

Characteristics of Multiple Random Variables

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

1 Simulation of Multiple Random Variables

Example Problem (1)

N Gaussian random variables

Definition

two statistically independent Gaussian random variables Y_1, Y_2 each with zero mean and unit variance, can be generated by the transformation

$$Y_1 = T_1(X_1, X_2) = \sqrt{-2\ln(X_1)} \cos(2\pi X_2)$$

$$Y_2 = T_2(X_1, X_2) = \sqrt{-2\ln(X_1)} \sin(2\pi X_2)$$

the joint density of Y_1 and Y_2

$$f_{Y_1 Y_2}(y_1, y_2) = \frac{e^{-Y_1^2/2}}{\sqrt{2\pi}} \frac{e^{-Y_2^2/2}}{\sqrt{2\pi}}$$

Example Problem (2)

N Gaussian random variables

Definition

$$[C_W] = \begin{bmatrix} \sigma_{W_1}^2 & \rho_W \sigma_{W_1} \sigma_{W_2} \\ \rho_W \sigma_{W_1} \sigma_{W_2} & \sigma_{W_2}^2 \end{bmatrix} = [T][T]^t$$

$$[T] = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}$$

the joint density of Y_1 and Y_2

$$T_{11} = \sigma_{W_1}, \quad T_{21} = \rho_W \sigma_{W_1} \sigma_{W_2}, \quad T_{22} = \sigma_{W_2} \sqrt{1 - \rho_W^2}$$

Example Problem (3)

 N Gaussian random variables

Definition

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} \cdot \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

the joint density of Y_1 and Y_2

$$W_1 = T_{11} Y_1 = \sigma_{W_1} Y_1$$

$$W_2 = T_{21} Y_1 + T_{22} Y_2 = \rho_W \sigma_{W_1} \sigma_{W_2} Y_1 + \sigma_{W_2} \sqrt{1 - \rho_W^2} Y_2$$

Example Problem (4)

 N Gaussian random variables

Definition

$$W_1 = \sigma_{W_1} Y_1$$

$$W_2 = \rho_W \sigma_{W_1} \sigma_{W_2} Y_1 + \sigma_{W_2} \sqrt{1 - \rho_W^2} Y_2$$

$$W_1 = \bar{W}_1 + \sigma_{W_1} Y_1$$

$$W_2 = \bar{W}_2 + \rho_W \sigma_{W_1} \sigma_{W_2} Y_1 + \sigma_{W_2} \sqrt{1 - \rho_W^2} Y_2$$

Example Problem (5)

 N Gaussian random variables

Definition

$$R_1 = T_1(W_1, W_2) = \sqrt{W_1^2 + W_2^2}$$

$$\Theta = T_2(W_1, W_2) = \tan^{-1}(W_2/W_1)$$

$$W_1 = T_1^{-1}(R, \Theta) = R \cos(\Theta)$$

$$W_2 = T_2^{-1}(R, \Theta) = R \sin(\Theta)$$

Example Problem (6)

N Gaussian random variables

Definition

$$f_{W_1 W_2}(w_1, w_2) \\ = \frac{1}{2\pi\sigma^2} \exp \left\{ - \left[(w_1 - \bar{W}_1)^2 + (w_2 - \bar{W}_2)^2 \right] / (2\sigma^2) \right\}$$

$$f_{R, \Theta}(r, \theta) \\ = \frac{ru(r)}{2\pi\sigma^2} \exp \left\{ - \left[(r \cos \theta - \bar{W}_1)^2 + (r \sin \theta - \bar{W}_2)^2 \right] / (2\sigma^2) \right\} \\ = \frac{ru(r)}{2\pi\sigma^2} \exp \left\{ - \frac{\left[r^2 + (\bar{W}_1^2 + \bar{W}_2^2) - 2r\bar{W}_1 \cos \theta - 2r\bar{W}_2 \sin \theta \right]^2}{2\sigma^2} \right\}$$

Example Problem (67)

 N Gaussian random variables

Definition

$$A_0 = \sqrt{\overline{W_1^2} + \overline{W_2^2}}$$

$$\theta_0 = \tan^{-1}(\overline{W_2}/\overline{W_1})$$

$$f_{R,\Theta}(r, \theta) = \frac{ru(r)}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2} [r^2 + A_0^2 - 2rA_0 \cos(\theta - \theta_0)]\right\}$$

