[m+3]$ $x_1[m+4]$
$y[m-0]$ $y[m+1]$ $y[m+2]$ $y[m+3]$ $y[m+4]$ $y[m+5]$

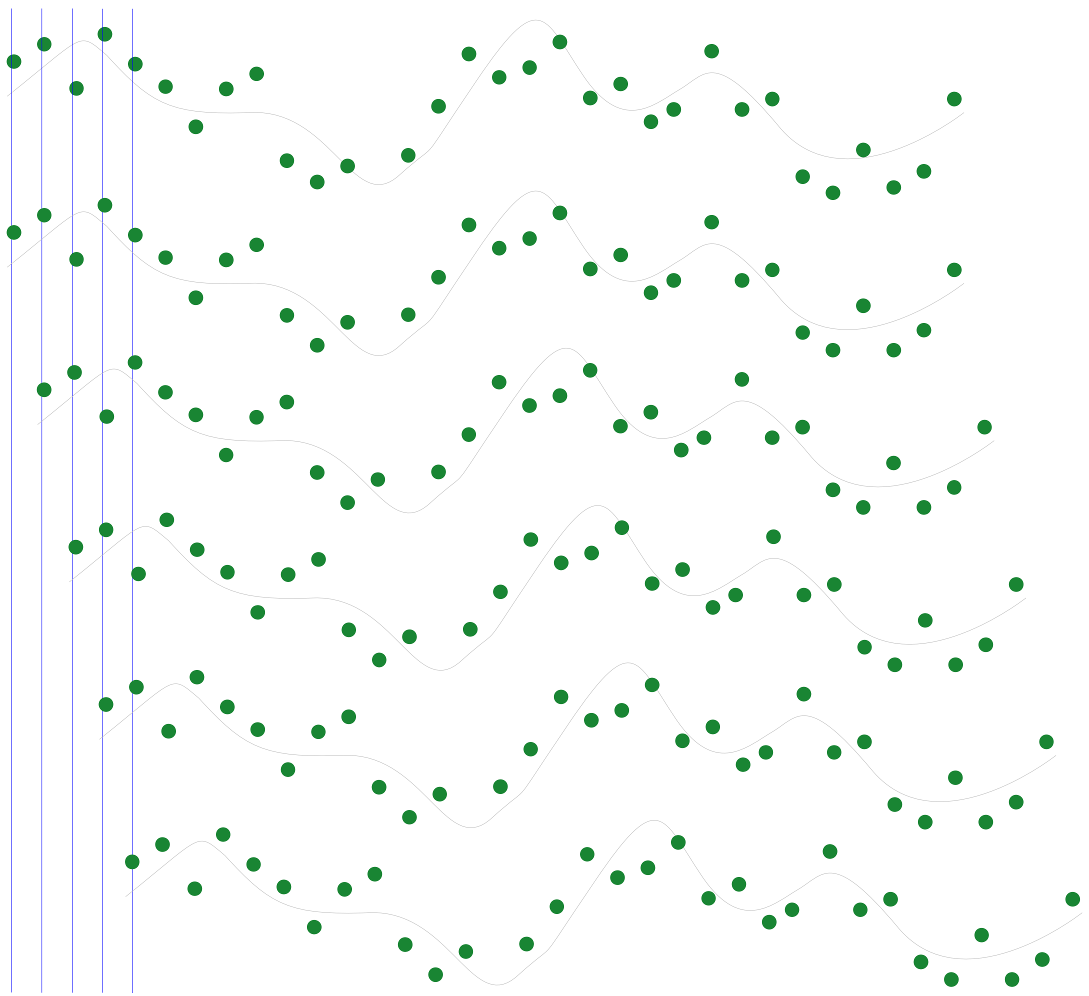
$$\begin{pmatrix} y[m-0] \\ y[m+1] \\ y[m+2] \\ y[m+3] \\ y[m+4] \end{pmatrix} = \begin{pmatrix} x_1[m-0] & x_1[m-1] & x_1[m-2] & x_1[m-3] & x_1[m-4] \\ x_1[m+1] & x_1[m-0] & x_1[m-1] & x_1[m-2] & x_1[m-3] \\ x_1[m+2] & x_1[m+1] & x_1[m-0] & x_1[m-1] & x_1[m-2] \\ x_1[m+3] & x_1[m+2] & x_1[m+1] & x_1[m-0] & x_1[m-1] \\ x_1[m+4] & x_1[m+3] & x_1[m+2] & x_1[m+1] & x_1[m-0] \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

Auto-correlation Matrix *Cross-correlation*

$$\begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} r_{xx}[0] & r_{xx}[1] & r_{xx}[2] & r_{xx}[3] & r_{xx}[4] \\ r_{xx}[1] & r_{xx}[0] & r_{xx}[1] & r_{xx}[2] & r_{xx}[3] \\ r_{xx}[2] & r_{xx}[1] & r_{xx}[0] & r_{xx}[1] & r_{xx}[2] \\ r_{xx}[3] & r_{xx}[2] & r_{xx}[1] & r_{xx}[0] & r_{xx}[1] \\ r_{xx}[4] & r_{xx}[3] & r_{xx}[2] & r_{xx}[1] & r_{xx}[0] \end{pmatrix}^{-1} \begin{pmatrix} r_{xy}[0] \\ r_{xy}[1] \\ r_{xy}[2] \\ r_{xy}[3] \\ r_{xy}[4] \end{pmatrix}$$

Noisy Sampled Signal

measurement 1 $x_1[m]$



Time Average

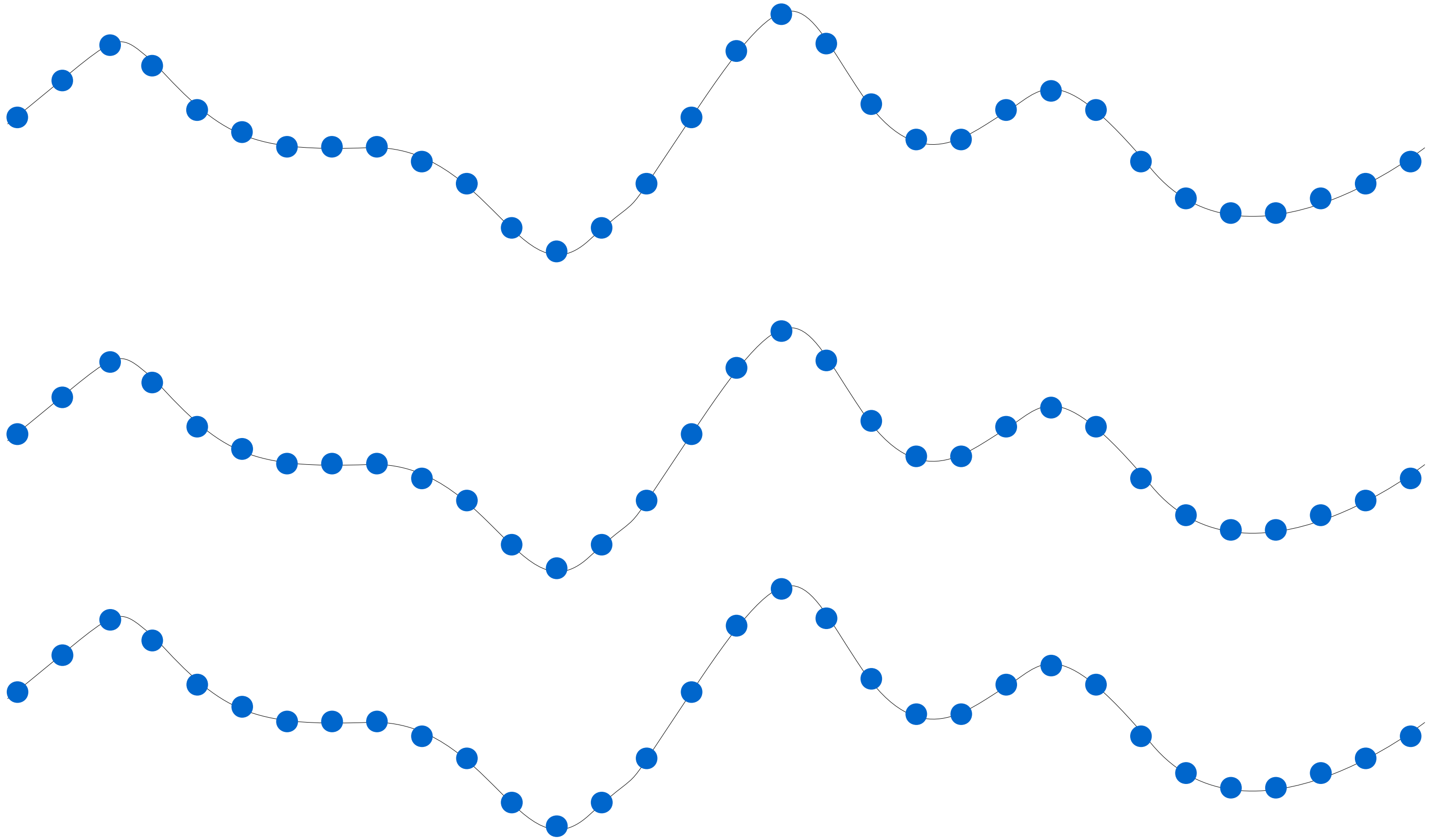
$$r_{xx}[k] = \frac{1}{N}\sum_{m=0}^{N-1} x_1[m]x_1[m+k]$$

$$r_{xy}[k] = \frac{1}{N}\sum_{m=0}^{N-1} x_1[m]y[m+k]$$

Young Won Lim
7/20/11

$\overline{x_m^2}$	$\overline{x_m x_{m-1}}$	$\overline{x_m x_{m-2}}$	$\overline{x_m x_{m-3}}$	$\overline{x_m x_{m-4}}$
$\overline{x_{m-1} x_m}$	$\overline{x_{m-1}^2}$	$\overline{x_{m-1} x_{m-2}}$	$\overline{x_{m-1} x_{m-3}}$	$\overline{x_{m-1} x_{1m-4}}$
$\overline{x_{m-2} x_m}$	$\overline{x_{m-2} x_{m-1}}$	$\overline{x_{m-2}^2}$	$\overline{x_{m-2} x_{m-3}}$	$\overline{x_{m-2} x_{m-4}}$
$\overline{x_{m-3} x_m}$	$\overline{x_{m-3} x_{m-1}}$	$\overline{x_{m-3} x_{m-2}}$	$\overline{x_{m-3}^2}$	$\overline{x_{m-3} x_{m-4}}$
$\overline{x_{m-4} x_m}$	$\overline{x_{m-4} x_{m-1}}$	$\overline{x_{m-4} x_{m-2}}$	$\overline{x_{m-4} x_{m-3}}$	$\overline{x_{m-4}^2}$

				$x_{1m} x_{1m-4}$
$x_{1m-1} x_{1m}$	x_{1m-1}^2	$x_{1m-1} x_{1m-2}$	$x_{1m-1} x_{1m-3}$	$x_{1m-1} x_{1m-4}$
$x_{1m-2} x_{1m}$	$x_{1m-2} x_{1m-1}$	x_{1m-2}^2	$x_{1m-2} x_{1m-3}$	$x_{1m-2} x_{1m-4}$
$x_{1m-3} x_{1m}$	$x_{1m-3} x_{1m-1}$	$x_{1m-3} x_{1m-2}$	x_{1m-3}^2	$x_{1m-3} x_{1m-4}$
$x_{1m-4} x_{1m}$	$x_{1m-4} x_{1m-1}$	$x_{1m-4} x_{1m-2}$	$x_{1m-4} x_{1m-3}$	x_{1m-4}^2



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[3] S.C. Chapra, Applied Numerical Methods W/ml Engineering And Science