DLTI Difference Equation

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Causal LTI Systems (1)

$$a_N y[n-N]+\cdots+a_1 y[n-1]+a_0 y[n] = b_M x[n-M]+\cdots+b_1 x[n-1]+b_0 x[n]$$

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

M = N

n ➡ n – N

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

$$y[n]+a_1 y[n-1]+\cdots+a_{N-1} y[n-N+1]+a_N y[n-N]$$
$$= b_0 x[n]+b_1 x[n-1]+\cdots+b_{N-1} x[n-N+1]+b_N x[n-N]$$

$$(E^N+a_1 E^{N-1}+\cdots+a_{N-1} E+a_N) y[n] = (b_0 E^M+b_{N-M+1} E^{M-1}+\cdots+b_{N-1} E+b_N) x[n]$$

$$Q(E) y(t) = P(E) x(t)$$

Causal LTI Systems (2)

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

Causal System: output cannot depend on future input values

Causality ⟺ $M \le N$

If M > N $y[n+N]$ would depend on $x[n+M]$

later instance

If M = N

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

Causal LTI Systems (3)

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

If M = N

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

advance operator from
advance operator from

n ➡ n – N

$$y[n]+a_1 y[n-1]+\cdots+a_{N-1} y[n-N+1]+a_N y[n-N]$$
$$= b_0 x[n]+b_1 x[n-1]+\cdots+b_{N-1} x[n-N+1]+b_N x[n-N]$$

The advance operator $E\, x[n] = x[n+1]$ $E^k\, x[n] = x[n+k]$

$$(E^N+a_1 E^{N-1}+\cdots+a_{N-1} E+a_N) y[n] = (b_0 E^M+b_{N-M+1} E^{M-1}+\cdots+b_{N-1} E+b_N) x[n]$$

$$Q(E) y[n] = P(E) x[n]$$

Zero Input Response $y_0(t)$ (1)

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) y[n] = (b_0 E^M + b_{N-M+1} E^{M-1} + \cdots + b_{N-1} E + b_N) x[n]$$

$$Q(E) y[n] = P(E) x[n]$$

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n] = 0$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) \cdot y_0[n] = 0$$

linear combination of $y_0[n]$ and advanced $y_0[n]$ is zero for all n

iff y0[n] and advanced y0[n] have the same form

only exponential function γ^n $\qquad E^k\{\gamma^n\} = \gamma^{n+k} = \gamma^k \gamma^n$

$$y_0[n] = c\,\gamma^n \qquad E^k\{y_0[n]\} = y_0[n+k] = c\,\gamma^{n+k}$$

$$c(\gamma^N + a_1 \gamma^{N-1} + \cdots + a_{N-1} \gamma + a_N) \cdot y_0[n] = 0$$

$$(\gamma^N + a_1 \gamma^{N-1} + \cdots + a_{N-1} \gamma + a_N) = 0 \iff Q(\gamma) = 0$$

Zero Input Response $y_0(t)$ (2)

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) y[n] = (b_0 E^M + b_{N-M+1} E^{M-1} + \cdots + b_{N-1} E + b_N) x[n]$$

$$Q(E) y[n] = P(E) x[n]$$

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N \; y[n] = 0$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) \cdot y_0[n] = 0$$

$$c(\gamma^N + a_1 \gamma^{N-1} + \cdots + a_{N-1} \gamma + a_N) \cdot y_0[n] = 0$$

$$(\gamma^N + a_1 \gamma^{N-1} + \cdots + a_{N-1} \gamma + a_N) = 0 \iff Q(\gamma) = 0$$

$$(\gamma - \gamma_1)(\gamma - \gamma_2) \; \cdots \; (\gamma - \gamma_N) = 0$$

$$y_0[n] = c_1 \gamma_1^n + c_2 \gamma_2^n + \cdots + c_N \gamma_N^n$$

γ_i characteristic roots

γ_i^n characteristic modes

ZIR: a linear combination of the characteristic modes of the system

Closed Form h[n] (1)

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) y[n] = (b_0 E^M + b_{N-M+1} E^{M-1} + \cdots + b_{N-1} E + b_N) x[n]$$

$$Q(E) y[n] = P(E) x[n]$$

h[n] : system response to input δ[n]

$Q(E) h[n] = P(E) \delta[n]$ *with initial condition*

$h[-1] = h[-2] = \cdots = h[-N] = 0$

When ***n < 0, h[n] = 0***

When ***n > 0****, h[n] must be made up of* ***characteristic modes***

When the input is zero, only the characteristic modes can be sustained

When ***n = 0****, it may have non-zero value* $\boldsymbol{A_0}$

$$h[n] = A_0 \delta[n] + y_c[n] u[n]$$

linear combination of ***the characteristic modes***

Closed Form h[n] (2)

$$y[n+N] + a_1\, y[n+N-1] + \cdots + a_{N-1}\, y[n+1] + a_N\, y[n]$$
$$= b_0\, x[n+M] + b_1\, x[n+M-1] + \cdots + b_{N-1}\, x[n+1] + b_N\, x[n]$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N)\, y[n] = (b_0 E^M + b_{N-M+1} E^{M-1} + \cdots + b_{N-1} E + b_N)\, x[n]$$

$$Q(E)\, y[n] = P(E)\, x[n]$$

$$Q(E)\, y(t) = P(E)\, x(t)$$

$$Q(E)\, h(t) = P(E)\, \delta(t)$$ *causal **h[n]***

$$h[-1] = h[-2] = \cdots = h[-N] = 0$$ *initial condition*

$$h[n] = A_0 \delta[n] + y_c[n]\, u[n]$$

$$Q(E)\, (A_0 \delta[n] + y_c[n]\, u[n]) = P(E)\, \delta(t)$$

$$Q(E)\, (y_c[n]\, u[n]) = 0$$
$$Q(E)\, (A_0 \delta[n]) = P(E)\, \delta(t)$$

y_c is made up of characteristic modes

$$A_0(\delta[n+N] + a_1\, \delta[n+N-1] + \cdots + a_{N-1}\, \delta[n+1] + a_N\, \delta[n])$$
$$= b_0\, \delta[n+M] + b_1\, \delta[n+M-1] + \cdots + b_{N-1}\, \delta[n+1] + b_N\, \delta[n]$$

n=0 $\quad A_0\, a_N = b_N \quad A_0 = \frac{a_N}{b_N}$

$$h[n] = \frac{b_N}{a_N} \delta[n] + y_c[n]\, u[n]$$

Closed Form h[n] (3)

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n]$$
$$= b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n]$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) y[n] = (b_0 E^M + b_{N-M+1} E^{M-1} + \cdots + b_{N-1} E + b_N) x[n]$$

$$Q(E) y[n] = P(E) x[n]$$

$$Q(E) y(t) = P(E) x(t)$$

$$Q(E) h(t) = P(E) \delta(t)$$

*causal **h[n]***

$$h[-1] = h[-2] = \cdots = h[-N] = 0$$

initial condition

$$h[n] = A_0 \delta[n] + y_c[n] u[n]$$

$$h[n] = \frac{b_N}{a_N} \delta[n] + y_c[n] u[n]$$

N unknown coefficients in $y_c[n]$
– determined from N values of h[n]

$$h[0], h[1], \cdots, h[N-1]$$

Example (1) – ZIR

$$y[n+2] - 0.6\,y[n+1] - 0.16\,y[n] = 5\,x[n+2]$$

$$(E^2 - 0.6\,E - 0.16)\;y(t) = 5\,E^2\;x[n]$$

initial condition $\quad y[-1] = 0,\;\; y[-2] = \frac{25}{4}$

input $\quad x(t) = 4^{-n}\,u[n]$

Characteristic polynomial

$$\gamma - 0.6\gamma - 0.16 = (\gamma+0.2)(\gamma-0.8)$$

Characteristic Equation $\quad (\gamma+0.2)(\gamma-0.8) = 0$

Characteristic Roots $\quad \gamma = -0.2, \quad \gamma = 0.8$

Zero Input Response $\quad y_0[n]$

$$y_0[n] = c_1(-0.2)^n + c_2(0.8)^n$$

$$y_0[n] = \frac{1}{5}(-0.2)^n + \frac{4}{5}(0.8)^n$$

$$y_0[-1] = -5c_1 + \frac{5}{4}c_2 = 0 \qquad c_1 = \frac{1}{5}$$

$$y_0[-2] = 25c_1 + \frac{25}{16}c_2 = \frac{25}{4} \qquad c_2 = \frac{4}{5}$$

Example (2) – ZSR

$$y[n]-0.6\,y[n-1]-0.16\,y[n-2] \;=\; 5\,x[n]$$

$$h[n]-0.6\,h[n-1]-0.16\,h[n-2] \;=\; 5\,\delta[n]$$

$$h[-1]=0,\;\; h[-2] \;=\; 0$$

Calculate iteratively $\quad h[0]=5,\;\; h[1] \;=\; 3,\;\cdots$

$$(E^2-0.6\,E-0.16)\;y(t) \;=\; 5\,E^2\;x[n]$$

$$y_c[n] \;=\; c_1(-0.2)^n \;+\; c_2(0.8)^n$$

$$h[n] \;=\; 0\cdot\delta[n] \;+\; [c_1(-0.2)^n+c_2(0.8)^n]\;u[n]$$

$$h[0]=5 \qquad h[0] \;=\; c_1+c_2 \qquad c_1 \;=\; 1$$

$$h[1]=1 \qquad h[1] \;=\; -0.2\,c_1+0.8\,c_2 \qquad c_2 \;=\; 4$$

$$h[n] \;=\; [(-0.2)^n+4\,(0.8)^n]\;u[n]$$

Closed form h[n]

$$y[n] \;=\; \sum_{m=0}^{n} x[m]\;h[n-m]$$

Zero State Response

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] B.P. Lathi, Linear Systems and Signals (2nd Ed)
[4] ocw.usu.edu, Electrical and Computer Engineering, Signals and Systems