Laurent Series and z-Transform - Geometric Series Causality B

20180706

2 formulas of z

① $$\frac{3}{2}\frac{-1}{(z-0.5)(z-2)} = \left(\frac{1}{z-0.5} - \frac{1}{z-2}\right)$$

$\updownarrow z^{-1}$

② $$\frac{3}{2}\frac{-z^2}{(z-2)(z-0.5)} = \left(\frac{0.5z}{(z-0.5)} - \frac{2z}{(z-2)}\right)$$

$$\boxed{\frac{3}{2}\frac{-1}{(z-0.5)(z-2)} \overset{z^{-1}}{\longleftrightarrow} \frac{3}{2}\frac{-z^2}{(z-2)(z-0.5)}}$$

$$\frac{3}{2}\frac{-1}{(z-0.5)(z-2)} = \frac{3}{2}\,\frac{2}{3}\left(\frac{1}{z-0.5}-\frac{1}{z-2}\right)$$

$$= \left(\frac{1}{z-0.5}-\frac{1}{z-2}\right)$$

$$\Big\downarrow z^{-1}$$

$$\frac{3}{2}\frac{-1}{(z^{-1}-0.5)(z^{-1}-2)} = \frac{3}{2}\,\frac{2}{3}\left(\frac{1}{z^{-1}-0.5}-\frac{1}{z^{-1}-2}\right)$$

$$= \left(\frac{2}{2z^{-1}-1}-\frac{0.5}{0.5z^{-1}-1}\right)$$

$$= \left(\frac{2z}{2-z}-\frac{0.5z}{0.5-z}\right)$$

$$= \left(\frac{-2z}{z-2}+\frac{0.5z}{z-0.5}\right)$$

$$= z\left(\frac{-2}{z-2}+\frac{0.5}{z-0.5}\right)$$

$$= z\left(\frac{-\frac{3}{2}z}{(z-2)(z-0.5)}\right)$$

$$= \frac{3}{2}\frac{-z^2}{(z-2)(z-0.5)}$$

$$\frac{3}{2}\frac{-z^2}{(z-2)(z-0.5)} = \frac{3}{2}\,\frac{2}{3}\left(\frac{0.5z}{(z-0.5)}-\frac{2z}{(z-2)}\right)$$

$f(z)$	$f(z^{-1})$
$g(z^{-1})$	$g(z)$

$X(z^{-1})$	$X(z)$
$Y(z)$	$Y(z^{-1})$

1

Laurent Series $a_n \leftrightarrow f(z)$ $b_n \leftrightarrow g(z)$

$n \geq 0$	$\lvert z \rvert < \rho$		$\lvert z \rvert > \rho^{-1}$	$n < 1$
causal a_n	$f(z)$	$\xleftrightarrow{z^{-1}}$	$f(z^{-1})$	anti-causal $-b_n$
	$\cdot \frac{\rho}{\rho} \frac{z^{-1}}{z^{-1}}$		$\cdot \frac{\rho}{\rho} \frac{z}{z}$	
$-a_n$ anti-causal	$g(z^{-1})$	$\xleftrightarrow{z^{-1}}$	$g(z)$	b_n causal
$n < 0$	$\lvert z \rvert > \rho$		$\lvert z \rvert < \rho^{-1}$	$n \geq 1$

Z- Transform $X(z) \leftrightarrow x_n$ $Y(z) \leftrightarrow y_n$

$g(z^{-1})$

$n < 1$	$\lvert z \rvert < \rho$		$\lvert z \rvert > \rho^{-1}$	$n \geq 0$
anti- causal $-y_n$	$X(z^{-1})$	$\xleftrightarrow{z^{-1}}$	$X(z)$	causal x_n
	$\cdot \frac{\rho}{\rho} \frac{z^{-1}}{z^{-1}}$		$\cdot \frac{\rho}{\rho} \frac{z}{z}$	
causal y_n	$Y(z)$	$\xleftrightarrow{z^{-1}}$	$Y(z^{-1})$	anti- causal $-x_n$
$n \geq 1$	$\lvert z \rvert > \rho$		$\lvert z \rvert < \rho^{-1}$	$n < 0$

$f(z)$	$f(z^{-1})$
$f(z)$	$f(z^{-1})$

$X(z^{-1})$	$X(z)$
$X(z^{-1})$	$X(z)$

Laurent Series $a_n \leftrightarrow f(z)$

$n \geq 0$	$\lvert z \rvert < p$		$\lvert z \rvert > p^{-1}$	$n < 1$
causal a_n	$\boxed{f(z)}$	$\xleftrightarrow{z^{-1}}$	$f(z^{-1})$	anti-causal a_{-n}
$-a_n$ anti-causal	$f(z)$	$\xleftrightarrow{z^{-1}}$	$\boxed{f(z^{-1})}$	$-a_{-n}$ causal
$n < 0$	$\lvert z \rvert > p$		$\lvert z \rvert < p^{-1}$	$n \geq 1$

Z- Transform $X(z) \leftrightarrow x_n$

$n < 1$	$\lvert z \rvert < p$		$\lvert z \rvert > p^{-1}$	$n \geq 0$
anti- causal x_{-n}	$X(z^{-1})$	$\xleftrightarrow{z^{-1}}$	$\boxed{X(z)}$	causal x_n
causal $-x_{-n}$	$\boxed{X(z^{-1})}$	$\xleftrightarrow{z^{-1}}$	$X(z)$	anti- causal $-x_n$
$n \geq 1$	$\lvert z \rvert > p$		$\lvert z \rvert < p^{-1}$	$n < 0$

$-\frac{p^{-1}}{1-p^{-1}z}$	$-\frac{p^{-1}}{1-p^{-1}z^{-1}}$
$\frac{z^{-1}}{1-pz^{-1}}$	$\frac{z}{1-pz}$

$-\frac{p^{-1}}{1-p^{-1}z}$	$-\frac{p^{-1}}{1-p^{-1}z^{-1}}$
$\frac{z^{-1}}{1-pz^{-1}}$	$\frac{z}{1-pz}$

Laurent Series $a_n \leftrightarrow f(z)$

	$n \geq 0$	$\lvert z\rvert < p$		$\lvert z\rvert > p^{-1}$	$n < 1$	
a_n	causal f(z) $-(p^{-1})^{n+1}$	$-\frac{p^{-1}}{1-p^{-1}z}$	$\xleftrightarrow{z^{-1}}$	$-\frac{p^{-1}}{1-p^{-1}z^{-1}}$	anti-causal f(z) $-(p^{-1})^{-n+1}$	a_{-n}
	$\updownarrow \cdot(-1)$	$\Vert$		$\Vert$	$\updownarrow \cdot(-1)$	
b_{-n}	$(p)^{-n-1}$ anti-causal f(z)	$\frac{z^{-1}}{1-pz^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{1-pz}$	$(p)^{n-1}$ causal f(z)	b_n
	$n < 0$	$\lvert z\rvert > p$		$\lvert z\rvert < p^{-1}$	$n \geq 1$	

Z- Transform $X(z) \leftrightarrow x_n$

	$n < 1$	$\lvert z\rvert < p$		$\lvert z\rvert > p^{-1}$	$n \geq 0$	
x_{-n}	anti-causal X(z) $-(p^{-1})^{-n+1}$	$-\frac{p^{-1}}{1-p^{-1}z}$	$\xleftrightarrow{z^{-1}}$	$-\frac{p^{-1}}{1-p^{-1}z^{-1}}$	causal X(z) $-(p^{-1})^{n+1}$	x_n
	$\updownarrow \cdot(-1)$	$\Vert$		$\Vert$	$\updownarrow \cdot(-1)$	
y_n	$(p)^{n-1}$ causal X(z)	$\frac{z^{-1}}{1-pz^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{1-pz}$	$(p)^{-n-1}$ anti-causal X(z)	y_{-n}
	$n \geq 1$	$\lvert z\rvert > p$		$\lvert z\rvert < p^{-1}$	$n < 0$	

① focusing functions

$f(z)$ $g(z)$	$Y(z)$ $X(z)$
$f(z)$ $g(z)$	$Y(z)$ $X(z)$

Laurent Series $a_n \leftrightarrow f(z)$

$$
\begin{tikzcd}
\boxed{f(z)} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{g(z)} = f(z^{-1}) \arrow[d, equal] \\
\boxed{f(z)} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{g(z)} = f(z^{-1})
\end{tikzcd}
$$

Z- Transform $X(z) \leftrightarrow x_n$

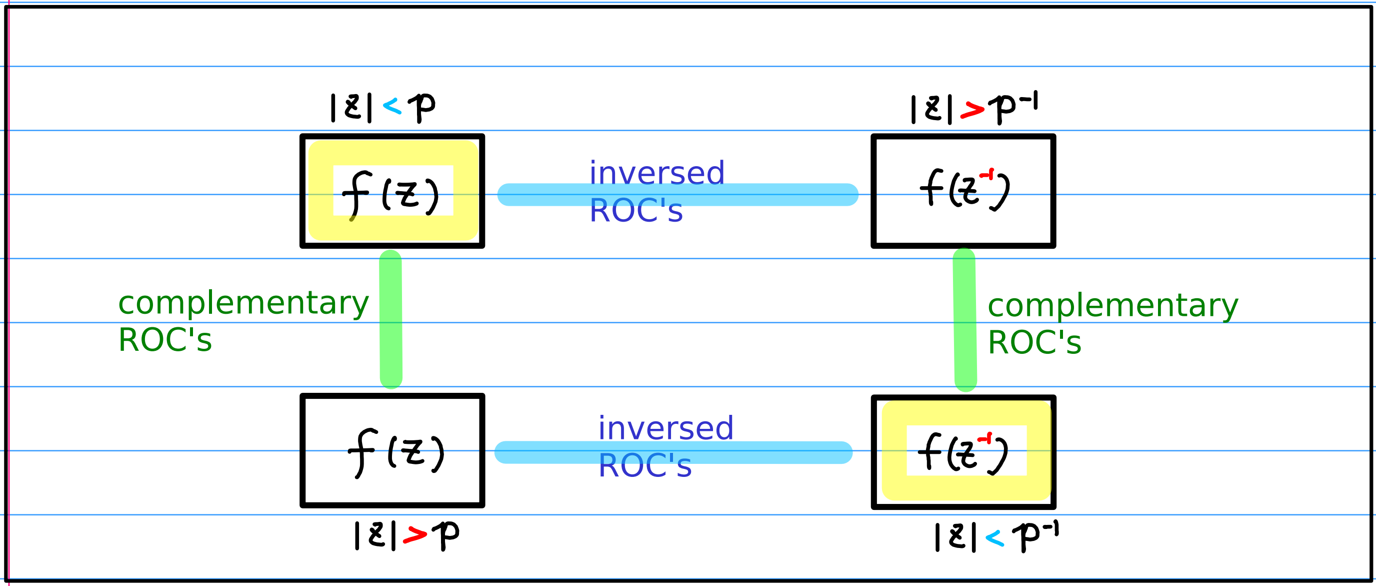
$$
\begin{tikzcd}
X(z^{-1}) = \boxed{Y(z)} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, equal] & \boxed{X(z)} \arrow[d, equal] \\
X(z^{-1}) = \boxed{Y(z)} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{X(z)}
\end{tikzcd}
$$

② focusing ROC's

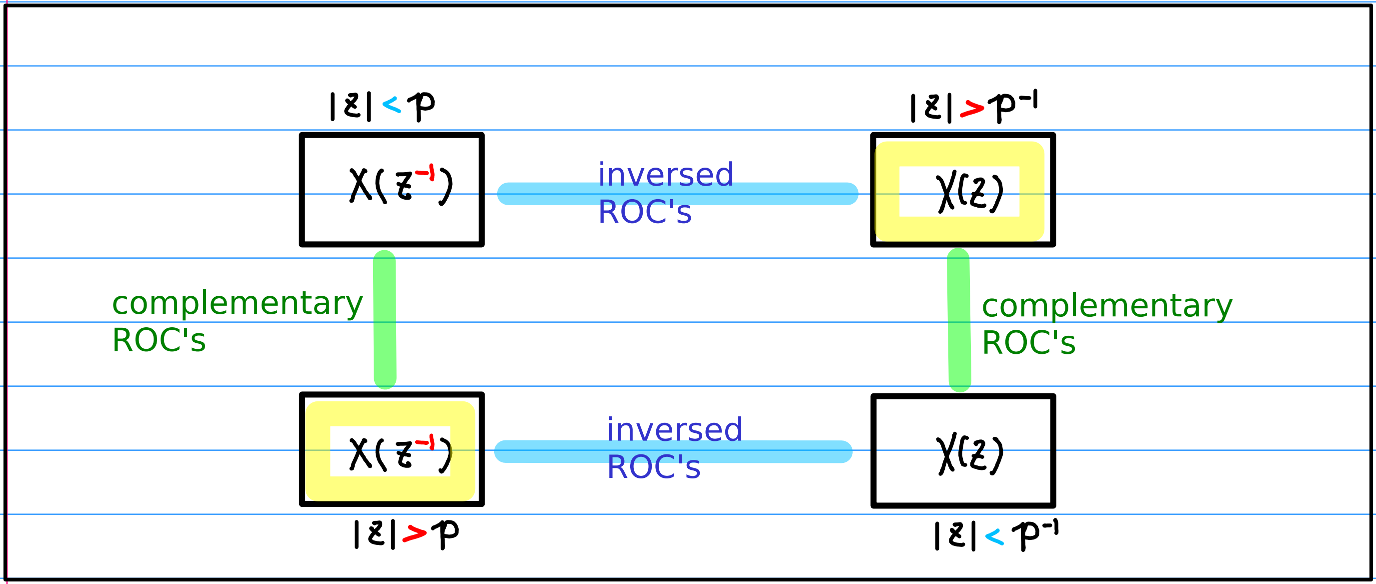
$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

Laurent Series $a_n \leftrightarrow f(z)$



Z- Transform $X(z) \leftrightarrow x_n$

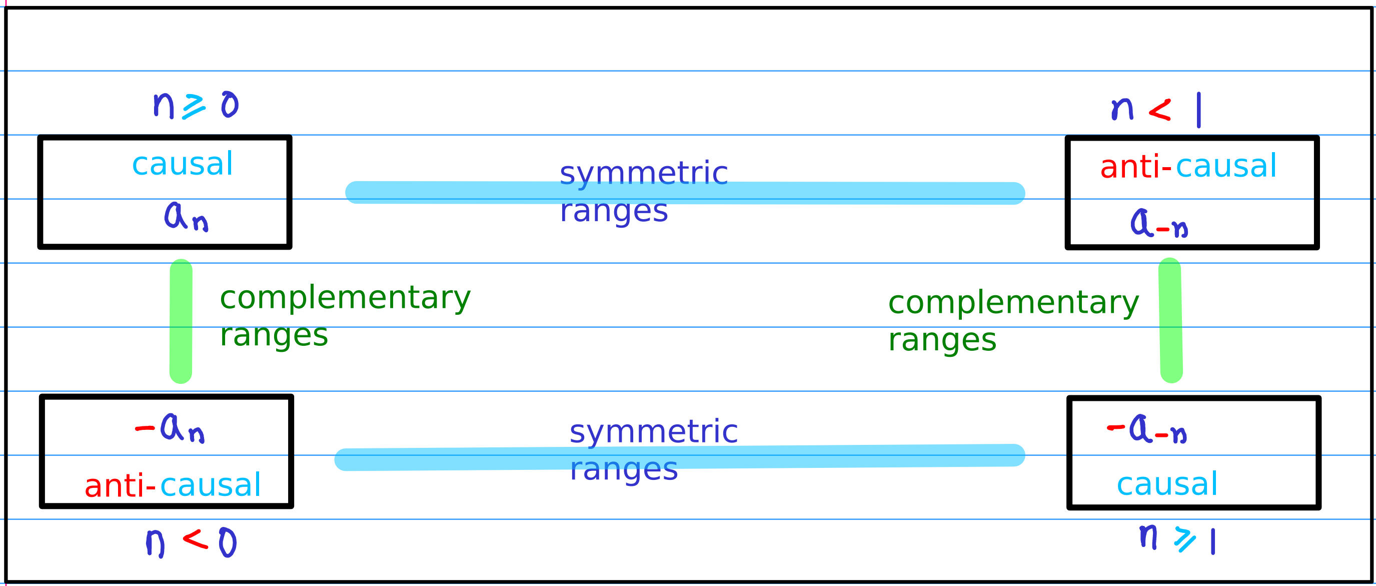


③ focusing ranges

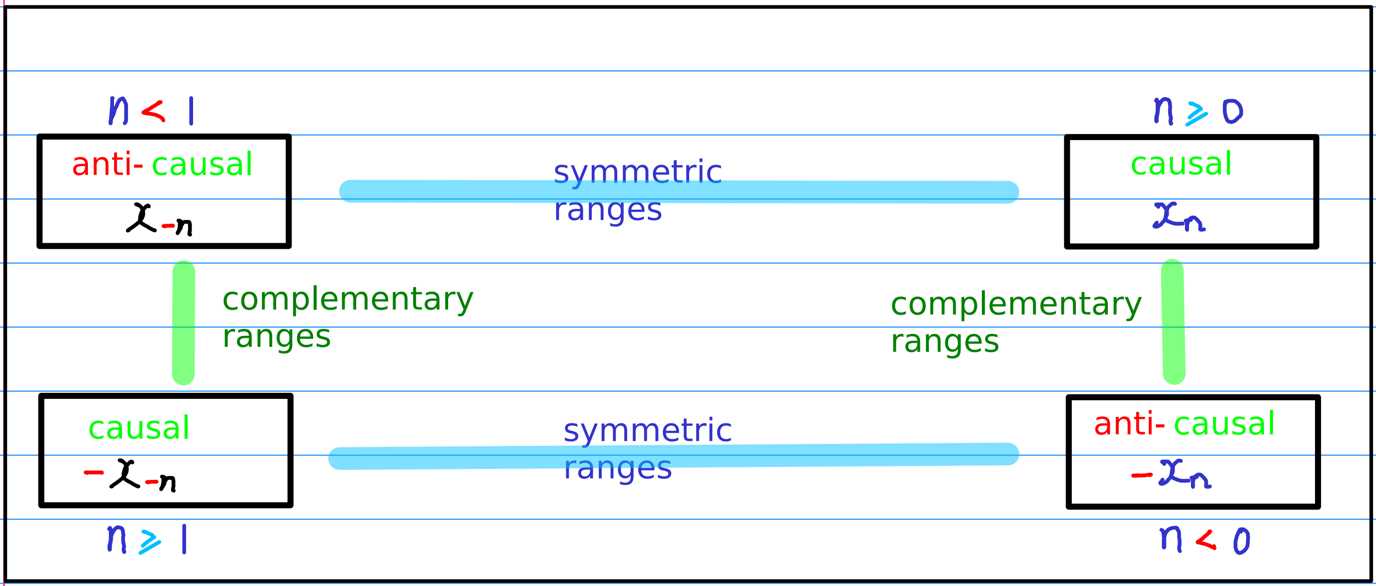
$n \geq 0$	$n < 1$
$n < 0$	$n \geq 1$

$n < 1$	$n \geq 0$
$n \geq 1$	$n < 0$

Laurent Series $a_n \leftrightarrow f(z)$



Z- Transform $X(z) \leftrightarrow x_n$



④ focusing sequences

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$(p^0, p^1, p^2, \cdots)$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$(p^0, p^1, p^2, \cdots)$

Laurent Series $a_n \leftrightarrow f(z)$

$n \geq 0$ causal

$\boxed{-(p^{-1}, p^{-2}, p^{-3}, \cdots)}$

n=0 1, 2, 3,....

n= -1,-2,-3,....

$\boxed{(p^0, p^1, p^2, \cdots)}$

$n < 0$ anti-causal

$n < 1$ anti-causal

$\boxed{-(p^{-1}, p^{-2}, p^{-3}, \cdots)}$

n=0 -1,-2,-3,....

n= 1, 2, 3,....

$\boxed{(p^0, p^1, p^2, \cdots)}$

$n \geq 1$ causal

Z- Transform $X(z) \leftrightarrow x_n$

$n < 1$ anti-causal

$\boxed{-(p^{-1}, p^{-2}, p^{-3}, \cdots)}$

n=0 -1,-2,-3,....

n= 1, 2, 3,....

$\boxed{(p^0, p^1, p^2, \cdots)}$

$n \geq 1$ causal

$n \geq 0$ causal

$\boxed{-(p^{-1}, p^{-2}, p^{-3}, \cdots)}$

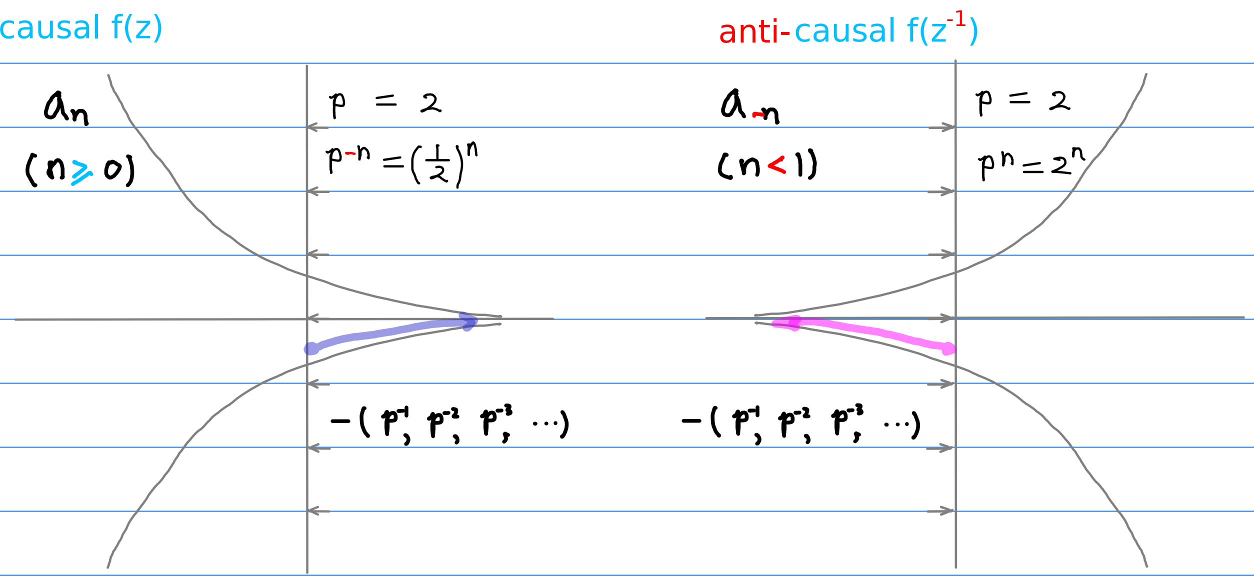
n=0 1, 2, 3,....

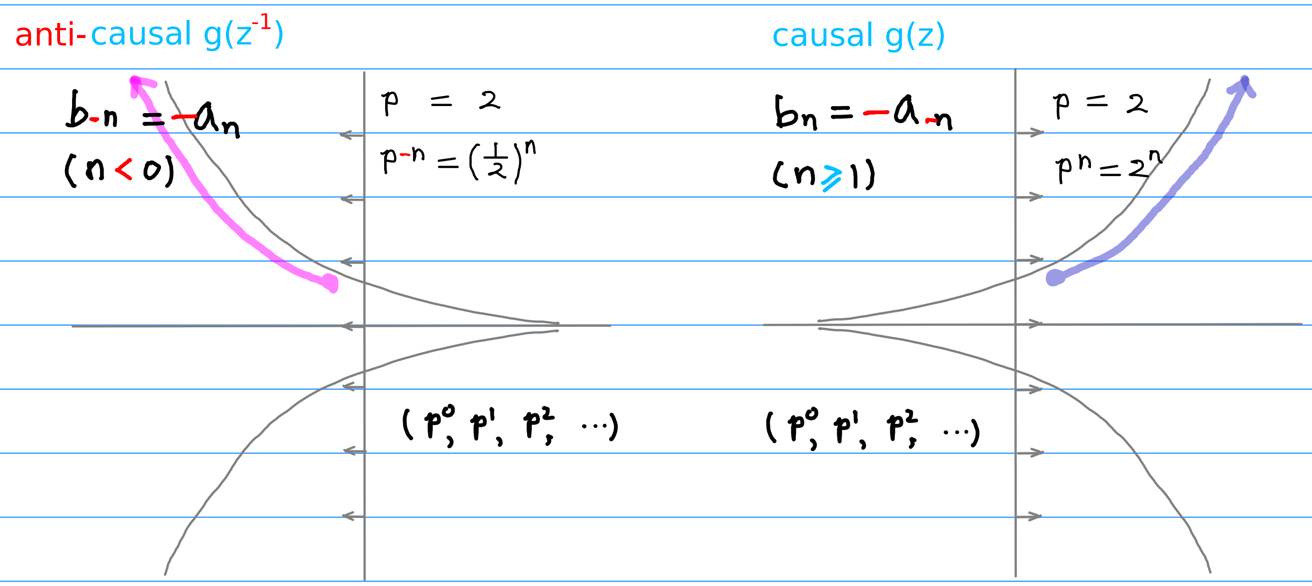
n= -1,-2,-3,....

$\boxed{(p^0, p^1, p^2, \cdots)}$

$n < 0$ anti-causal

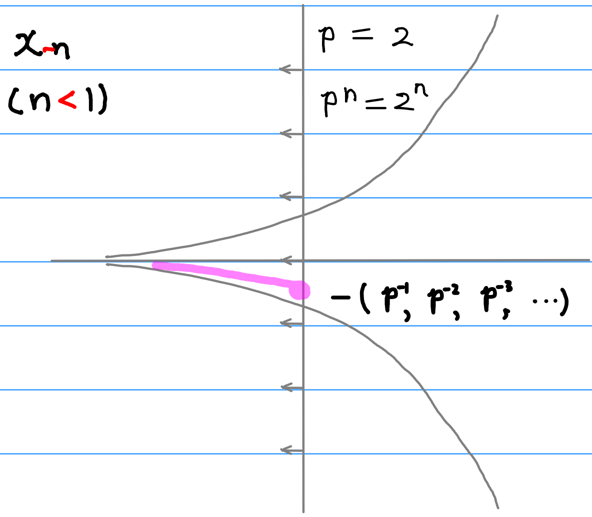
a_n, b_n Laurent Series graphs of $f(z)$, $g(z)$



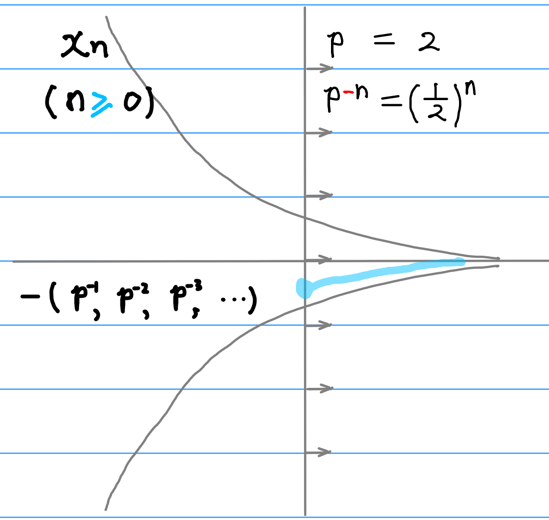


x_n, y_n graphs of z-transform $X(z)$, $Y(z)$

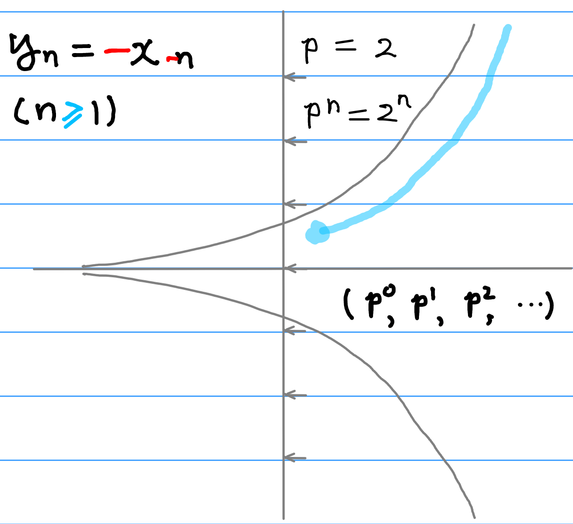
anti-causal $X(z^{-1})$



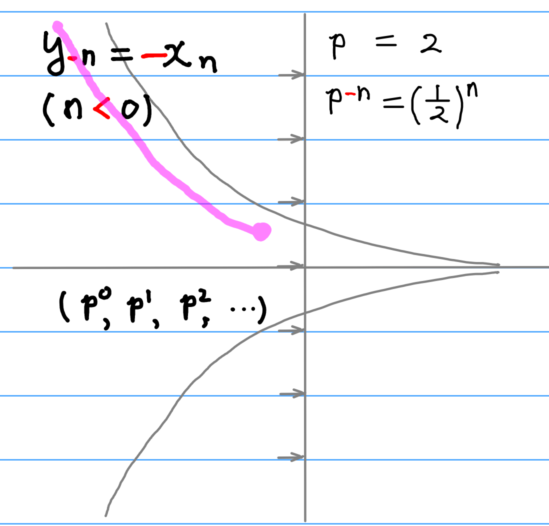
causal $X(z)$



causal $Y(z)$

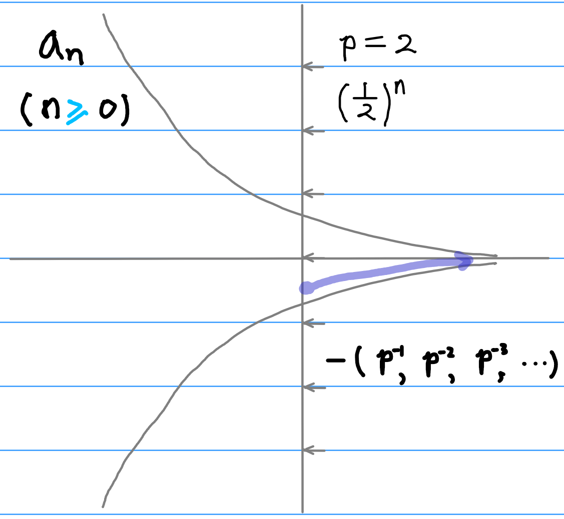


anti-causal $Y(z^{-1})$

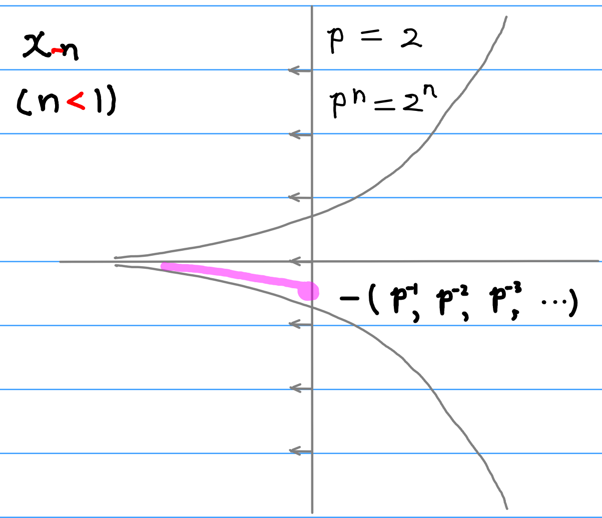


a_n, x_n graphs of $f(z)$, $X(z)$

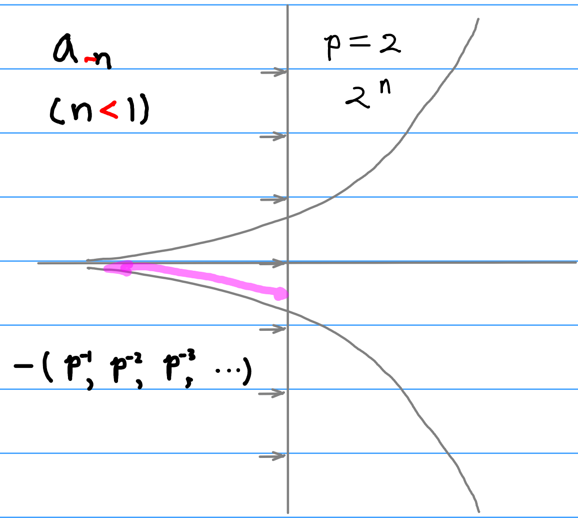
causal $f(z)$



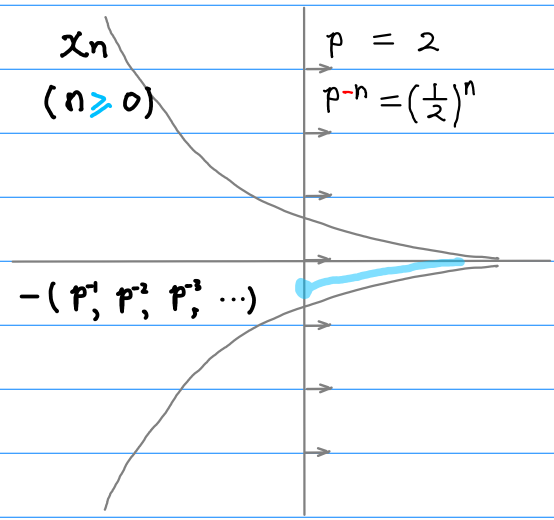
anti-causal $X(z^{-1})$



anti-causal $f(z^{-1})$

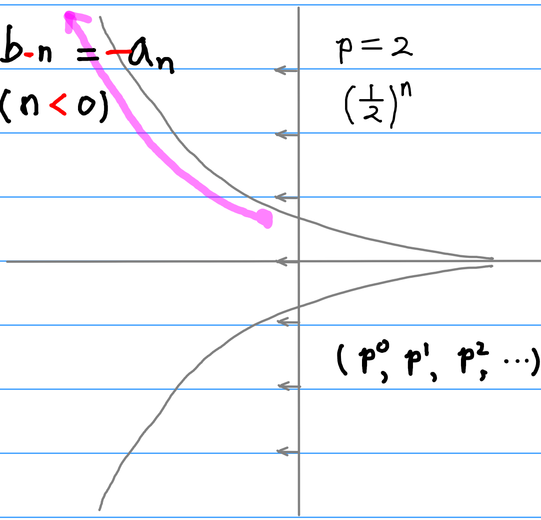


causal $X(z)$



b_n, y_n graphs of $g(z)$, $Y(z)$

anti-causal $g(z^{-1})$



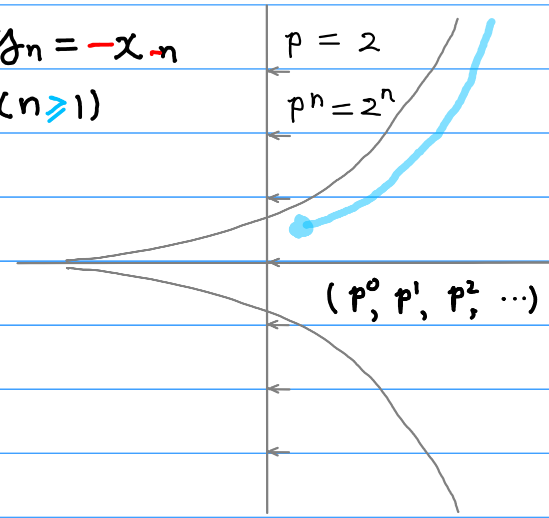
$b_{-n} = -a_n$ $(n < 0)$

$p = 2$

$\left(\frac{1}{2}\right)^n$

$(p^0, p^1, p^2, \cdots)$

causal $Y(z)$



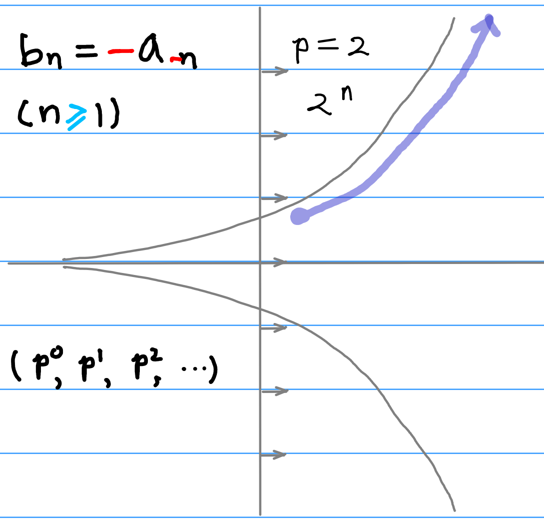
$y_n = -x_{-n}$ $(n \geq 1)$

$p = 2$

$p^n = 2^n$

$(p^0, p^1, p^2, \cdots)$

causal $g(z)$



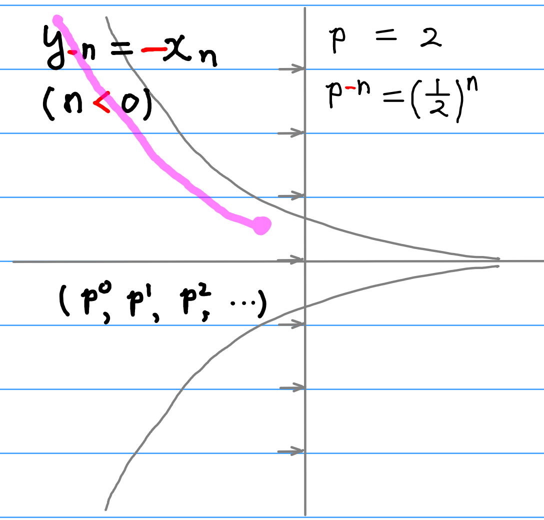
$b_n = -a_{-n}$ $(n \geq 1)$

$p = 2$

2^n

$(p^0, p^1, p^2, \cdots)$

anti-causal $Y(z^{-1})$



$y_{-n} = -x_n$ $(n < 0)$

$p = 2$

$p^{-n} = \left(\frac{1}{2}\right)^n$

$(p^0, p^1, p^2, \cdots)$

$f(z)$ ①

causal $f(z)$ $(|z|<p)$ anti-causal $f(z)$ $(|z|>p)$

$f(z) \leftrightarrow a_n$ $(n \geq 0)$ $g(z^{-1}) \leftrightarrow b_{-n}$ $(n<0)$

$n \geq 0$ $|z|<p$ $n = 0, 1, 2, \cdots$ $-p^{-n-1} z^n$

a_n $\boxed{\text{causal } f(z) \quad -(p^{-1})^{n+1}}$ $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $\boxed{-(p^{-1}+p^{-2}z^{1}+p^{-3}z^{2}+\cdots) = \sum_{n=0}^{\infty} -(p^{-1})^{n+1} z^n \quad (n \geq 0)}$

$\cdot(-1)$ $\cdot \dfrac{p}{p}\dfrac{z^{-1}}{z^{-1}}$

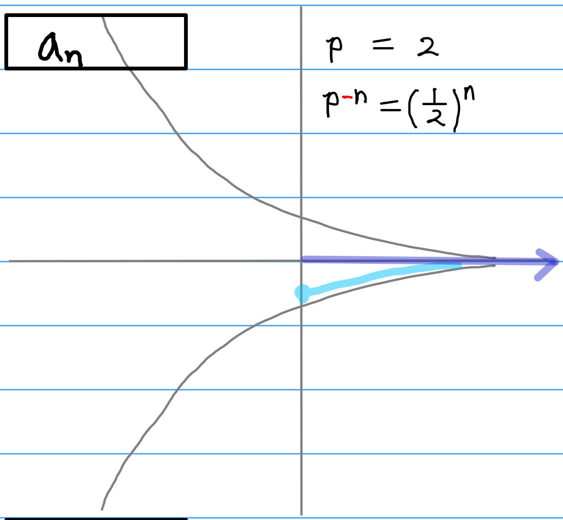
b_{-n} $\boxed{(p)^{-n-1} \quad \text{anti- causal } f(z)}$ $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ $\boxed{p^{0}z^{-1}+p^{1}z^{-2}+p^{2}z^{-3}+\cdots = \sum_{n=-1}^{-\infty} (p)^{-n-1} z^n \quad (n<0)}$

$n<0$ $|z|>p$ $n = -1, -2, -3, \cdots$ $p^{-n-1} z^n$

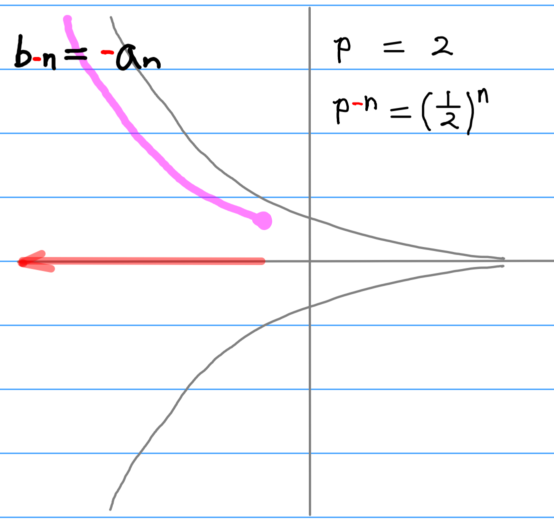
causal $n = 0, +1, +2, +3, \ldots.$ anti-causal $n = -1, -2, -3, \ldots.$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$ $(p^{0}, p^{1}, p^{2}, \cdots)$

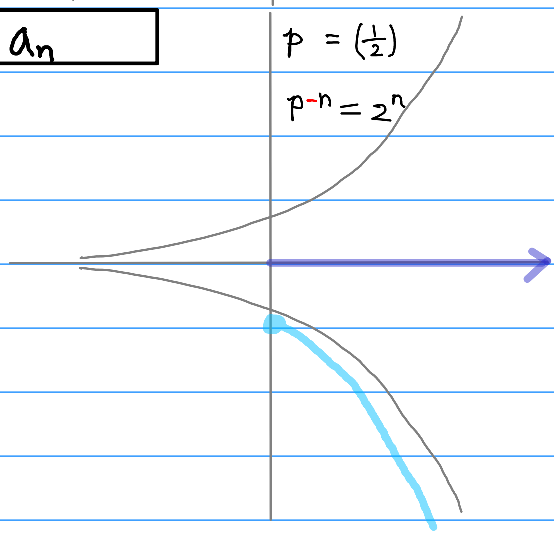
$\boxed{a_n}$ $p = 2$ $p^{-n} = \left(\frac{1}{2}\right)^n$



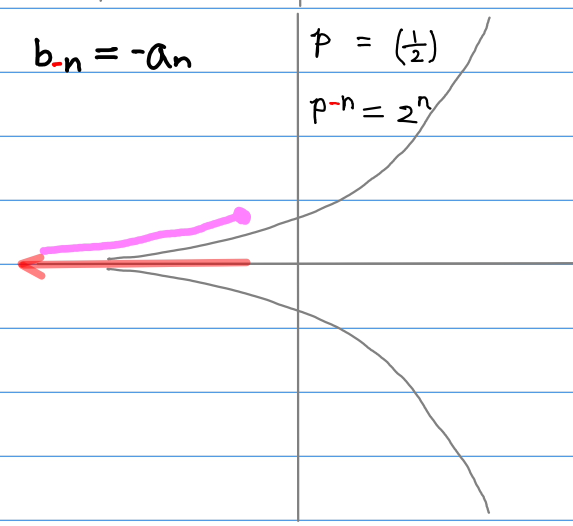
$b_{-n} = -a_n$ $p = 2$ $p^{-n} = \left(\frac{1}{2}\right)^n$



$\boxed{a_n}$ $p = \left(\frac{1}{2}\right)$ $p^{-n} = 2^n$



$b_{-n} = -a_n$ $p = \left(\frac{1}{2}\right)$ $p^{-n} = 2^n$



$f(z)$

2

anti-causal $g(z)$ $(|z|>p^{-1})$ causal $g(z)$ $(|z|<p^{-1})$

$f(z^{-1}) \leftrightarrow a_{-n}$ $(n<1)$ $g(z) \leftrightarrow b_n$ $(n \geq 1)$

$n = 0, -1, -2, \cdots$ $-p^{n-1}z^n$ $|z|>p^{-1}$ $n<1$

$$\boxed{-(p^{-1}+p^{-2}z^{-1}+p^{-3}z^{-2}+\cdots) = \sum_{n=0}^{-\infty} -(p)^{n-1}z^n \quad (n<1)}$$

$$\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} \qquad \boxed{\begin{array}{c}\text{anti- causal } g(z)\\ -(p^{-1})^{-n+1}\end{array}} \; a_{-n}$$

$\cdot\frac{p}{p}\frac{z}{z}$ $\cdot(-1)$

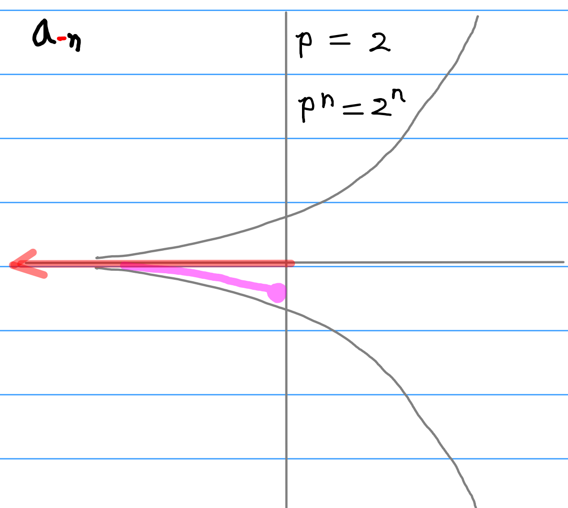
$$\boxed{p^0z^1+p^1z^2+p^2z^3+\cdots = \sum_{n=1}^{\infty}(p)^{n-1}z^n \quad (n\geq 1)}$$

$$\boxed{\frac{z}{1-pz}} \qquad \boxed{\begin{array}{c}(p)^{n-1}\\ \text{causal } g(z)\end{array}} \; b_n$$

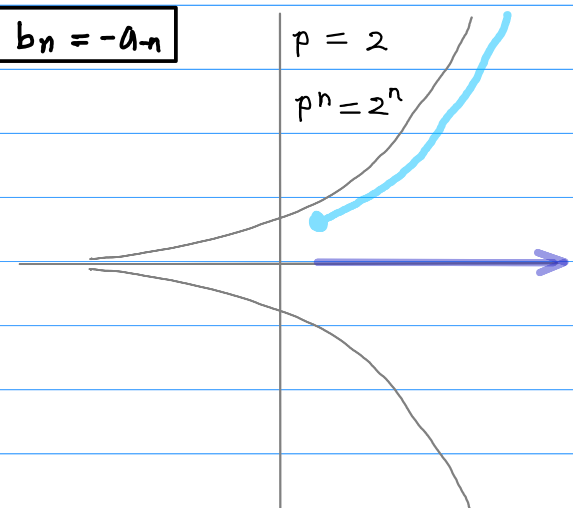
$n=1, 2, 3, \cdots$ $p^{n-1}z^n$ $|z|<p^{-1}$ $n \geq 1$

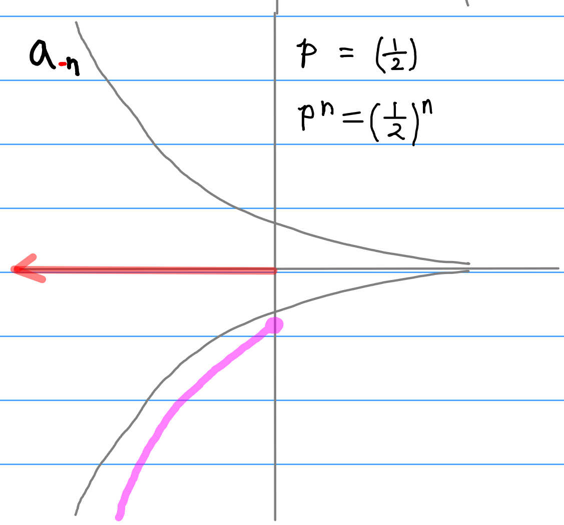
anti-causal $n=0,-1,-2,-3,\ldots.$ causal $n=+1,+2,+3,\ldots.$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$ $(p^0, p^1, p^2, \cdots)$

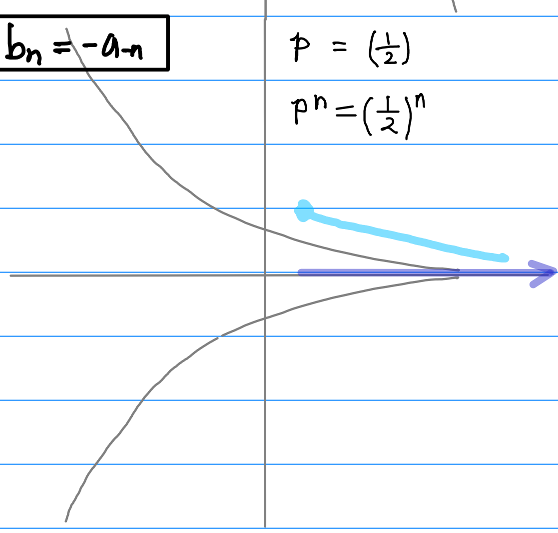


$\boxed{b_n = -a_{-n}}$





$\boxed{b_n = -a_{-n}}$



$f(z)$ 3

$|z| < p$

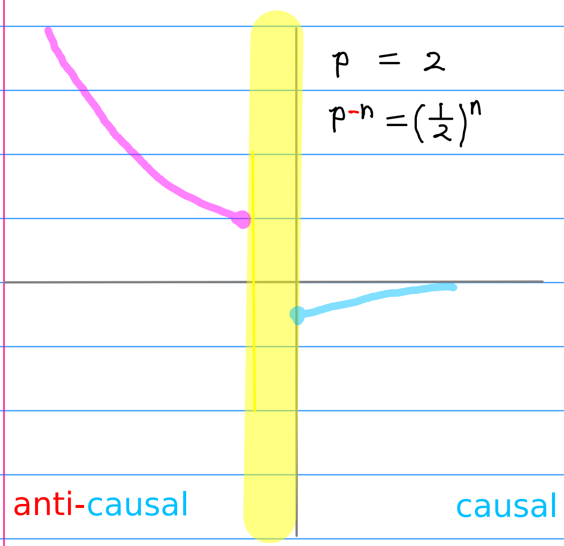
$|z| > p$

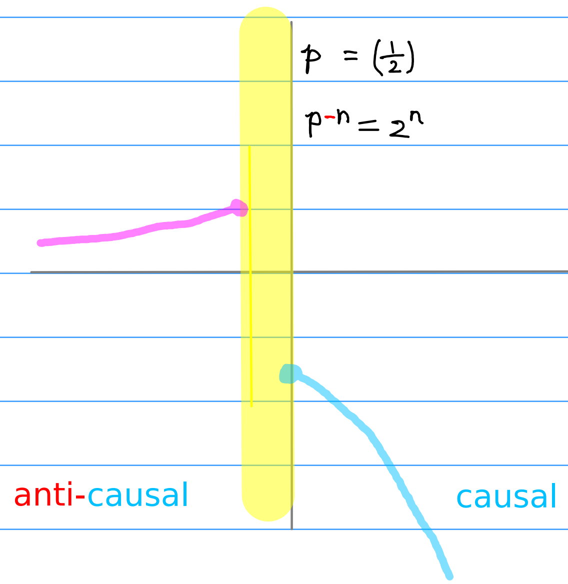
$|z| < p^{-1}$

$|z| > p^{-1}$

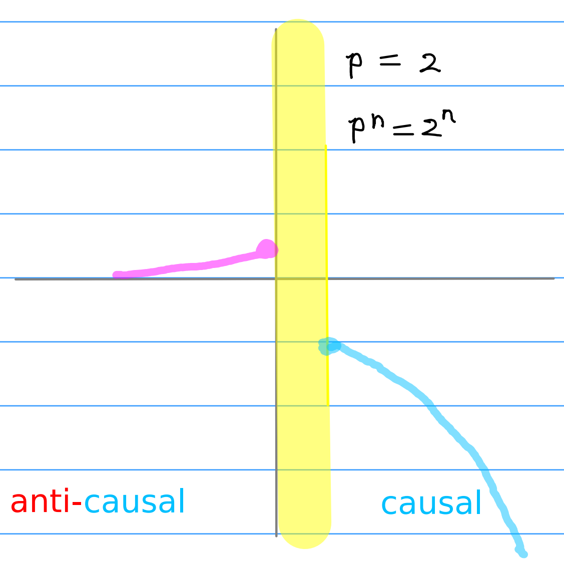
complementary ROC's

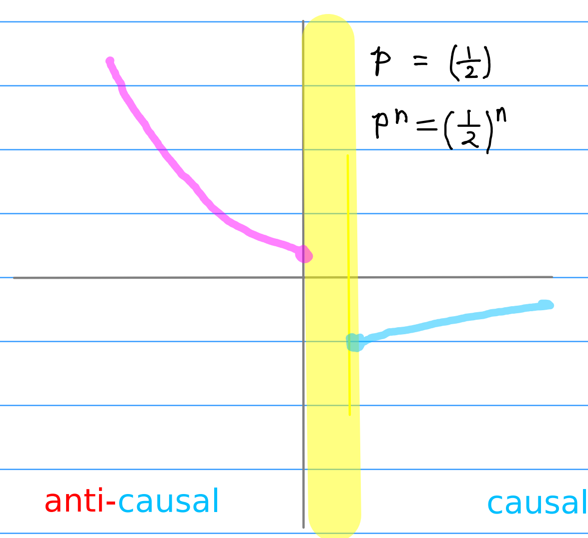
$f(z)$ $(\lvert z\rvert < p)$	a_n $(n \geq 0)$	$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$	causal	n=0,+1,+2,+3,....
$f(z)$ $(\lvert z\rvert > p)$	$-a_n$ $(n < 0)$	$(p^{0}, p^{1}, p^{2}, \cdots)$	anti-causal	n=-1,-2,-3,....





$g(z)$ $(\lvert z\rvert < p^{-1})$	b_n $(n \geq 1)$	$-(p^{0}, p^{1}, p^{2}, \cdots)$	causal	n=+1,+2,+3,....
$g(z)$ $(\lvert z\rvert > p^{-1})$	$-b_n$ $(n < 1)$	$(p^{-1}, p^{-2}, p^{-3}, \cdots)$	anti-causal	n=0,-1,-2,-3,....





$X(z)$ ①

anti-causal $Y(z)$ $(|z|<p)$ $\qquad$ causal $Y(z)$ $(|z|>p)$

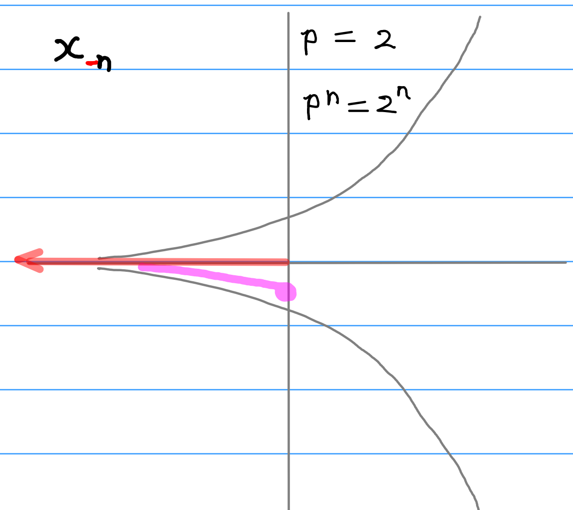
$X(z^{-1}) \leftrightarrow x_{-n}$ $(n<1)$ $\qquad$ $Y(z) \leftrightarrow y_n$ $(n \geq 0)$

	$n<1$	$\lvert z\rvert<p$	$n=0,-1,-2,\cdots$ $\qquad$ $-p^{n-1}z^{-n}$
x_{-n}	anti-causal Y(z) $\boxed{-(p^{-1})^{-n+1}}$	$\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$	$-(p^{-1}+p^{-2}z^{1}+p^{-3}z^{2}+\cdots) = -\sum_{n=0}^{-\infty}(p^{-1})^{-n+1}z^{-n}$ $\quad n<1$
	$\updownarrow \cdot(-1)$	$\Vert$ $\quad \cdot\dfrac{p}{p}\dfrac{z^{-1}}{z^{-1}}$	
y_n	$\boxed{(p)^{n-1}}$ causal Y(z)	$\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$	$p^{0}z^{-1}+p^{1}z^{-2}+p^{2}z^{-3}+\cdots = \sum_{n=1}^{\infty}(p)^{n-1}z^{-n}$ $\quad n\geq 1$
	$n \geq 1$	$\lvert z\rvert>p$	$n=1,2,3,\cdots$ $\qquad$ $p^{n-1}z^{-n}$

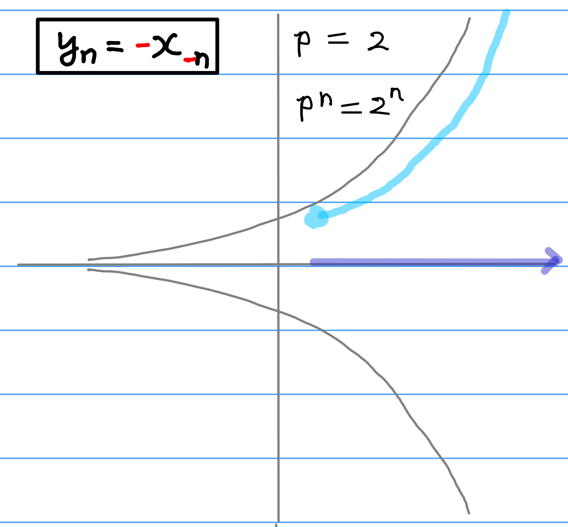
anti-causal $\qquad$ n=0,-1,-2,-3,.... $\qquad$ causal $\qquad$ n=+1,+2,+3,....

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$ $\qquad$ $(p^{0}, p^{1}, p^{2}, \cdots)$

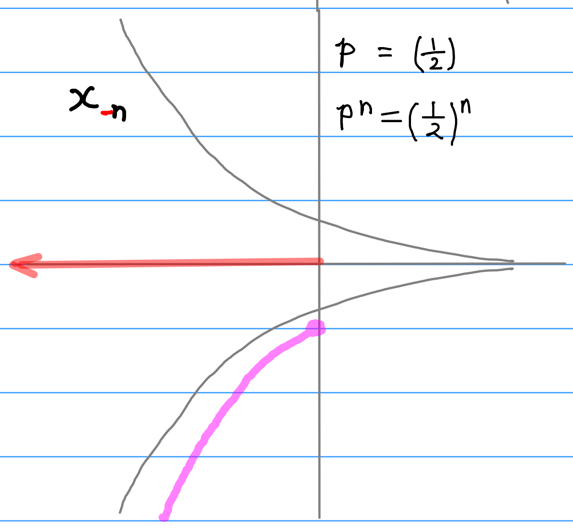
x_{-n}



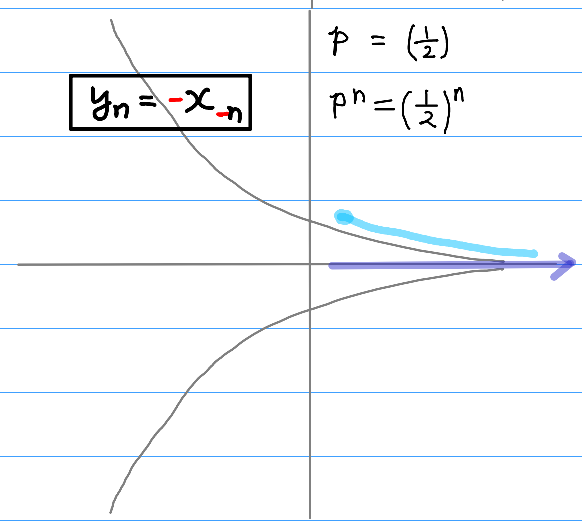
$\boxed{y_n = -x_{-n}}$



x_{-n}



$\boxed{y_n = -x_{-n}}$



X(z) ②

causal X(z) ($|z| < p^{-1}$)   anti-causal X(z) ($|z| > p^{-1}$)

$X(z) \leftrightarrow x_n \quad (n \geq 1)$ $Y(z^{-1}) \leftrightarrow -y_n \quad (n < 1)$

n = 0, 1, 2, ⋯ $-p^{-n-1} z^{-n}$

$$\boxed{-(p^{-1} + p^{-2}z^{-1} + p^{-3}z^{-2} + \cdots) = \sum_{n=0}^{\infty} -(p^{-1})^{n+1} z^{-n} \quad (n \geq 0)}$$

$|z| > p^{-1}$ $n \geq 0$

$$\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} \quad \boxed{\text{causal } X(z) \quad -(p^{-1})^{n+1}} \quad x_n$$

$\cdot \frac{p}{p}\frac{z}{z}$ $\cdot(-1)$

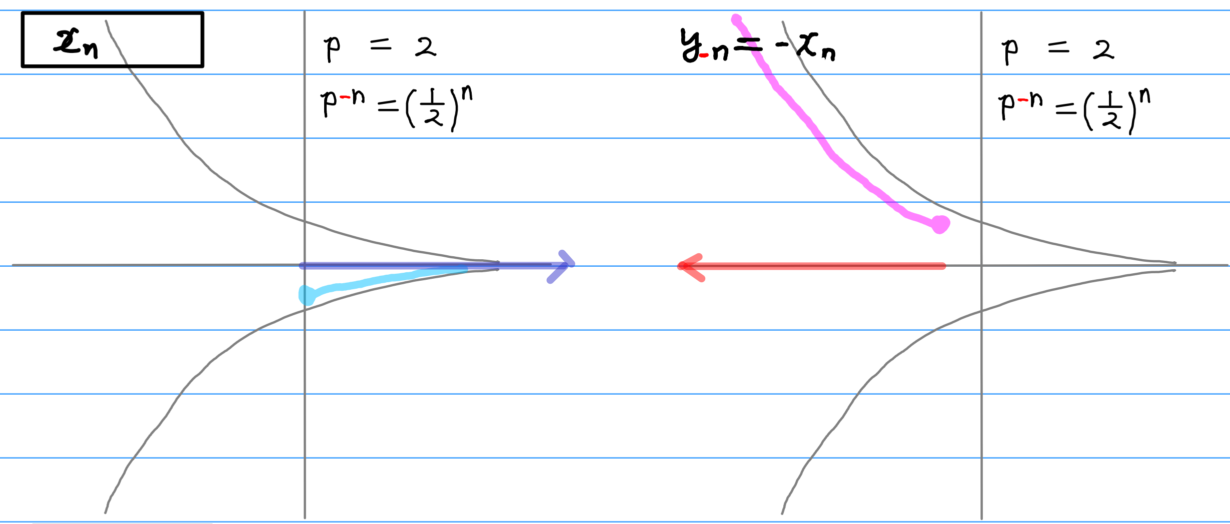
$$\boxed{p^0 z^1 + p^1 z^2 + p^2 z^3 + \cdots = \sum_{n=-1}^{-\infty} (p^{-1})^{n+1} z^{-n} \quad (n < 0)}$$

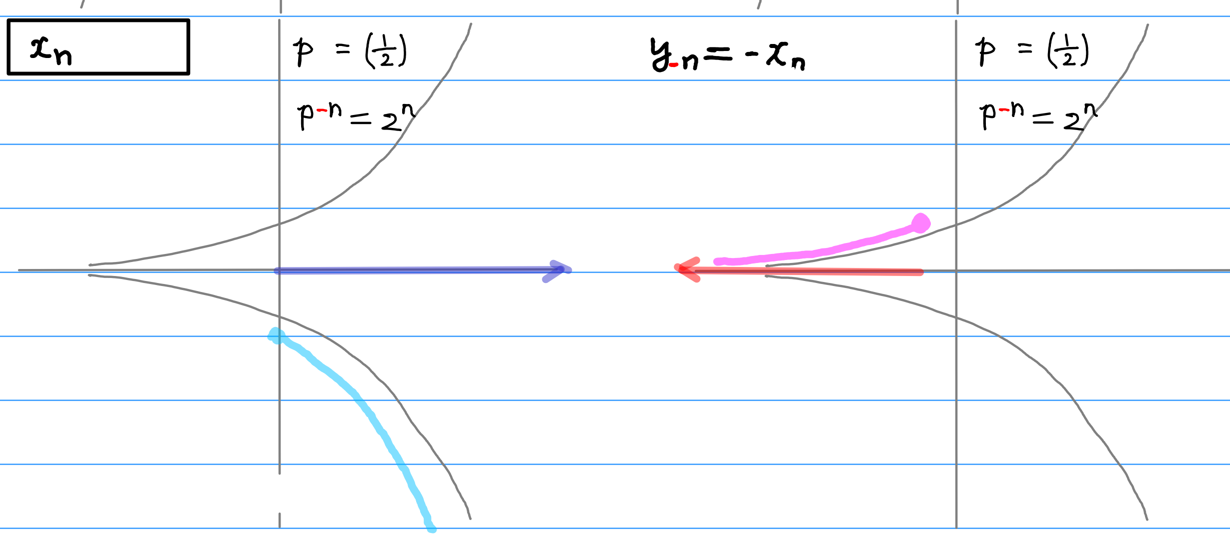
$$\boxed{\frac{z}{1-pz}} \quad \boxed{(p)^{-n-1} \quad \text{anti-causal } X(z)} \quad y_{-n}$$

n = -1, -2, -3, ⋯ $p^{-n-1} z^{-n}$ $|z| < p^{-1}$ $n < 0$

causal   n=0,+1,+2,+3,....   anti-causal   n=-1,-2,-3,....

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$ $(p^0, p^1, p^2, \cdots)$

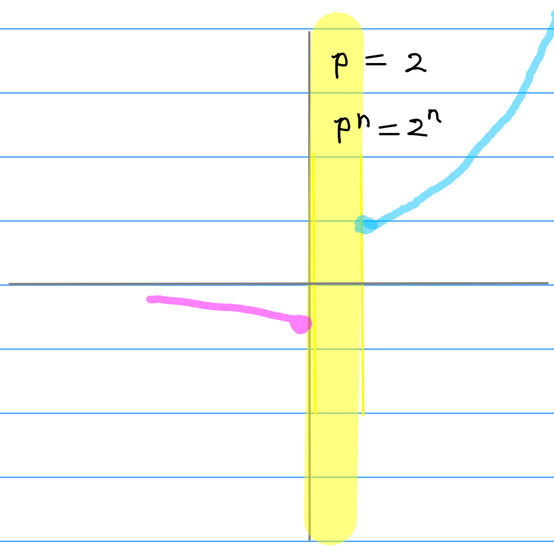


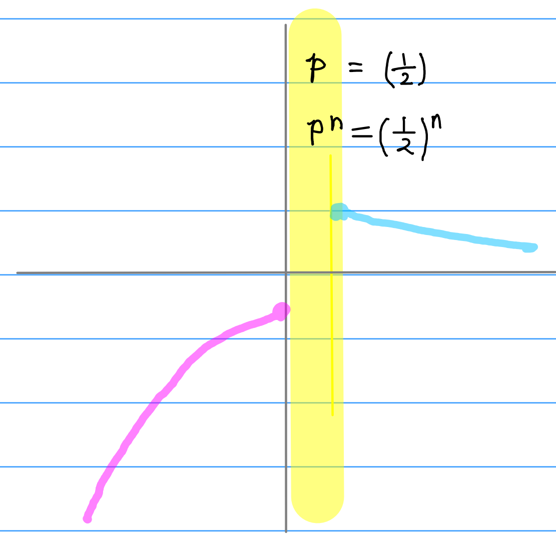


$X(z)$ ③

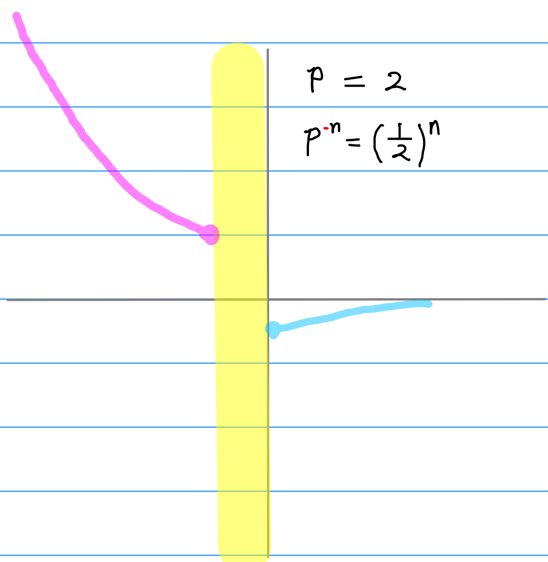
$\boxed{\begin{array}{l} |z| < p \\ |z| > p \end{array}}$ $\boxed{\begin{array}{l} |z| < p^{-1} \\ |z| > p^{-1} \end{array}}$ complementary ROC's

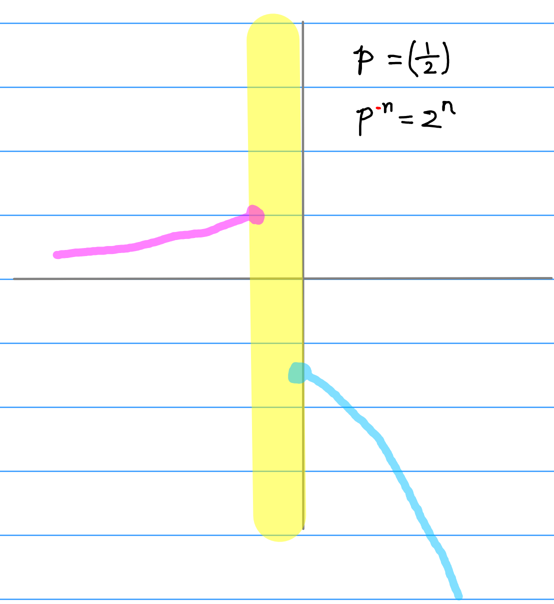
$Y(z)$ $(\lvert z\rvert < p^{-1})$	$-y_n$	$(n < 1)$	$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$	anti-causal	n= 0, -1,-2,-3,....
$Y(z)$ $(\lvert z\rvert > p^{-1})$	y_n	$(n \geq 1)$	$(p^{0}, p^{1}, p^{2}, \cdots)$	causal	n= 1, 2, 3,....





$X(z)$ $(\lvert z\rvert > p)$	x_n	$(n \geq 0)$	$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$	causal	n= 0 1, 2, 3,....
$X(z)$ $(\lvert z\rvert < p)$	$-x_n$	$(n < 0)$	$(p^{0}, p^{1}, p^{2}, \cdots)$	anti-causal	n= -1,-2,-3,....





$f(z)$

④

causal $f(z)$ $(|z|<p)$ anti-causal $g(z)$ $(|z|>p^{-1})$

$f(z) \leftrightarrow a_n$ $(n \geq 0)$ $f(z^{-1}) \leftrightarrow a_{-n}$ $(n<1)$

$n \geq 0$ $|z|<p$ $n = 0, 1, 2, \cdots$ $-p^{-n-1} z^n$

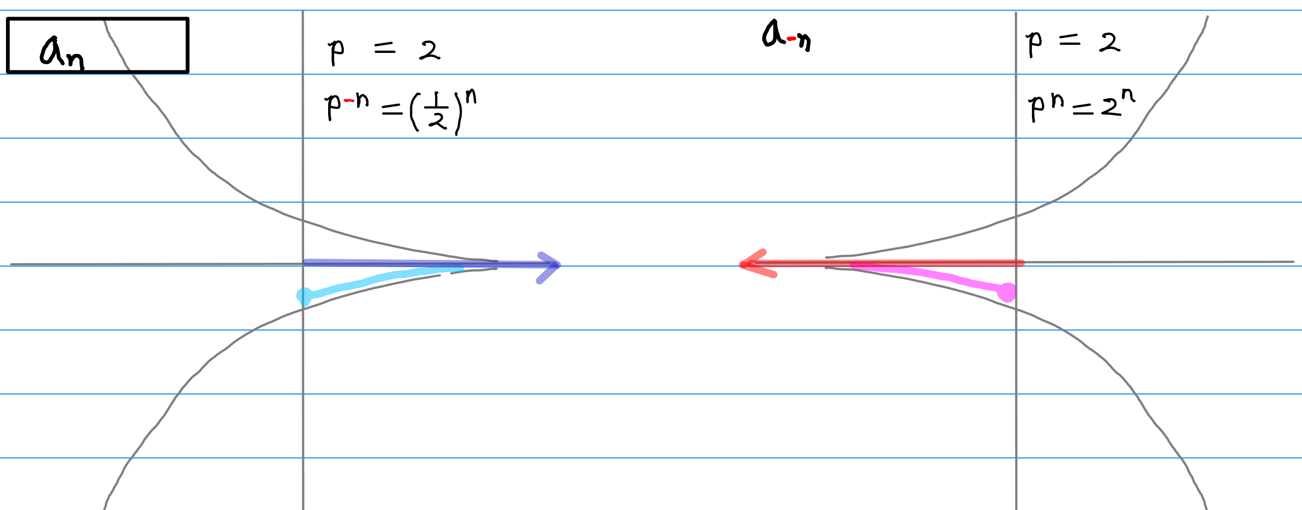
a_n causal f(z) $-(p^{-1})^{n+1}$ $-\frac{p^{-1}}{1-p^{-1}z}$ $-(p^{-1}+p^{-2}z^{1}+p^{-3}z^{2}+\cdots) = \sum_{n=0}^{\infty} -(p^{-1})^{n+1} z^n$ $(n \geq 0)$

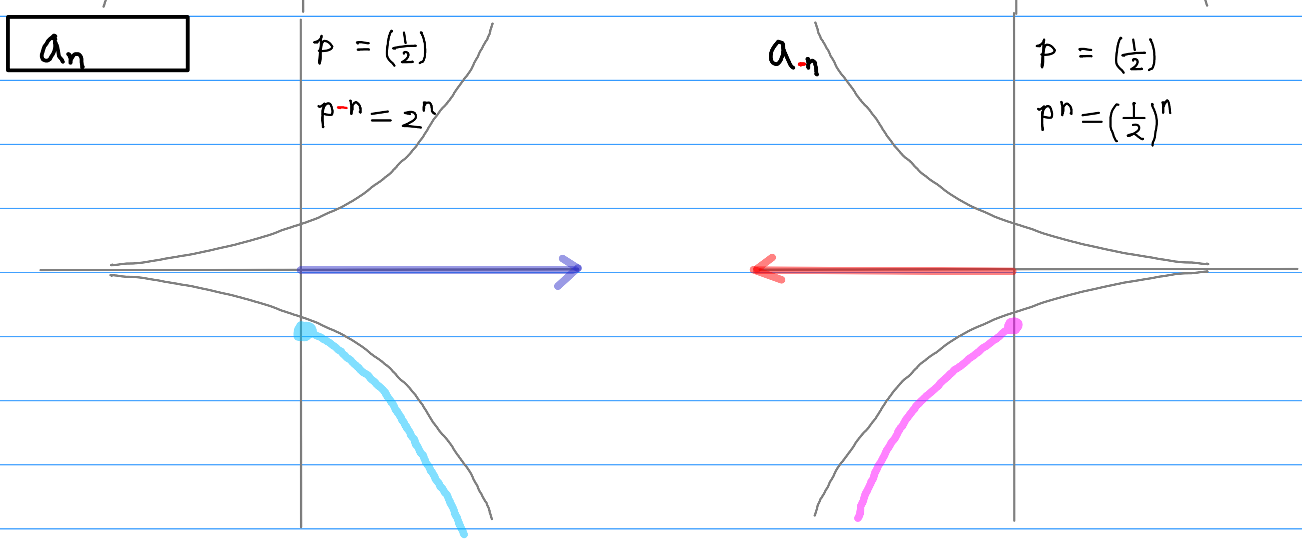
$n = 0, -1, -2, \cdots$ $-p^{n-1} z^n$ $|z|>p^{-1}$ $n<1$

$-(p^{-1}+p^{-2}z^{-1}+p^{-3}z^{-2}+\cdots) = \sum_{n=0}^{-\infty} -(p)^{n-1} z^n$ $(n<1)$ $-\frac{p^{-1}}{1-p^{-1}z^{-1}}$ anti- causal g(z) $-(p^{-1})^{-n+1}$ a_{-n}

causal n=0,+1,+2,+3,.... $-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

anti-causal n=0,-1,-2,-3,.... $-(p^{-1}, p^{-2}, p^{-3}, \cdots)$





$f(z)$

⑤

anti-causal $f(z)$ $(|z| > p)$

$g(z^{-1}) \leftrightarrow b_{-n}$ $(n < 0)$

causal $g(z)$ $(|z| < p^{-1})$

$g(z) \leftrightarrow b_n$ $(n \geq 1)$

---

b_{-n} $\boxed{(p)^{-n-1}}$ anti-causal f(z) $\boxed{\dfrac{z^{-1}}{1 - pz^{-1}}}$ $\boxed{p^0 z^{-1} + p^1 z^{-2} + p^2 z^{-3} + \cdots = \sum_{n=-1}^{-\infty} (p)^{-n-1} z^n \quad (n<0)}$

$n < 0$ $\quad |z| > p$ $\quad n = -1, -2, -3, \cdots$ $\quad p^{-n-1} z^n$

$\boxed{p^0 z^1 + p^1 z^2 + p^2 z^3 + \cdots = \sum_{n=1}^{\infty} (p)^{n-1} z^n \quad (n \geq 1)}$ $\boxed{\dfrac{z}{1 - pz}}$ $\boxed{(p)^{n-1}}$ causal f(z) b_n

$n = 1, 2, 3, \cdots$ $\quad p^{n-1} z^n$ $\quad |z| < p^{-1}$ $\quad n \geq 1$

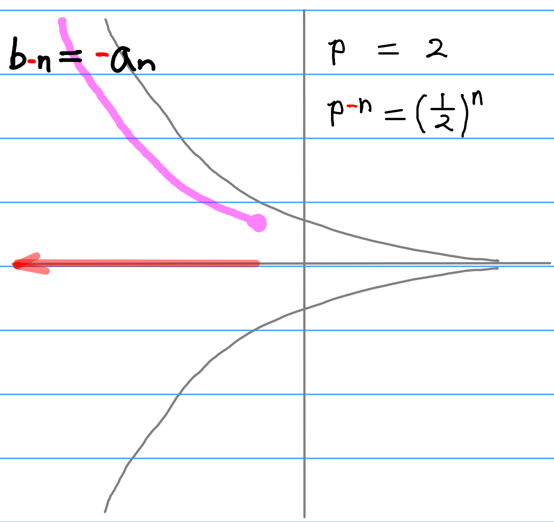
---

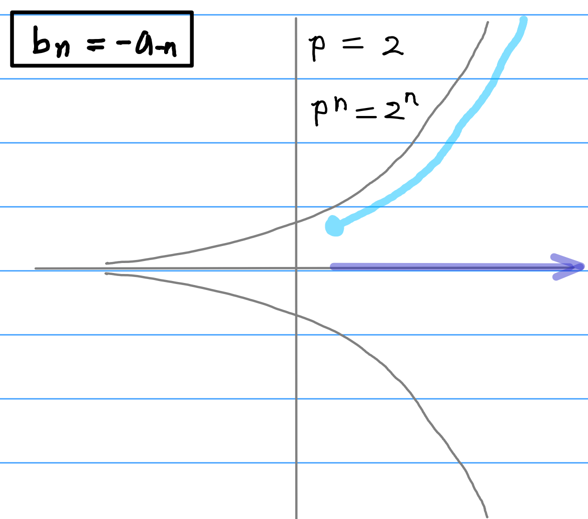
anti-causal $\quad$ n=-1,-2,-3,....

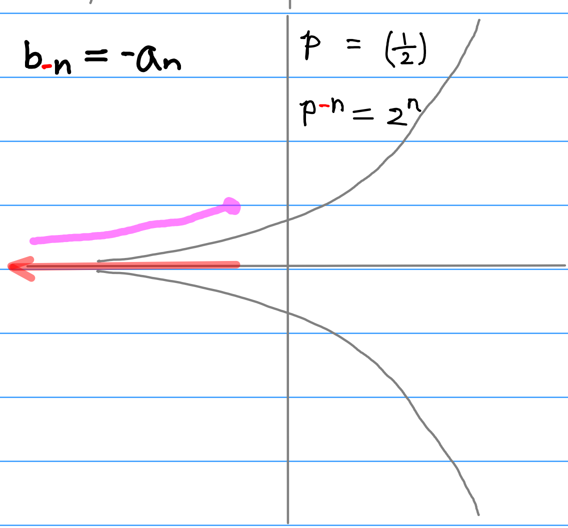
$(p^0, p^1, p^2, \cdots)$

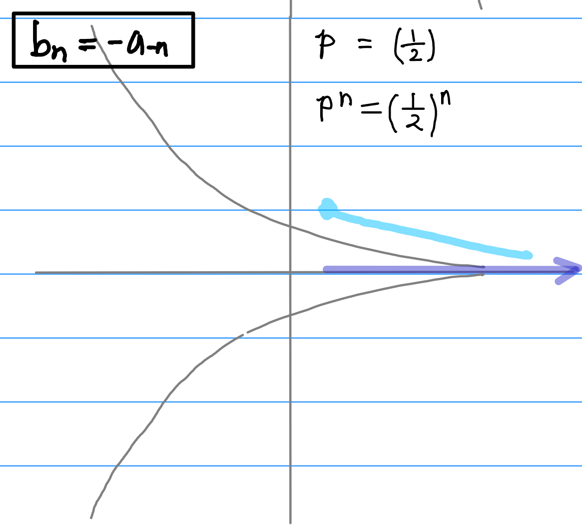
causal $\quad$ n=+1,+2,+3,....

$(p^0, p^1, p^2, \cdots)$









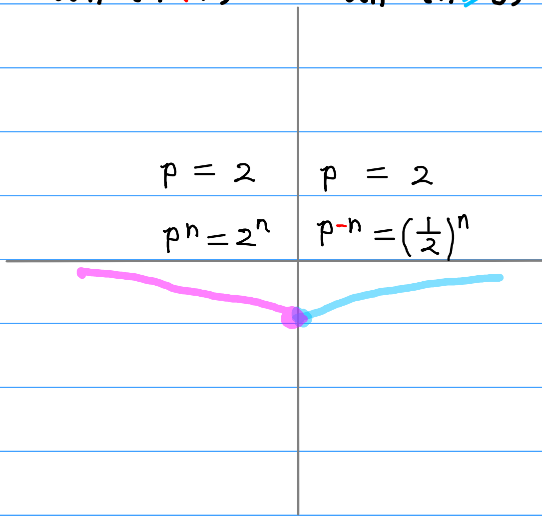
$f(z)$

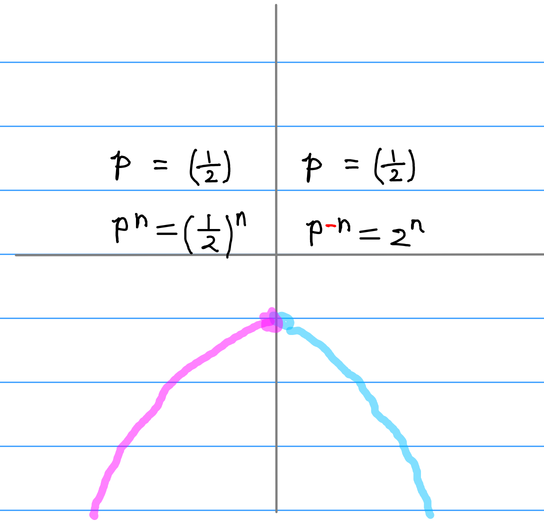
⑥

$\boxed{\begin{array}{l}|z|<p\\|z|>p^{-1}\end{array}}$ $\boxed{\begin{array}{l}|z|>p\\|z|<p^{-1}\end{array}}$

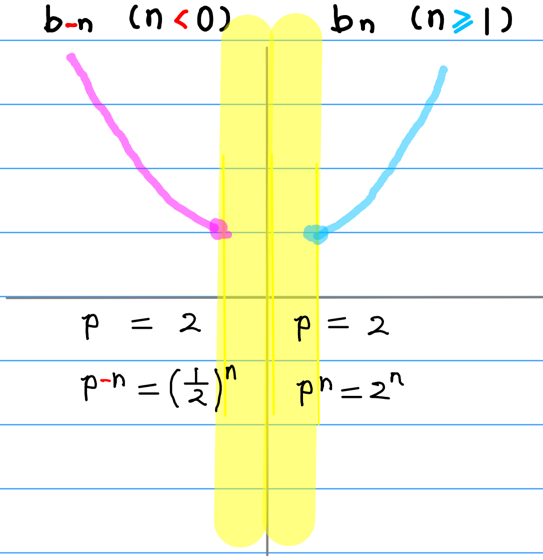
inversed ROC's

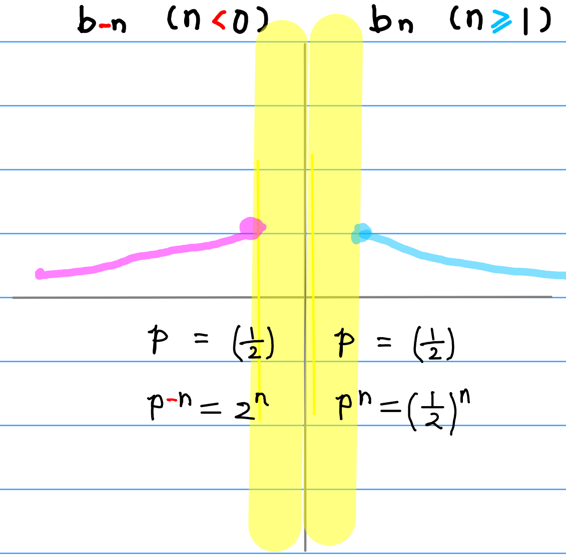
anti-causal	causal	anti-causal	causal
$g(z)$ $(\lvert z\rvert>p^{-1})$	$f(z)$ $(\lvert z\rvert<p)$	$g(z)$ $(\lvert z\rvert>p^{-1})$	$f(z)$ $(\lvert z\rvert<p)$
a_{-n} $(n<1)$	a_n $(n\geq 0)$	a_{-n} $(n<1)$	a_n $(n\geq 0)$





anti-causal	causal	anti-causal	causal
$f(z)$ $(\lvert z\rvert>p)$	$g(z)$ $(\lvert z\rvert<p^{-1})$	$f(z)$ $(\lvert z\rvert>p)$	$g(z)$ $(\lvert z\rvert<p^{-1})$
b_{-n} $(n<0)$	b_n $(n\geq 1)$	b_{-n} $(n<0)$	b_n $(n\geq 1)$





x_n		a_{-n}
y_n		b_{-n}
causal		anti-causal
$n \geq 0$	$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$	$n < 1$
$n \geq 1$	$(p^{-2}, p^{-3}, p^{-4}, \cdots)$	$n < 0$
anti-causal		causal
$n < 1$	$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$	$n \geq 0$
$n < 0$	$(p^{-2}, p^{-3}, p^{-4}, \cdots)$	$n \geq 1$

$X(z)$ ④

anti-causal $Y(z)$ $(|z|<p)$ — causal $X(z)$ $(|z|<p^{-1})$

$X(z^{-1}) \leftrightarrow x_{-n}$ $(n<1)$ — $X(z) \leftrightarrow x_n$ $(n \geq 1)$

$n<1$ $\quad |z|<p$ $\quad n=0,-1,-2,\cdots$ $\quad -p^{n-1}z^{-n}$

x_{-n} $\boxed{\text{anti-causal } Y(z) \;\; -(p^{-1})^{-n+1}}$ $\boxed{-\frac{p^{-1}}{1-p^{-1}z}}$

$$\boxed{-(p^{-1}+p^{-2}z^{1}+p^{-3}z^{2}+\cdots) = -\sum_{n=0}^{-\infty}(p^{-1})^{-n+1}z^{-n} \qquad n<1}$$

$n=0,1,2,\cdots$ $\quad -p^{-n-1}z^{-n}$ $\quad |z|>p^{-1}$ $\quad n \geq 0$

$$\boxed{-(p^{-1}+p^{-2}z^{-1}+p^{-3}z^{-2}+\cdots) = \sum_{n=0}^{\infty} -(p^{-1})^{n+1}z^{-n} \qquad (n \geq 0)}$$

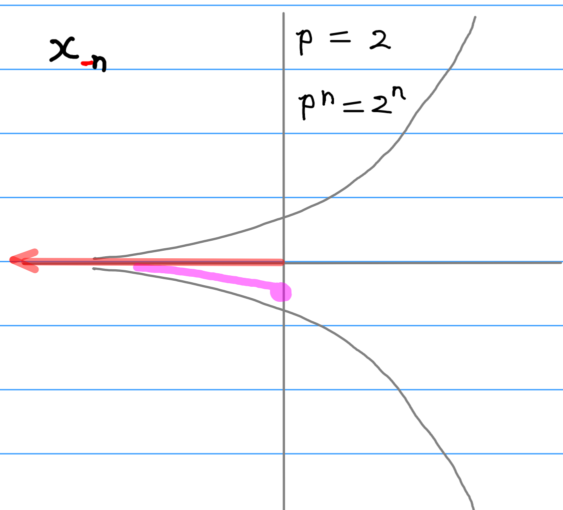
$\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}$ $\boxed{\text{causal } X(z) \;\; -(p^{-1})^{n+1}}$ x_n

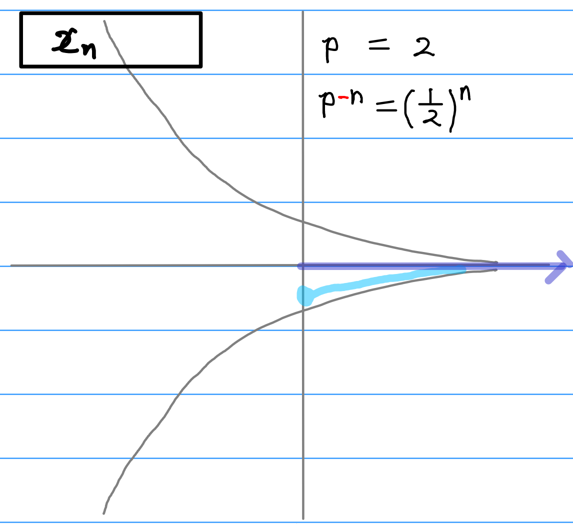
anti-causal $\quad n=0,-1,-2,-3,\ldots.$

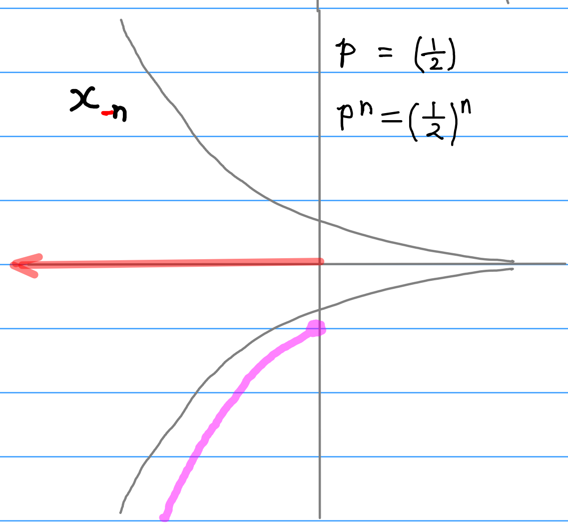
$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

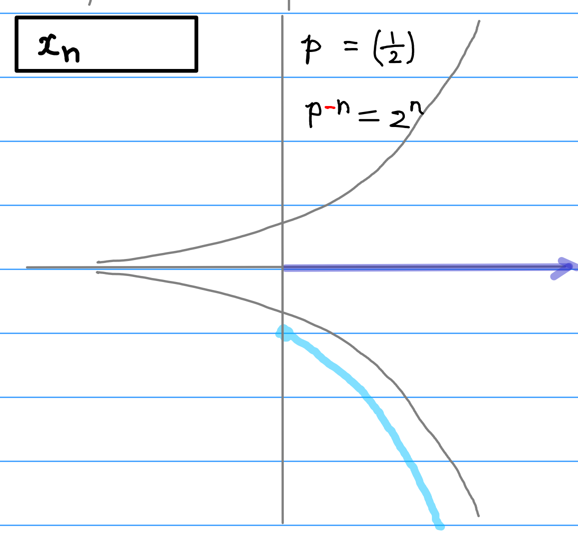
causal $\quad n=0,+1,+2,+3,\ldots.$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$









$X(z)$ 5

causal $Y(z)$ $(|z|>p)$ anti-causal $X(z)$ $(|z|>p^{-1})$

$Y(z) \leftrightarrow y_n$ $(n \geq 0)$ $Y(z^{-1}) \leftrightarrow -y_n$ $(n<1)$

y_n $\boxed{(p)^{n-1}}$ causal Y(z) $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ $\boxed{p^0z^{-1}+p^1z^{-2}+p^2z^{-3}+\cdots = \sum_{n=1}^{\infty}(p)^{n-1}z^{-n} \quad n \geq 1}$

$n \geq 1$ $|z|>p$ $n=1,2,3,\cdots$ $p^{n-1}z^{-n}$

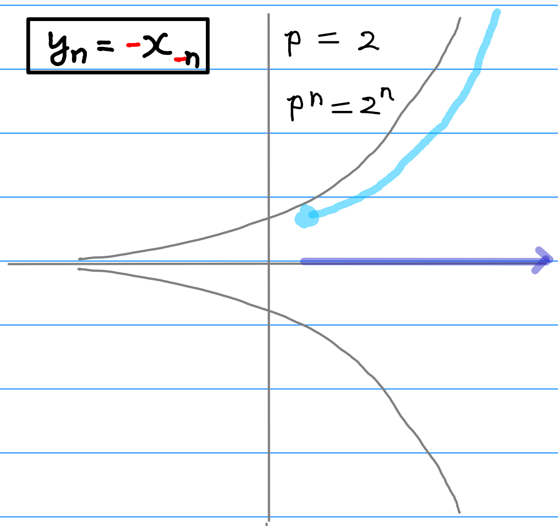
$\boxed{p^0z^1+p^1z^2+p^2z^3+\cdots = \sum_{n=-1}^{-\infty}(p^{-1})^{n+1}z^{-n} \quad (n<0)}$ $\boxed{\dfrac{z}{1-pz}}$ $\boxed{(p)^{-n-1}}$ anti-causal X(z) y_{-n}

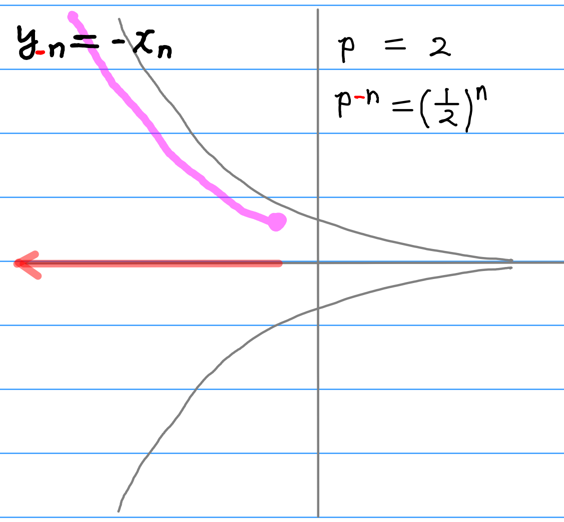
$n=-1,-2,-3,\cdots$ $p^{-n-1}z^{-n}$ $|z|<p^{-1}$ $n<0$

causal n=+1,+2,+3,.... $(p^0, p^1, p^2, \cdots)$

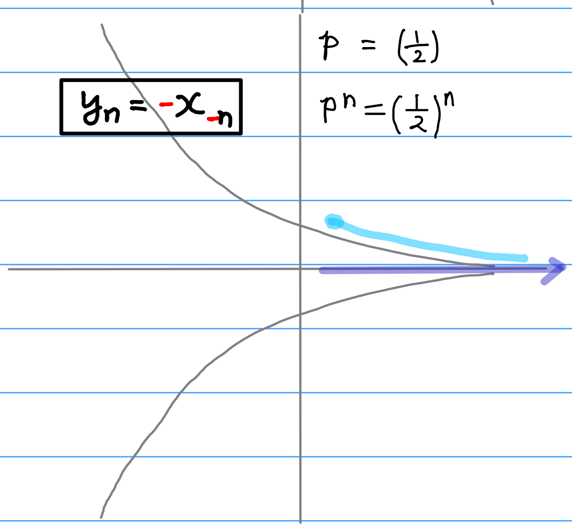
anti-causal n=-1,-2,-3,.... $(p^0, p^1, p^2, \cdots)$

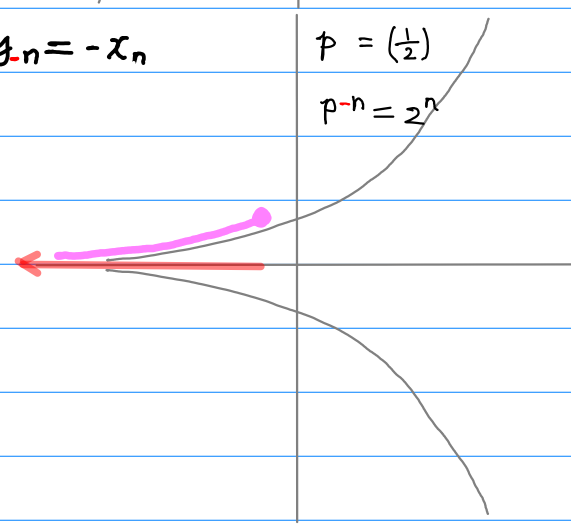
$\boxed{y_n = -x_{-n}}$





$\boxed{y_n = -x_{-n}}$





$X(z)$ ⑥

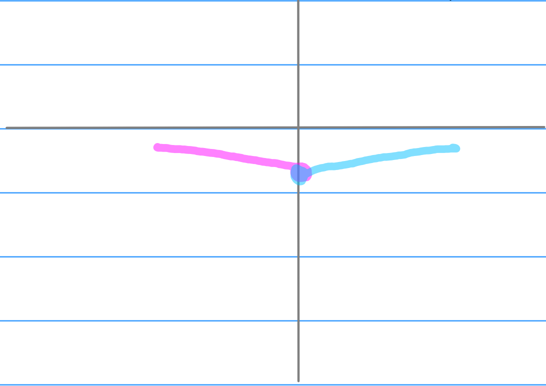
$\boxed{\begin{array}{l}|z| > p \\ |z| < p^{-1}\end{array}}$ $\boxed{\begin{array}{l}|z| < p \\ |z| > p^{-1}\end{array}}$ inversed ROC's

$Y(z)$ $(|z| < p^{-1})$ $X(z)$ $(|z| > p)$

$-y_n$ $(n < 1)$ x_n $(n \geq 0)$

$p = 2$ | $p = 2$

$p^n = 2^n$ | $p^{-n} = \left(\frac{1}{2}\right)^n$

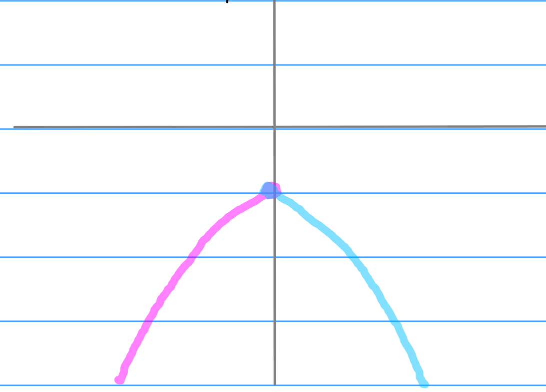


$Y(z)$ $(|z| < p^{-1})$ $X(z)$ $(|z| > p)$

$-y_n$ $(n < 1)$ x_n $(n \geq 0)$

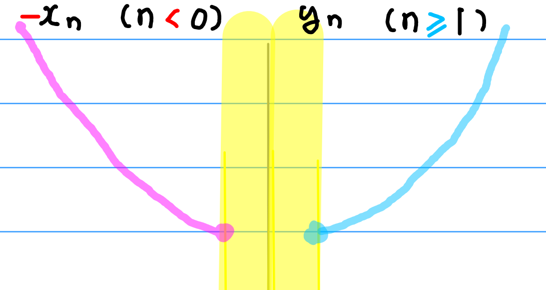
$p = \left(\frac{1}{2}\right)$ | $p = \left(\frac{1}{2}\right)$

$p^n = \left(\frac{1}{2}\right)^n$ | $p^{-n} = 2^n$



$X(z)$ $(|z| < p)$ $Y(z)$ $(|z| > p^{-1})$

$-x_n$ $(n < 0)$ y_n $(n \geq 1)$

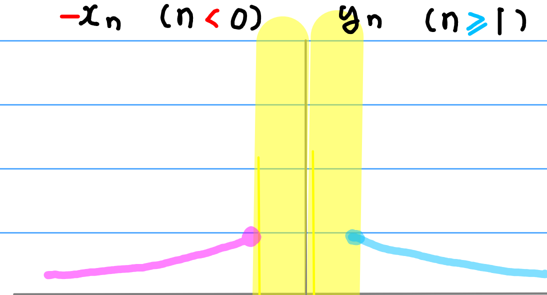


$p = 2$ | $p = 2$

$p^{-n} = \left(\frac{1}{2}\right)^n$ | $p^n = 2^n$

$X(z)$ $(|z| < p)$ $Y(z)$ $(|z| > p^{-1})$

$-x_n$ $(n < 0)$ y_n $(n \geq 1)$



$p = \left(\frac{1}{2}\right)$ | $p = \left(\frac{1}{2}\right)$

$p^{-n} = 2^n$ | $p^n = \left(\frac{1}{2}\right)^n$

getting causal sequence

$$\begin{tikzcd}
\boxed{\dfrac{1}{z - p}} \arrow[r, leftrightarrow, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, leftrightarrow, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1 - p^{-1} z}} = f(z) \leftrightarrow \boxed{?} \\
\boxed{\dfrac{z^{-1}}{1 - p z^{-1}}} = Y(z) \leftrightarrow \boxed{?} &
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{\dfrac{1}{z^{-1} - p}} \arrow[r, leftrightarrow, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, leftrightarrow, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1 - p^{-1} z^{-1}}} = X(z) \leftrightarrow \boxed{?} \\
\boxed{\dfrac{z}{1 - p z}} = g(z) \leftrightarrow \boxed{?} &
\end{tikzcd}$$

Getting causal sequence w/o memorizing

$$\boxed{-\frac{p^{-1}}{1-p^{-1}z}} = f(z) \longleftrightarrow \boxed{-(p^{-1})^{n+1}}$$

$$\boxed{\frac{z}{1-pz}} = g(z) \longleftrightarrow \boxed{(p)^{n-1}}$$

Left shift

$$\boxed{\frac{z^{-1}}{1-pz^{-1}}} = Y(z) \longleftrightarrow \boxed{(p)^{n-1}}$$

Left shift

$$\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} = X(z) \longleftrightarrow \boxed{-(p^{-1})^{n+1}}$$

Getting anti-causal sequence

$$
\begin{tikzcd}
\boxed{\dfrac{1}{z-p}} \arrow[r, leftrightarrow, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, leftrightarrow, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \\
\boxed{\dfrac{z^{-1}}{1-pz^{-1}}} &
\end{tikzcd}
$$

$$-\frac{p^{-1}}{1-p^{-1}z} = f(z) \longleftrightarrow -(p^{-1})^{n+1}$$

$$X(z^{-1}) \longleftrightarrow \boxed{?}$$

$$\frac{z^{-1}}{1-pz^{-1}} = g(z^{-1}) \longleftrightarrow \boxed{?}$$

$$Y(z) \longleftrightarrow (p)^{n-1}$$

$$
\begin{tikzcd}
\boxed{\dfrac{1}{z^{-1}-p}} \arrow[r, leftrightarrow, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, leftrightarrow, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \\
\boxed{\dfrac{z}{1-pz}} &
\end{tikzcd}
$$

$$-\frac{p^{-1}}{1-p^{-1}z^{-1}} = f(z^{-1}) \longleftrightarrow \boxed{?}$$

$$X(z) \longleftrightarrow -(p^{-1})^{n+1}$$

$$\frac{z}{1-pz} = g(z) \longleftrightarrow (p)^{n-1}$$

$$Y(z^{-1}) \longleftrightarrow \boxed{?}$$

① $z \leftarrow z^{-1}$ ② $a_n \leftarrow a_{-n}$

$$\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\frac{1}{z-p}} \arrow[r, leftrightarrow, "\frac{p^{-1}}{p^{-1}}"] \arrow[d, leftrightarrow, "\frac{z^{-1}}{z^{-1}}"'] & \boxed{-\frac{p^{-1}}{1-p^{-1}z}} = f(z) \arrow[d, dash, "z^{-1}"] \\
\boxed{\frac{z^{-1}}{1-pz^{-1}}} = g(z^{-1}) \arrow[d, dash, "z^{-1}"'] & \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} = f(z^{-1}) \\
\boxed{\frac{z}{1-pz}} = g(z) & \boxed{\frac{1}{z^{-1}-p}} \arrow[u, leftrightarrow, "\frac{p^{-1}}{p^{-1}}"'] \arrow[l, leftrightarrow, "\frac{z^{-1}}{z^{-1}}"]
\end{tikzcd}$$

$f(z) \leftrightarrow -(p^{-1})^{n+1}$

$X(z^{-1}) \leftrightarrow \boxed{-(p^{-1})^{-n+1}}$

$g(z^{-1}) \leftrightarrow \boxed{(p)^{-n-1}}$

$Y(z) \leftrightarrow (p)^{n-1}$

$f(z^{-1}) \leftrightarrow \boxed{-(p^{-1})^{-n+1}}$

$X(z) \leftrightarrow -(p^{-1})^{n+1}$

$g(z) \leftrightarrow (p)^{n-1}$

$Y(z^{-1}) \leftrightarrow \boxed{(p)^{-n-1}}$

Getting anti-causal sequence w/o memorizing

$$\begin{tikzcd}
f(z^{-1}) = \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \arrow[r, "z^{-1}"] \arrow[d, leftrightarrow] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} = f(z) \arrow[d, leftrightarrow] \\
a_{-n} = \boxed{-(p^{-1})^{-n+1}} & \boxed{-(p^{-1})^{n+1}} = a_n \arrow[l, "-n"']
\end{tikzcd}$$

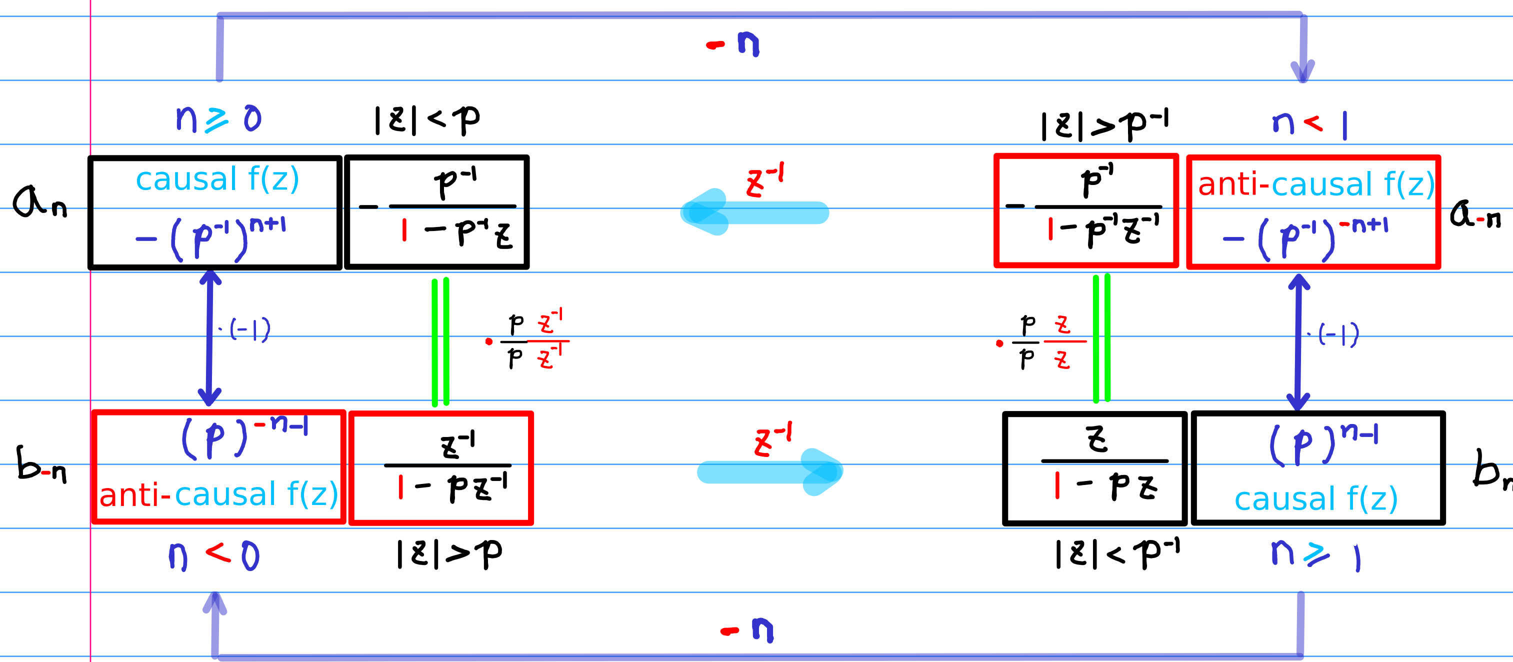
$$\begin{tikzcd}
g(z^{-1}) = \boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \arrow[r, "z^{-1}"] \arrow[d, leftrightarrow] & \boxed{\dfrac{z}{1-pz}} = g(z) \arrow[d, leftrightarrow] \\
b_{-n} = \boxed{(p)^{-n-1}} & \boxed{(p)^{n-1}} = b_n \arrow[l, "-n"']
\end{tikzcd}$$

$$\begin{tikzcd}
Y(z^{-1}) = \boxed{\dfrac{z}{1-pz}} \arrow[r, "z^{-1}"] \arrow[d, leftrightarrow] & \boxed{\dfrac{z^{-1}}{1-pz^{-1}}} = Y(z) \arrow[d, leftrightarrow] \\
y_{-n} = \boxed{(p)^{-n-1}} & \boxed{(p)^{n-1}} = y_n \arrow[l, "-n"']
\end{tikzcd}$$

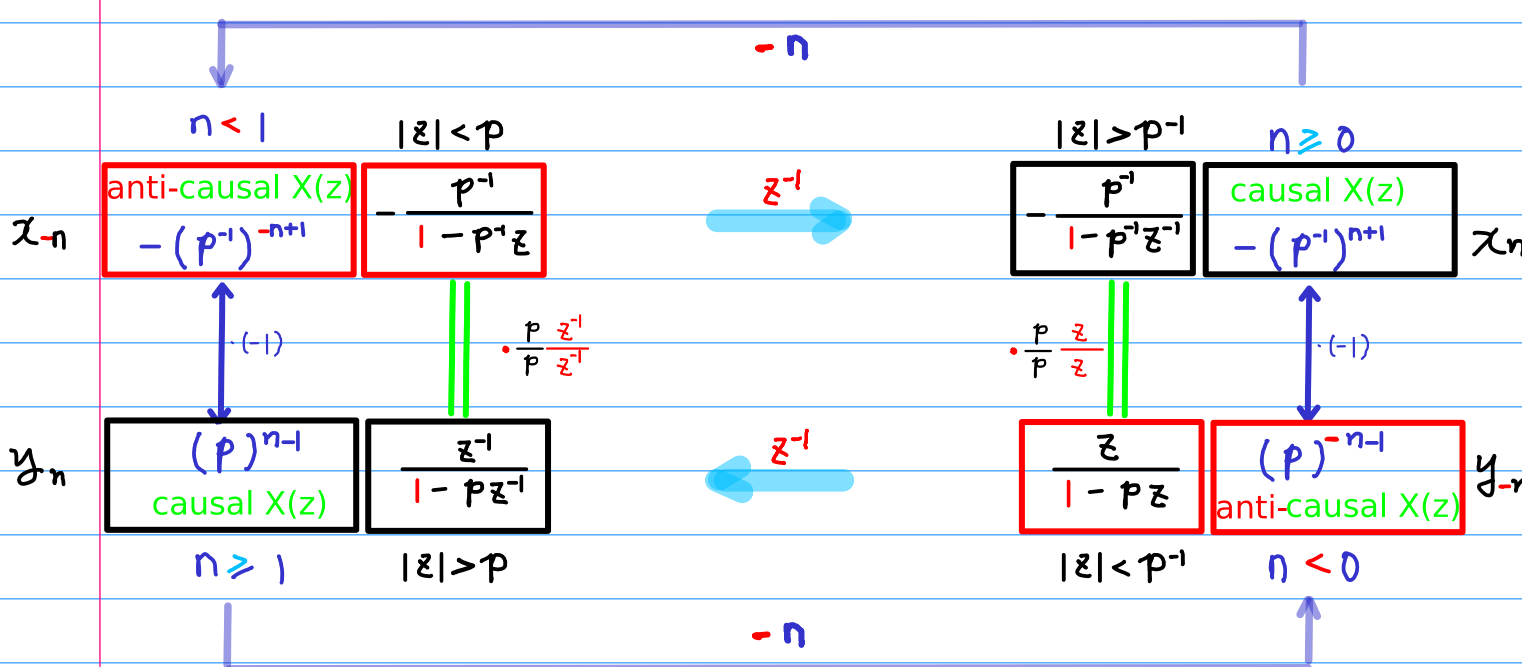
$$\begin{tikzcd}
X(z^{-1}) = \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \arrow[r, "z^{-1}"] \arrow[d, leftrightarrow] & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} = X(z) \arrow[d, leftrightarrow] \\
x_{-n} = \boxed{-(p^{-1})^{-n+1}} & \boxed{-(p^{-1})^{n+1}} = x_n \arrow[l, "-n"']
\end{tikzcd}$$

getting anti-causal sequence

Laurent Series



z-Transform



$f(z^{-1}) \quad g(z^{-1})$

1. $\boxed{z^{-1} \rightarrow z}$ $\qquad f(z),\ g(z)$
2. $f(z) \leftrightarrow a_n \qquad g(z) \leftrightarrow b_n$
3. $\boxed{n \rightarrow -n}$ $\qquad a_{-n},\ b_{-n}$

$X(z^{-1}) \quad Y(z^{-1})$

1. $\boxed{z^{-1} \rightarrow z}$ $\qquad X(z),\ Y(z)$
2. $X(z) \leftrightarrow x_n \qquad Y(z) \leftrightarrow y_n$
3. $\boxed{n \rightarrow -n}$ $\qquad x_{-n},\ x_{-n}$

$$f(z^{-1}) = \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} \qquad g(z^{-1}) = \boxed{\frac{z^{-1}}{1-pz^{-1}}}$$

anti-causal

$$f(z) = \boxed{-\frac{p^{-1}}{1-p^{-1}z}} \qquad g(z) = \boxed{\frac{z}{1-pz}}$$

$$Y(z^{-1}) = \boxed{\frac{z}{1-pz}} \qquad X(z^{-1}) = \boxed{-\frac{p^{-1}}{1-p^{-1}z}}$$

anti-causal

$$Y(z) = \boxed{\frac{z^{-1}}{1-pz^{-1}}} \qquad X(z) = \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}$$

$f(z^{-1})$ $g(z^{-1})$

① $\boxed{z^{-1} \to z}$ $f(z)$, $g(z)$

② $f(z) \leftrightarrow a_n$ $g(z) \leftrightarrow b_n$

③ $\boxed{n \to -n}$ a_{-n}, b_{-n}

$$f(z^{-1}) = \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} \qquad g(z^{-1}) = \boxed{\frac{z^{-1}}{1-pz^{-1}}}$$

① $$f(z) = \boxed{-\frac{p^{-1}}{1-p^{-1}z}} \qquad g(z) = \boxed{\frac{z}{1-pz}}$$

② $$a_n = \boxed{-(p^{-1})^{n+1}} \qquad b_n = \boxed{(p)^{n-1}}$$

③ $$a_{-n} = \boxed{-(p^{-1})^{-n+1}} \qquad b_{-n} = \boxed{(p)^{-n-1}}$$

$X(z^{-1})$ $Y(z^{-1})$

① $\boxed{z^{-1} \to z}$ $X(z)$, $Y(z)$

② $X(z) \leftrightarrow x_n$ $Y(z) \leftrightarrow y_n$

③ $\boxed{n \to -n}$ x_{-n}, x_{-n}

$$Y(z^{-1}) = \boxed{\frac{z}{1 - pz}} \qquad X(z^{-1}) = \boxed{-\frac{p^{-1}}{1 - p^{-1}z}}$$

① $$Y(z) = \boxed{\frac{z^{-1}}{1 - pz^{-1}}} \qquad X(z) = \boxed{-\frac{p^{-1}}{1 - p^{-1}z^{-1}}}$$

② $$y_n = \boxed{(p)^{n-1}} \qquad x_n = \boxed{-(p^{-1})^{n+1}}$$

③ $$y_{-n} = \boxed{-(p^{-1})^{-n+1}} \qquad x_{-n} = \boxed{(p)^{-n-1}}$$

① $\frac{-1}{(z-1)(z-2)}$

② $\frac{-0.5z^2}{(z-1)(z-0.5)}$

$-\frac{1}{1-z}+\frac{0.5}{1-0.5z}$

$f(z)$ $|z|<1$ causal

$X(z)$ $|z|<1$ anti-causal

$+\frac{z^{-1}}{1-z^{-1}}-\frac{z^{-1}}{1-2z^{-1}}$

$f(z)$ $|z|>1$ anti-causal

$X(z)$ $|z|>1$ causal

$+\frac{z}{1-z}-\frac{z}{1-2z}$

$f(z)$ $|z|<0.5$ causal

$X(z)$ $|z|<0.5$ anti-causal

$-\frac{1}{1-z^{-1}}+\frac{1}{1-0.5z^{-1}}$

$f(z)$ $|z|>2$ anti-causal

$X(z)$ $|z|>2$ causal

$$-\frac{1}{1-z}+\frac{0.5}{1-0.5z}$$

$f(z) \quad |z|<1$

$$+\frac{z}{1-z}-\frac{z}{1-2z}$$

$f(z) \quad |z|<0.5$

$\cdot z \quad n-1$

$$+\frac{1}{1-z}-\frac{1}{1-2z}$$

$g(z) \quad |z|<0.5$

$$+\frac{z^{-1}}{1-z^{-1}}-\frac{z^{-1}}{1-2z^{-1}}$$

$X(z) \quad |z|>1$

$\cdot z^{-1} \quad n-1$

$$+\frac{1}{1-z^{-1}}-\frac{1}{1-2z^{-1}}$$

$V(z) \quad |z|>1$

$$-\frac{1}{1-z^{-1}}+\frac{1}{1-0.5z^{-1}}$$

$X(z) \quad |z|>2$

$$-\frac{1}{1-z} + \frac{0.5}{1-0.5z}$$

$X(z) \quad |z| < 1$

$z^{-1} \quad -n$

$$-\frac{1}{1-z^{-1}} + \frac{1}{1-0.5z^{-1}}$$

$V(z) \quad |z| > 2$

$$+\frac{z}{1-z} - \frac{z}{1-2z}$$

$X(z) \quad |z| < 0.5$

$z^{-1} \quad -n$

$$+\frac{z^{-1}}{1-z^{-1}} - \frac{z^{-1}}{1-2z^{-1}}$$

$V(z) \quad |z| > 1$

$\cdot z^{-1} \quad n-1$

$$+\frac{1}{1-z^{-1}} - \frac{1}{1-2z^{-1}}$$

$W(z) \quad |z| > 1$

$$+\frac{z^{-1}}{1-z^{-1}} - \frac{z^{-1}}{1-2z^{-1}}$$

$f(z) \quad |z| > 1$

$z^{-1} \quad -n$

$$+\frac{z}{1-z} - \frac{z}{1-2z}$$

$g(z) \quad |z| < 0.5$

$\cdot z \quad n-1$

$$+\frac{1}{1-z} - \frac{1}{1-2z}$$

$h(z) \quad |z| < 0.5$

$$-\frac{1}{1-z^{-1}} + \frac{1}{1-0.5z^{-1}}$$

$f(z) \quad |z| > 2$

$z^{-1} \quad -n$

$$-\frac{1}{1-z} + \frac{0.5}{1-0.5z}$$

$g(z) \quad |z| < 1$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$n = 0, 1, 2, \cdots$

causal $n \geq 0$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$n = 0, -1, -2, \cdots$

anti-causal $n < 1$

$(p^{0}, p^{1}, p^{2}, \cdots)$

$n = 1, 2, 3, \cdots$

anti-causal $n < 0$

$(p^{0}, p^{1}, p^{2}, \cdots)$

$n = -1, -2, -3, \cdots$

causal $n \geq 1$

x_n $\quad a_{-n}$

y_n $\quad b_{-n}$

causal		anti-causal
$n \geq 0$	$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$	$n < 1$
$n \geq 1$	$(p^{-2}, p^{-3}, p^{-4}, \cdots)$	$n < 0$
anti-causal		**causal**
$n < 1$	$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$	$n \geq 0$
$n < 0$	$(p^{-2}, p^{-3}, p^{-4}, \cdots)$	$n \geq 1$