ODE Background: Complex Variables (4A)

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Complex Numbers

Complex Numbers

$$i = \sqrt{-1}$$

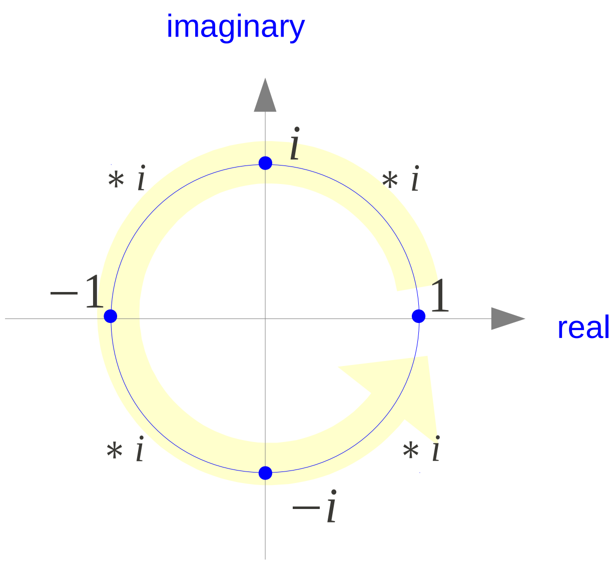
$$i^2 = -1$$

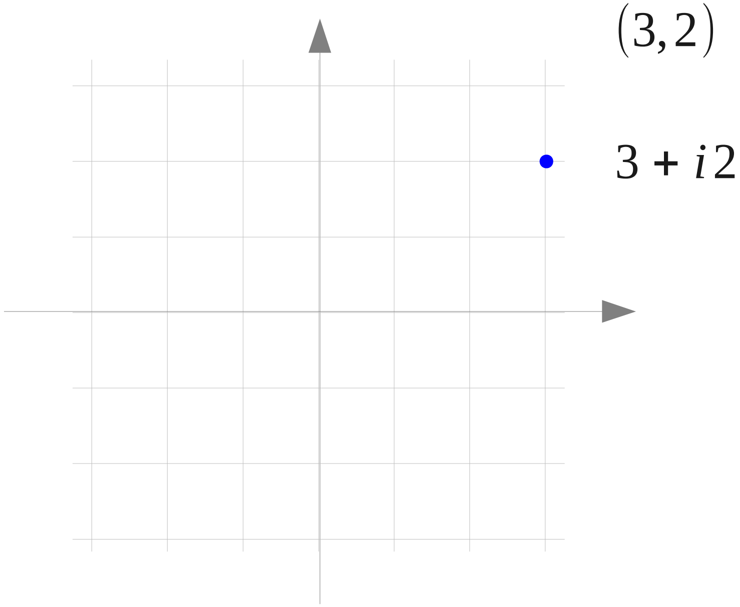
$$i^3 = -i \qquad i^2 \cdot i = -i$$

$$i^4 = +1 \qquad i^2 \cdot i^2 = +1$$

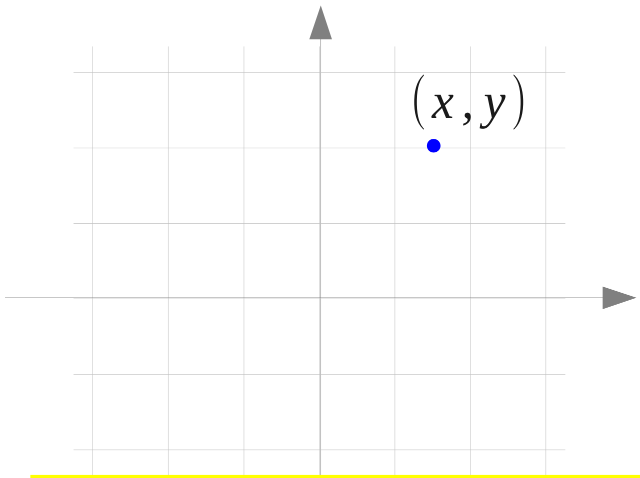
$(3, 2)$ **two** real numbers
2-d coordinate

$3 + i2$ **one** complex number
with real part of 3
and imaginary part of 2





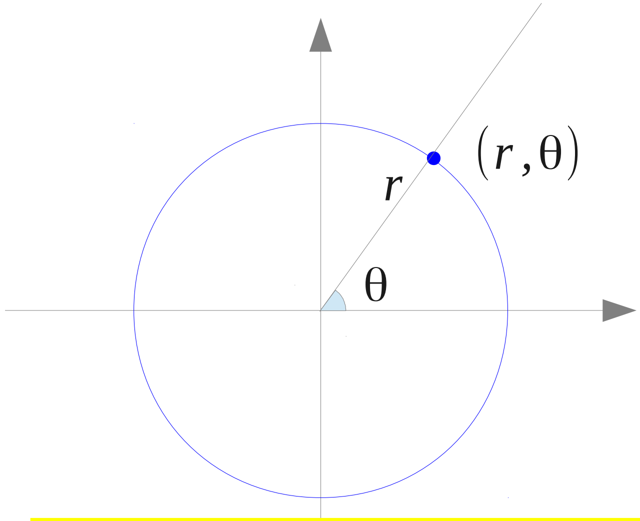
Coordinate Systems



(a) Cartesian Coordinate System

$$x = r\cos\theta$$

$$y = r\sin\theta$$

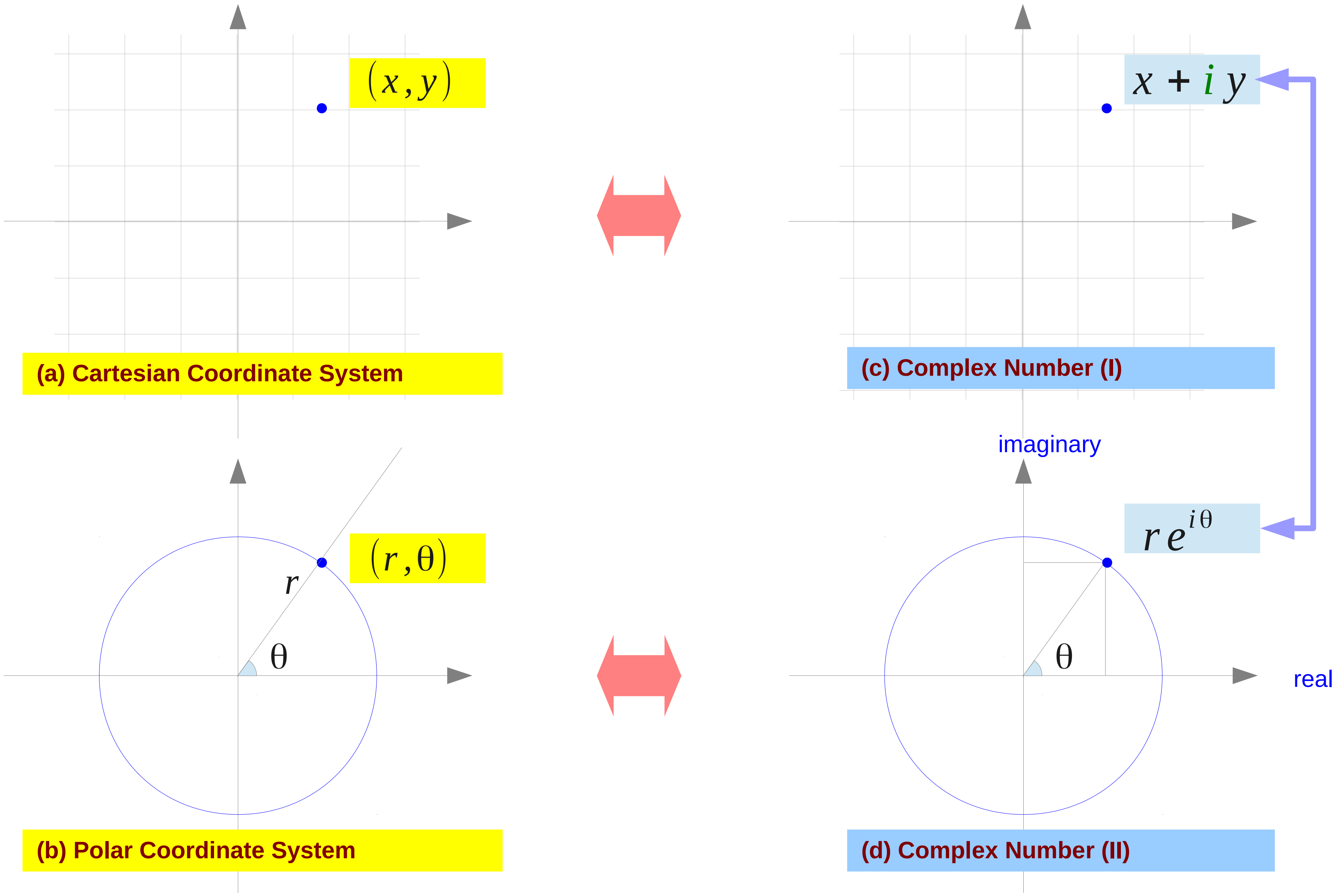


(b) Polar Coordinate System

$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

Coordinate Systems and Complex Numbers



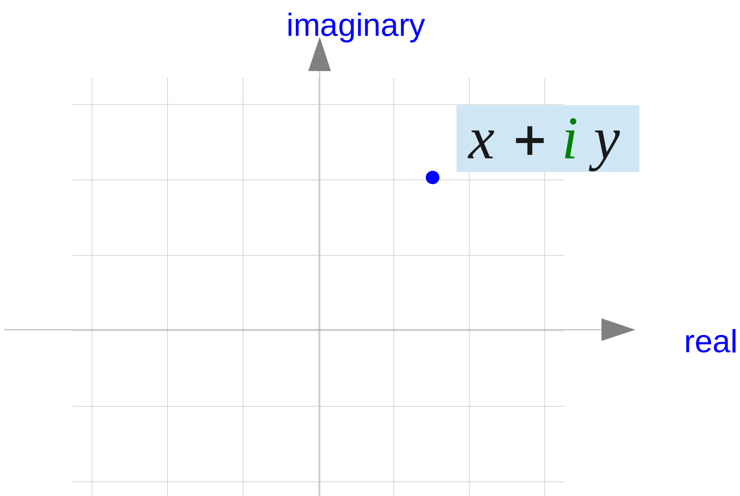
Complex Numbers

$$x = r\cos\theta$$
$$y = r\sin\theta$$

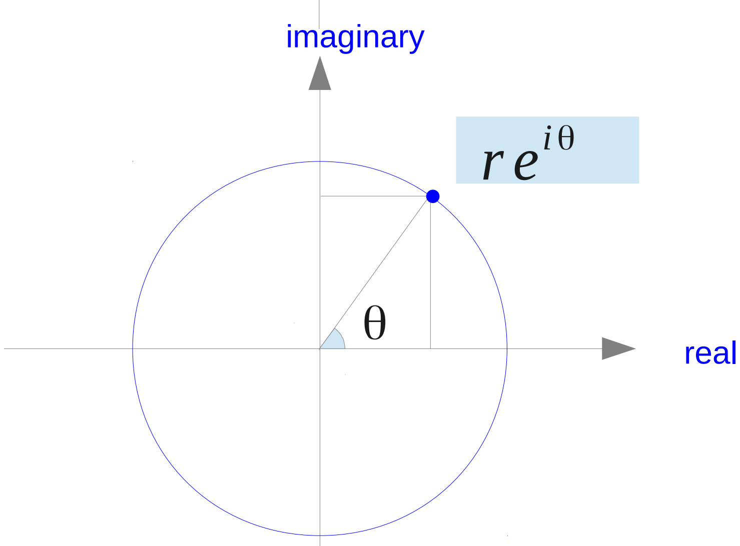
$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$

$$x + iy = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$



(c) Complex Number (I)



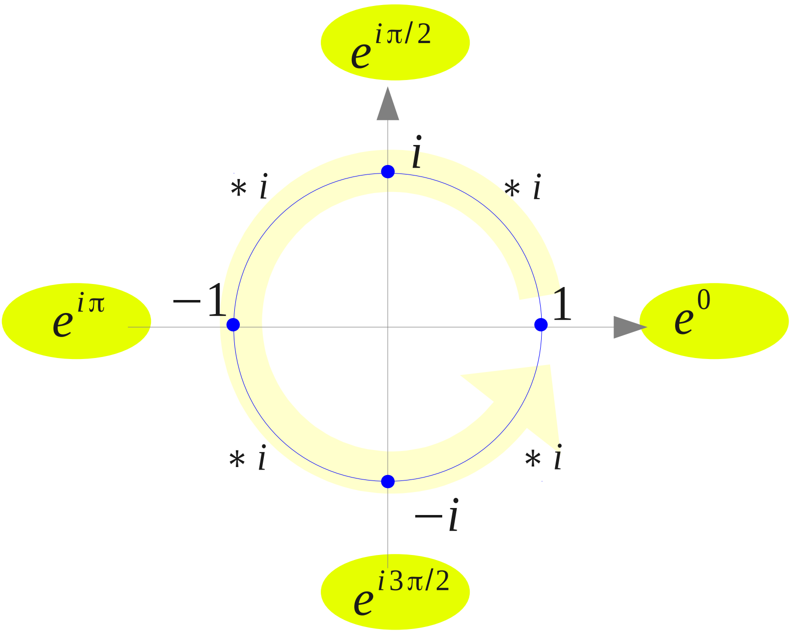
(d) Complex Number (II)

Euler's Formula (1)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$x \Rightarrow \Re\{e^{i\theta}\} = \cos\theta$

$y \Rightarrow \Im\{e^{i\theta}\} = \sin\theta$



$$e^{0} = +1 + 0i = +1 = e^{-i2\pi}$$

$$e^{i\pi/2} = 0 + 1i = +i = e^{-i3\pi/2}$$

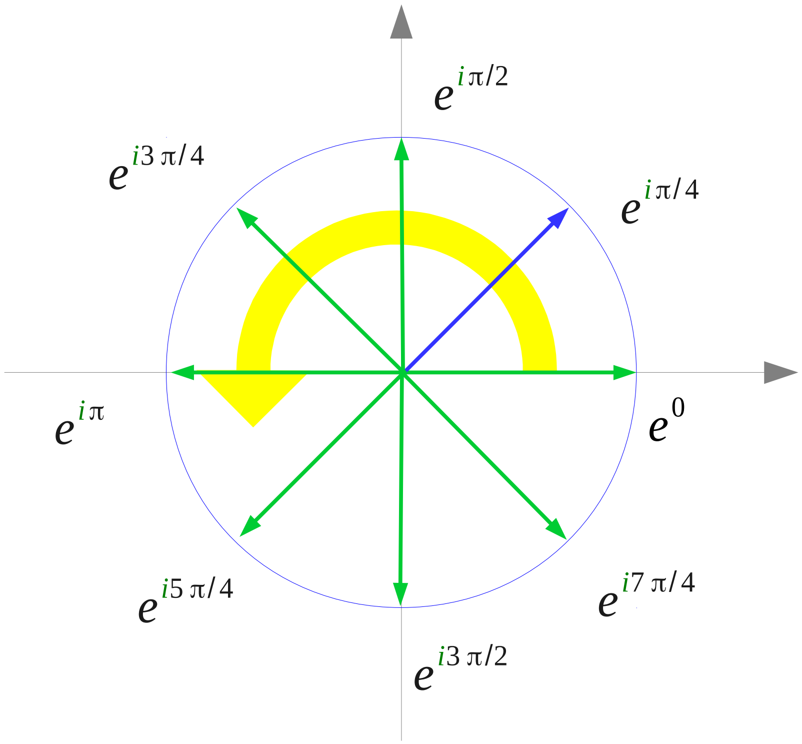
$$e^{i\pi} = -1 + 0i = -1 = e^{-i\pi}$$

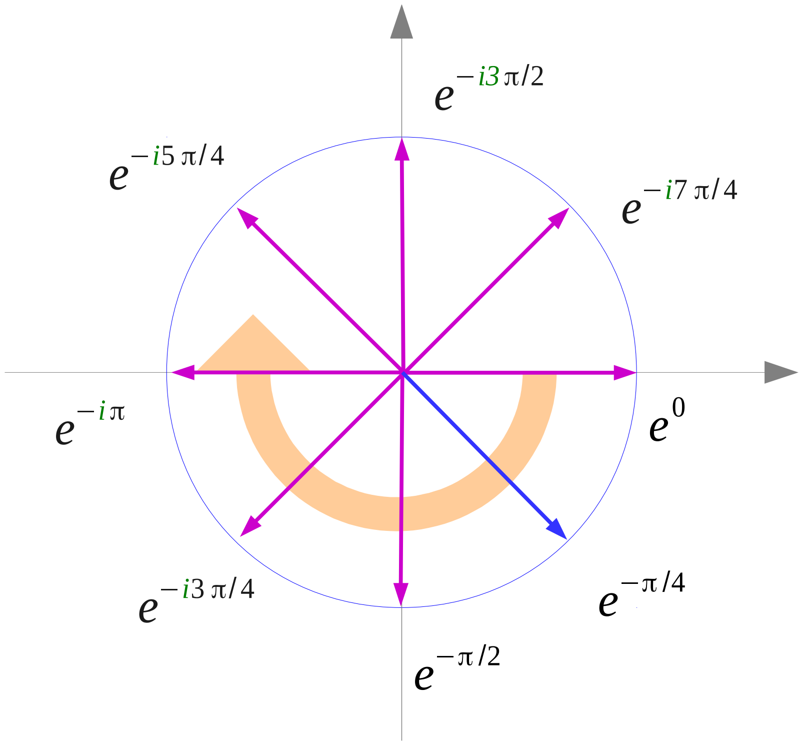
$$e^{i3\pi/2} = 0 - 1i = -i = e^{-i\pi/2}$$

Euler's Formula (2)

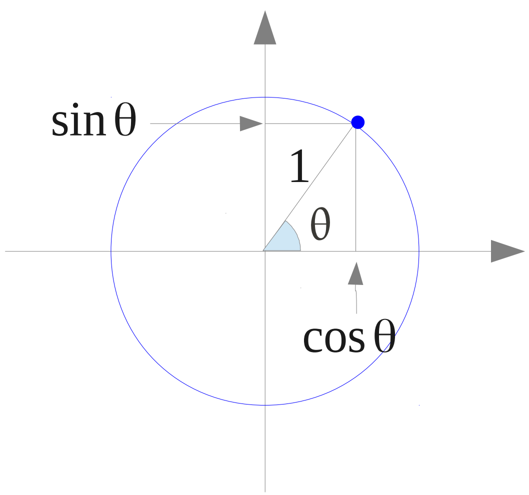
$$e^{+i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

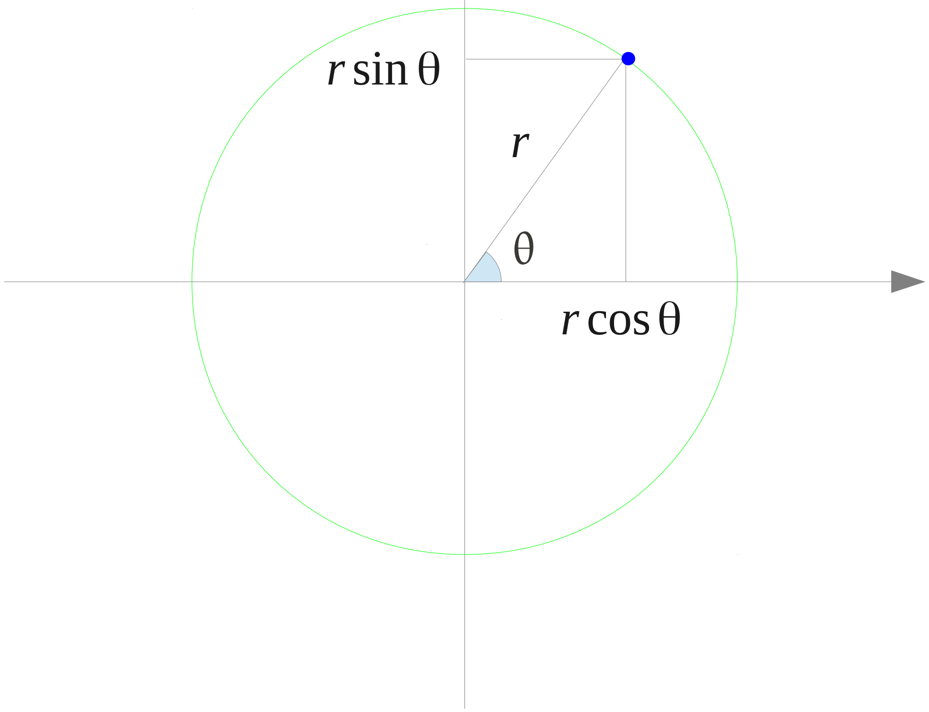




Absolute Values and Arguments

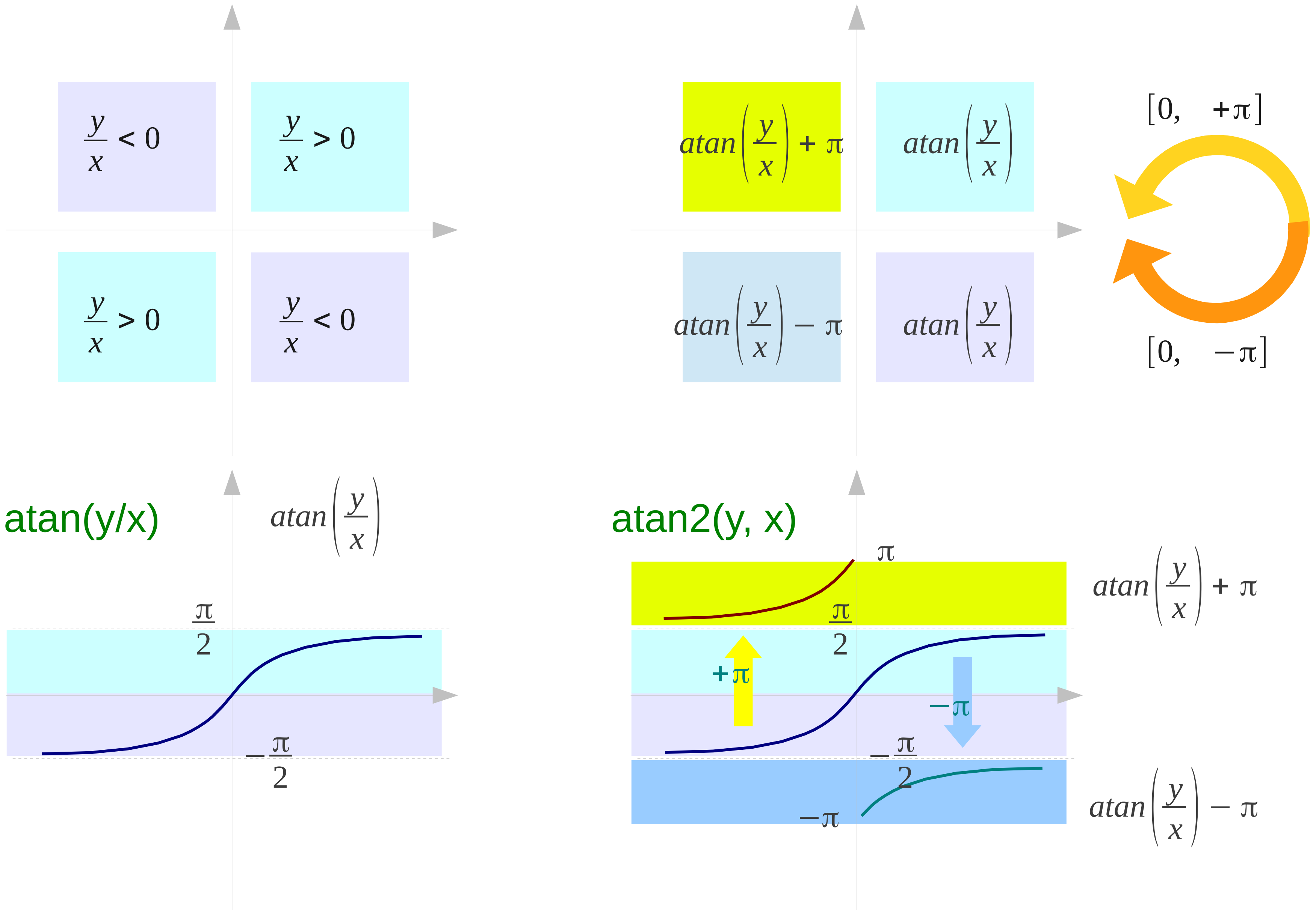


$$z = r e^{i\theta} = r(\cos\theta + i\sin\theta)$$



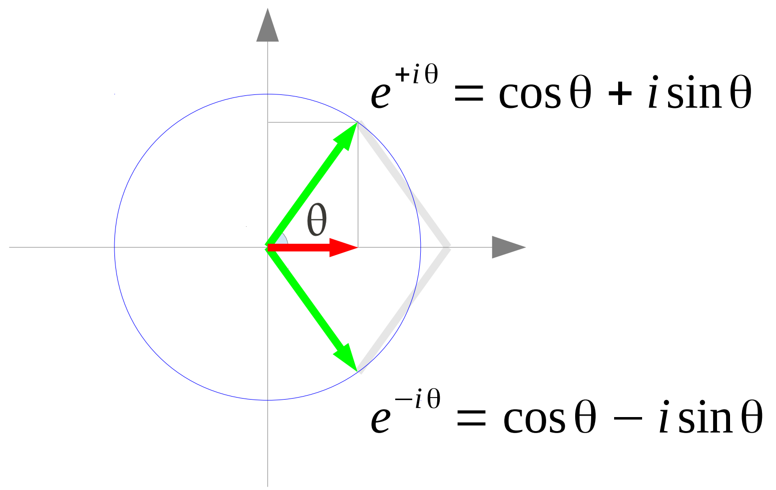
absolute value $\lvert z\rvert = r$	➡ $\lvert r e^{i\theta}\rvert = \lvert r\rvert\,\lvert e^{i\theta}\rvert = r\sqrt{\cos^2\theta + \sin^2\theta}$
argument, phase $arg(z) = \theta$	➡ $arg(r e^{i\theta})$

Computing Complex Number Argument



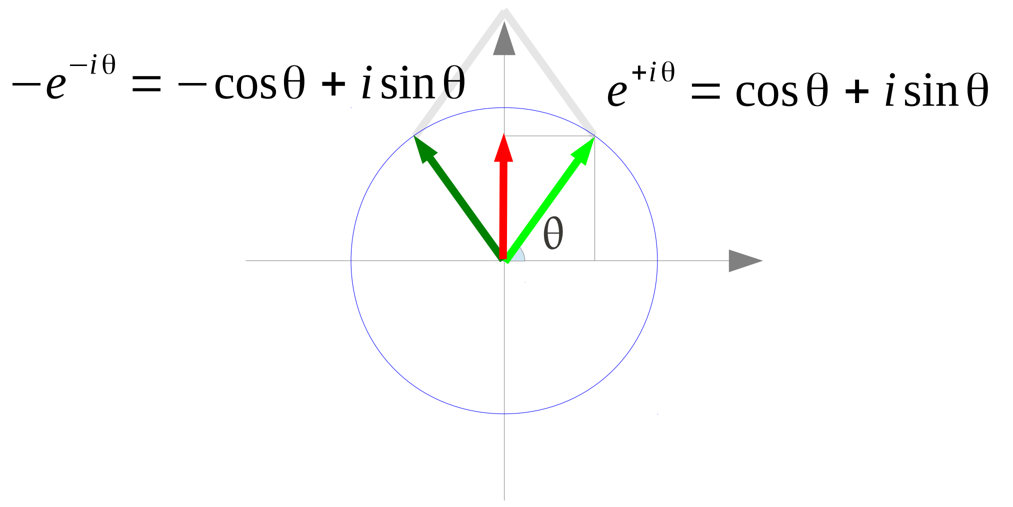
$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$



$$\Re\{e^{i\theta}\} = \boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

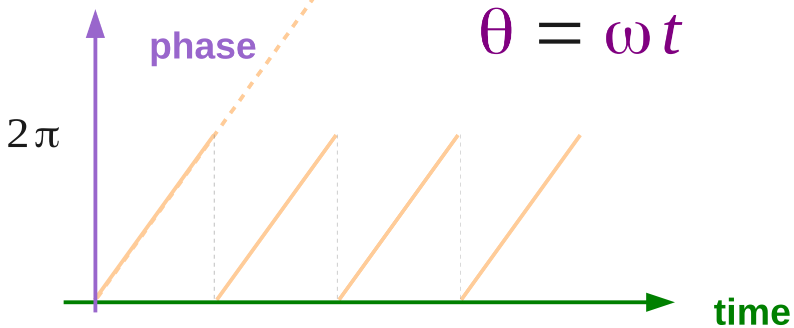
$$\Im\{e^{i\theta}\} = \boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

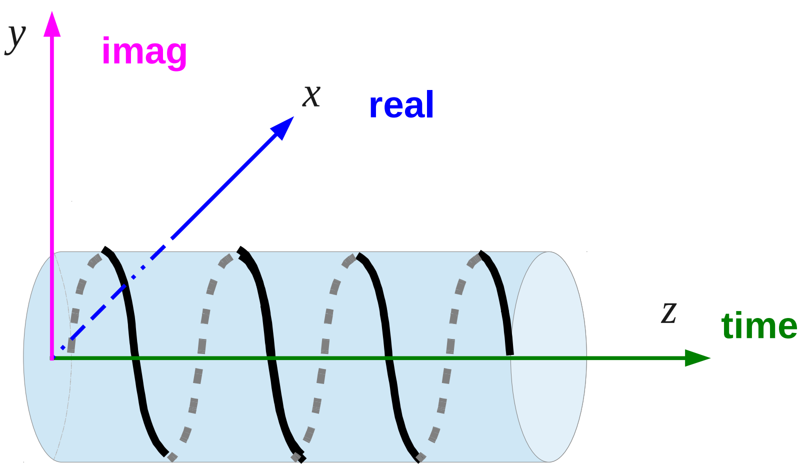


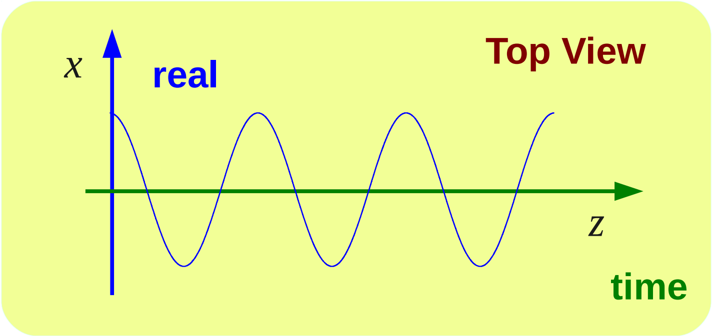
Complex Exponential

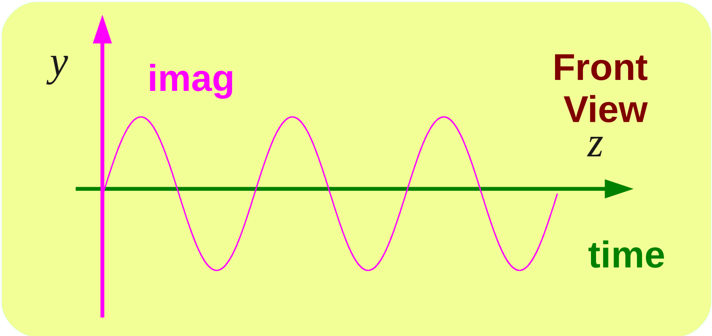
$$e^{+j\omega t} = \cos\omega t + j\sin\omega t$$

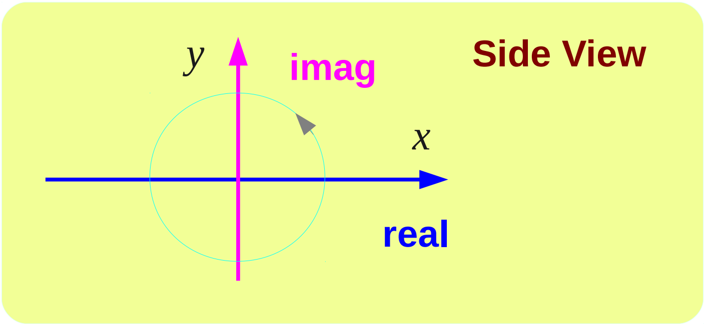
$$\theta = \omega t$$



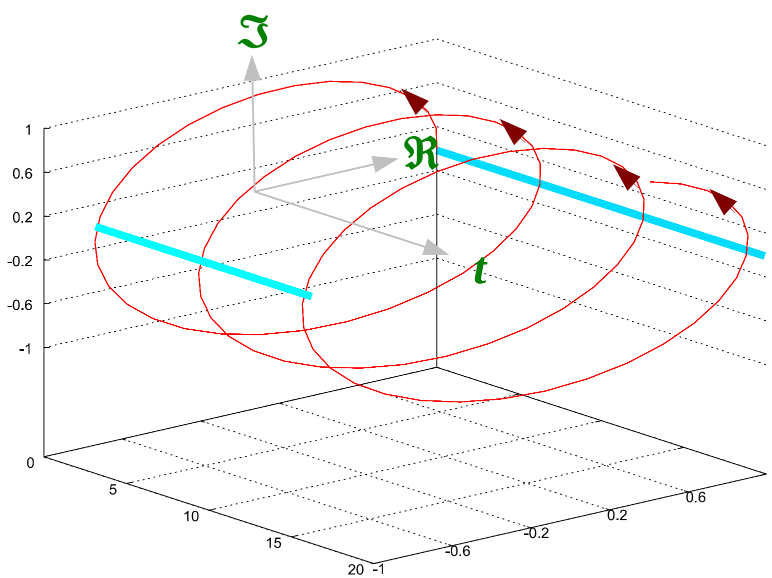


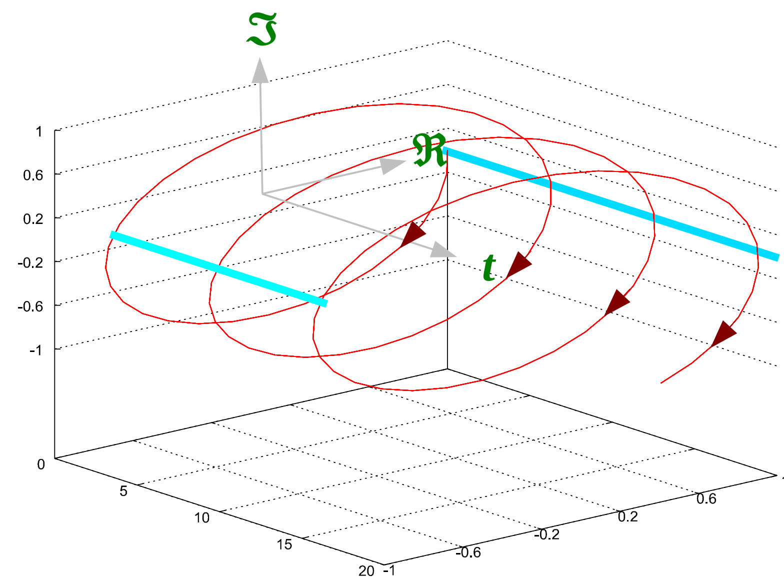






Conjugate Complex Exponential





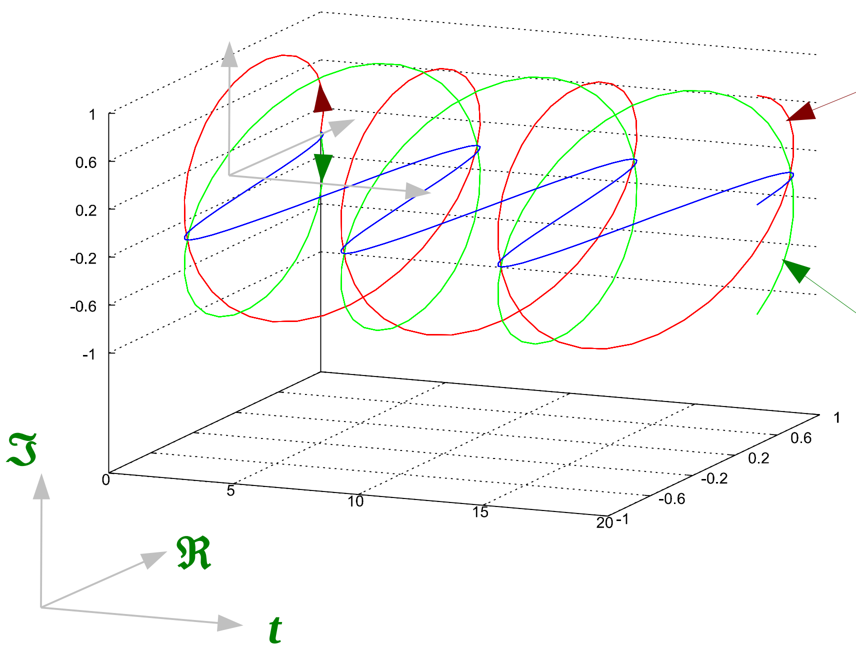
$$e^{+j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

$$e^{+jt} = \cos(t) + j\sin(t) \qquad (\omega_0 = 1)$$

$$e^{-j\omega t} = \cos(\omega_0 t) - j\sin(\omega_0 t)$$

$$e^{-jt} = \cos(t) - j\sin(t) \qquad (\omega_0 = 1)$$

Cos(ω_0t)



$$e^{+jt} = \cos(t) + j\sin(t)$$

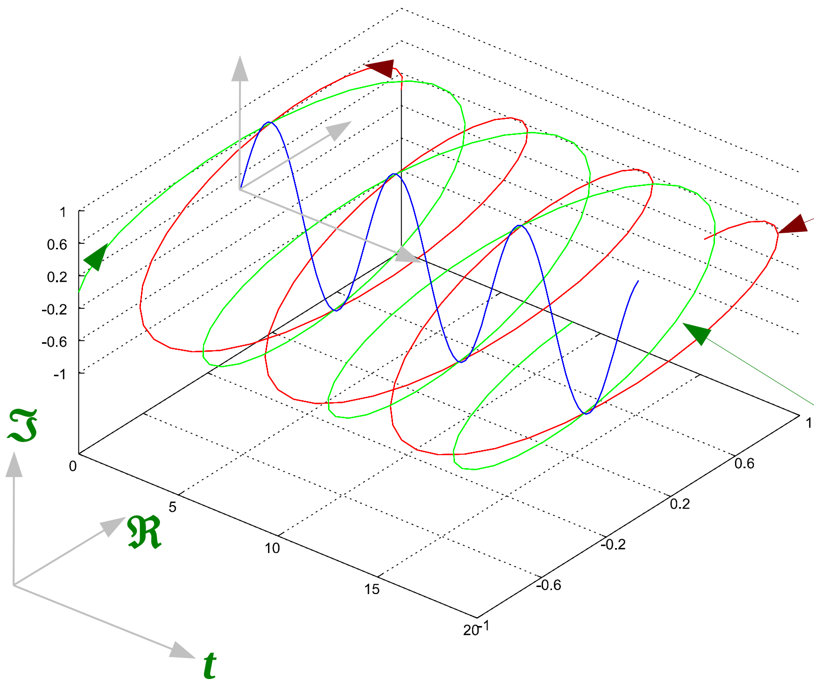
$$(e^{+jt} + e^{-jt}) = 2\cos(t)$$

$$e^{-jt} = \cos(t) - j\sin(t)$$

$$x(t) = A\cos(\omega_0 t)$$

$$= \frac{A}{2} e^{+j\omega_0 t} + \frac{A}{2} e^{-j\omega_0 t}$$

$\sin(\omega_0 t)$



$$e^{+jt} = \cos(t) + j\sin(t)$$

$$(e^{+jt} - e^{-jt}) = 2j\sin(t)$$

$$-e^{-jt} = -\cos(t) + j\sin(t)$$

$$x(t) = A\sin(\omega_0 t)$$

$$= \frac{A}{2j} e^{+j\omega_0 t} - \frac{A}{2j} e^{-j\omega_0 t}$$

Complex Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

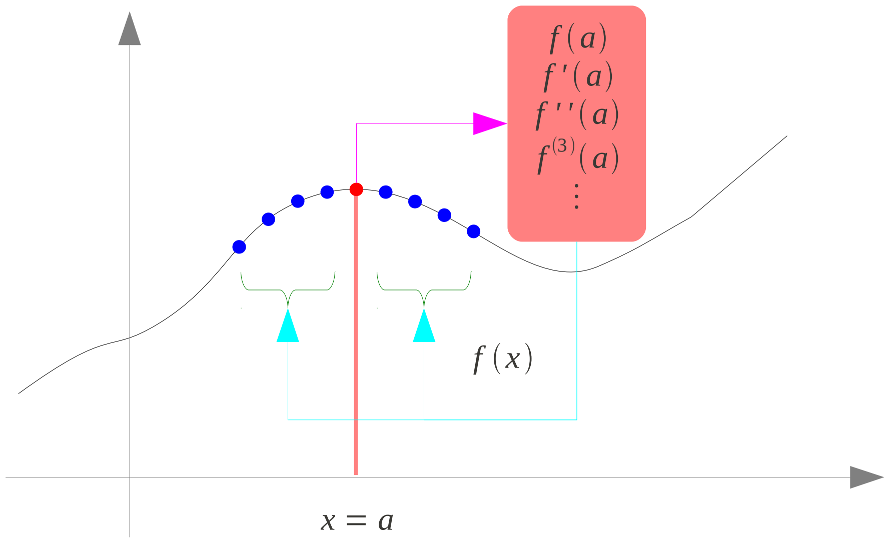
$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Taylor Series

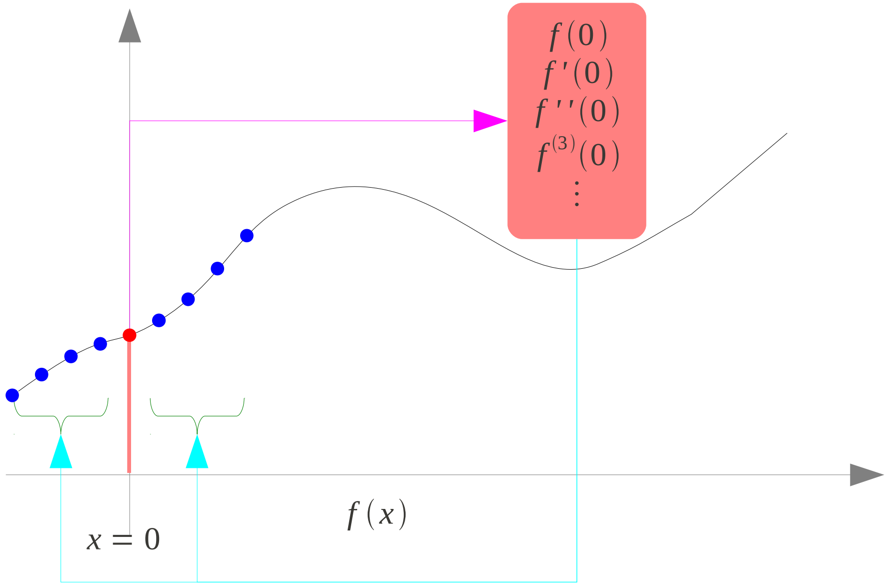
$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$



Maclaurin Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$



Power Series Expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Complex Arithmetic

$$z = a + ib$$
$$w = c + id$$
$$z+w = (a+c) + i(b+d)$$

$$z = a + ib$$
$$w = c + id$$
$$z-w = (a-c) + i(b-d)$$

$$z = a + ib$$
$$w = c + id$$
$$zw = (ac-bd) + i(ad+bc)$$

$$z = a + ib$$
$$w = c + id$$
$$\frac{z}{w} = \left(\frac{a+ib}{c+id}\right)$$
$$= \left(\frac{a+ib}{c+id}\right)\left(\frac{c-id}{c-id}\right)$$
$$= \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{-ad+bc}{c^2+d^2}\right)$$

Complex Conjugate

$$z = x + iy = \Re\{z\} + i\Im\{z\}$$

$$\bar{z} = x - iy = \Re\{z\} - i\Im\{z\}$$

$$\Re\{z\} = \frac{1}{2}(z + \bar{z})$$

$$\Im\{z\} = \frac{1}{2i}(z - \bar{z})$$

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\overline{z - w} = \bar{z} - \bar{w}$$

$$\overline{zw} = \bar{z}\,\bar{w}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$$

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

$$\frac{1}{z} = \frac{1}{z}\cdot\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{x^2 + y^2} = \frac{\bar{z}}{|z|^2}$$

Complex Power (1)

$$a = e^{\log_e a} = e^{\ln a}$$

$$a^{b} = \left(e^{\log_e a}\right)^{b} = \left(e^{\ln a}\right)^{b} = e^{b \ln a}$$

$$a^{ib} = \left(e^{\log_e a}\right)^{ib} = \left(e^{\ln a}\right)^{ib} = e^{ib \ln a}$$

$$= \cos(b \ln a) + i \sin(b \ln a)$$

$$a^{c+ib} = a^{c}\left(e^{\log_e a}\right)^{ib} = a^{c}\left(e^{\ln a}\right)^{ib} = a^{c} e^{ib \ln a}$$

Complex Power (2)

$$a = e^{\ln a}$$

$$a^{b} = e^{b \ln a}$$

$$a^{ib} = e^{ib \ln a} = [\cos(b \ln a) + i \sin(b \ln a)]$$

$$a^{c+ib} = a^{c} e^{ib \ln a} = a^{c} [\cos(b \ln a) + i \sin(b \ln a)]$$

Matrices

Determinant (1)

Determinant of order 2

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \qquad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \qquad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

Determinant (2)

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} \overset{+}{a_1} & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} & \overset{-}{a_2} & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \qquad \begin{bmatrix} & & \overset{+}{a_3} \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = + a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Determinant (3)

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} & -a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \qquad \begin{bmatrix} a_1 & & a_3 \\ & +b_2 & \\ c_1 & & c_3 \end{bmatrix} \qquad \begin{bmatrix} a_1 & & a_3 \\ b_1 & & b_3 \\ & -c_2 & \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

Determinant – Rule of Sarrus

Determinant of order 3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Recursive Method

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32})$$

$$- a_{12}(a_{21}a_{33} - a_{23}a_{31})$$

$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Determinant of order 3 only

Rule of Sarrus

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$$+ a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$$+ a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$$+ a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Solving Linear Equations

A set of linear equations

If the inverse matrix exists

$$\begin{aligned} a\,x + b\,y &= e \\ c\,x + d\,y &= f \end{aligned} \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{vmatrix} e & b \\ f & d \end{vmatrix} = de - bf \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} de - bf \\ -ce + af \end{bmatrix}$$

$$\begin{vmatrix} a & e \\ c & f \end{vmatrix} = af - ce$$

Cramer's Rule

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Cramer's Rule

Determinant of order 3

$$a_1 x + a_2 y + a_3 z = d$$
$$b_1 x + b_2 y + b_3 z = e$$
$$c_1 x + c_2 y + c_3 z = f$$

$$\Leftrightarrow$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} d & a_2 & a_3 \\ e & b_2 & b_3 \\ f & c_2 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d & a_3 \\ b_1 & e & b_3 \\ c_1 & f & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & a_2 & d \\ b_1 & b_2 & e \\ c_1 & c_2 & f \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

Linear Independence

Wronskian

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f'_1 & f'_2 & \cdots & f'_n \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

$$W(f_1(x),\ f_2(x),\ \cdots\ ,\ f_n(x)) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ \dfrac{df_1}{dx} & \dfrac{df_2}{dx} & \cdots & \dfrac{df_n}{dx} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{d^{(n-1)} f_1}{dx^{(n-1)}} & \dfrac{d^{(n-1)} f_2}{dx^{(n-1)}} & \cdots & \dfrac{d^{(n-1)} f_n}{dx^{(n-1)}} \end{vmatrix}$$

Linear Independent Functions and Wronskian

$$C_1 y_1 + C_2 y_2 = 0 \quad \Rightarrow \quad C_1 = C_2 = 0$$

always zero means all coefficients must be zero

y_1 and y_2 .are linearly independent functions

$$C_1 y_1 + C_2 y_2 = 0$$

$$C_1 y_1' + C_2 y_2' = 0$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If the inverse matrix exists

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 \quad \Leftrightarrow \quad \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the only solution: trivial

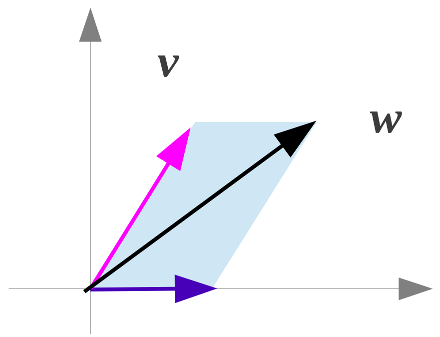
$$W(y_1, y_2) \neq 0$$

Linear Dependent (1)

$\{\boldsymbol{u}\ ,\ \boldsymbol{v}\ ,\ \boldsymbol{w}\}$ linearly dependent

$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$

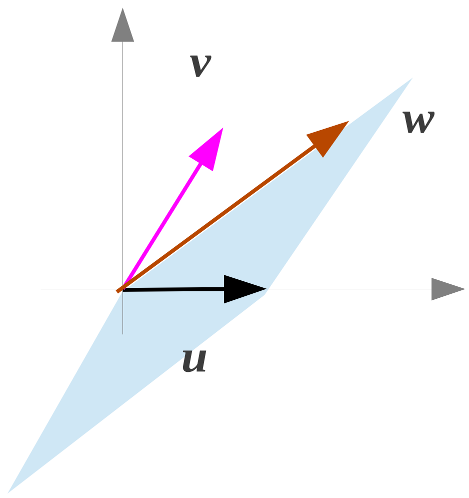
w in terms of u & v



$\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w} = \boldsymbol{0}$

$\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{v}$

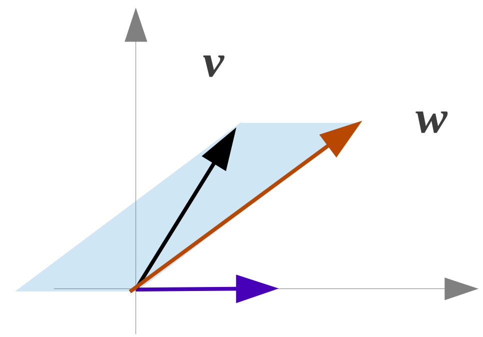
u in terms of w & v



$\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w} = \boldsymbol{0}$

$\boldsymbol{v} = \boldsymbol{w} - \boldsymbol{u}$

v in terms of w & u



$\boldsymbol{u} + \boldsymbol{v} - \boldsymbol{w} = \boldsymbol{0}$

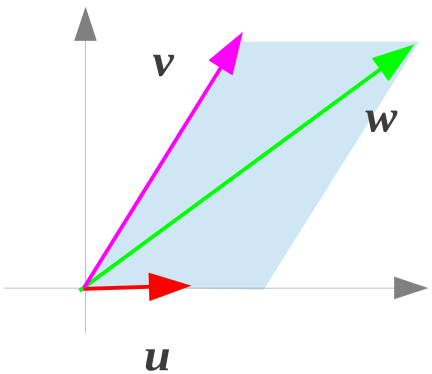
Linear Dependent (2)

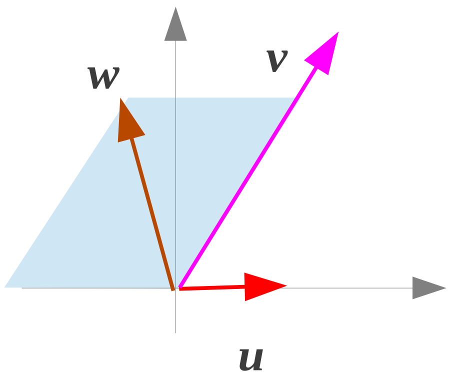
$\{\boldsymbol{u}\ ,\ \boldsymbol{v}\ ,\ \boldsymbol{w}\}$ linearly dependent

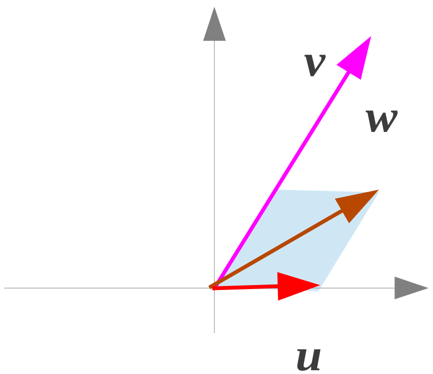
w1 in terms of u & v

w2 in terms of u & v

w3 in terms of u & v







$$k_1\boldsymbol{u} + k_2\boldsymbol{v} + k_3\boldsymbol{w} = \boldsymbol{0}$$

$$(k_1 = 0) \wedge (k_2 = 0) \wedge (k_3 = 0)$$
$$(k_1 \neq 0) \vee (k_2 \neq 0) \vee (k_3 \neq 0)$$

$$m_1\boldsymbol{u} + m_2\boldsymbol{v} + m_3\boldsymbol{w} = \boldsymbol{0}$$

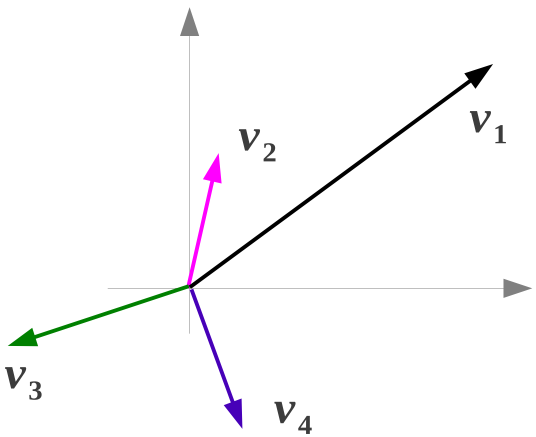
$$(m_1 = 0) \wedge (m_2 = 0) \wedge (m_3 = 0)$$
$$(m_1 \neq 0) \vee (m_2 \neq 0) \vee (m_3 \neq 0)$$

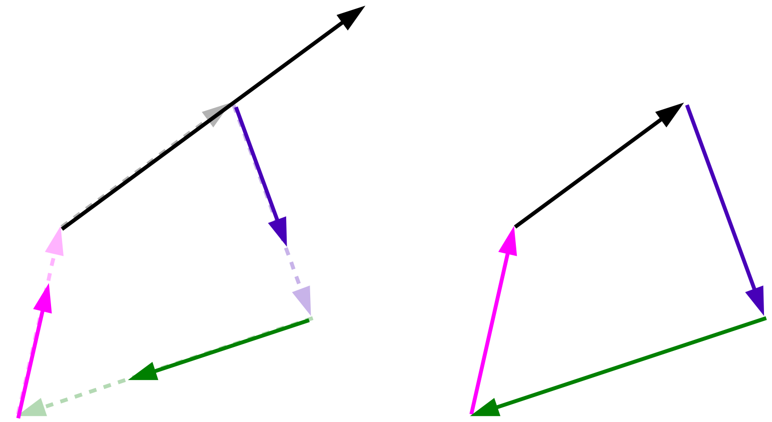
$$n_1\boldsymbol{u} + n_2\boldsymbol{v} + n_3\boldsymbol{w} = \boldsymbol{0}$$

$$(n_1 = 0) \wedge (n_2 = 0) \wedge (n_3 = 0)$$
$$(n_1 \neq 0) \vee (n_2 \neq 0) \vee (n_3 \neq 0)$$

Linear Dependent (3)

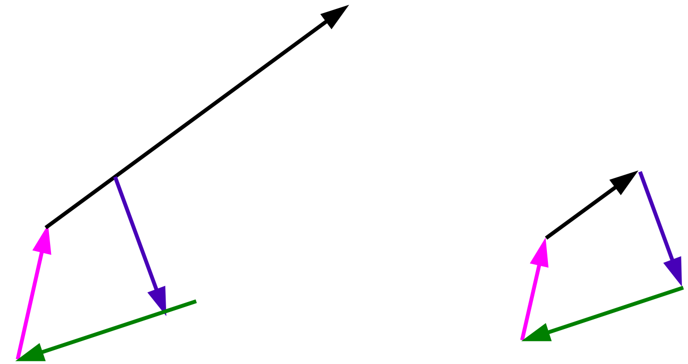
$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ linearly dependent





$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + k_3\boldsymbol{v}_3 + k_4\boldsymbol{v}_4 = \boldsymbol{0}$$

$$(k_1 \neq 0) \lor (k_2 \neq 0) \lor (k_3 \neq 0) \lor (k_4 \neq 0)$$



$$m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 + m_3\boldsymbol{v}_3 + m_4\boldsymbol{v}_4 = \boldsymbol{0}$$

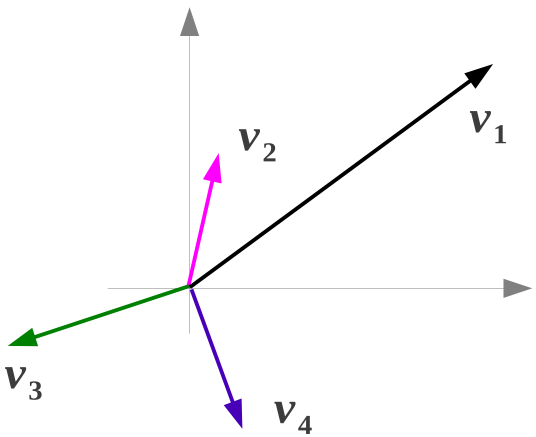
$$(m_1 \neq 0) \lor (m_2 \neq 0) \lor (m_3 \neq 0) \lor (m_4 \neq 0)$$

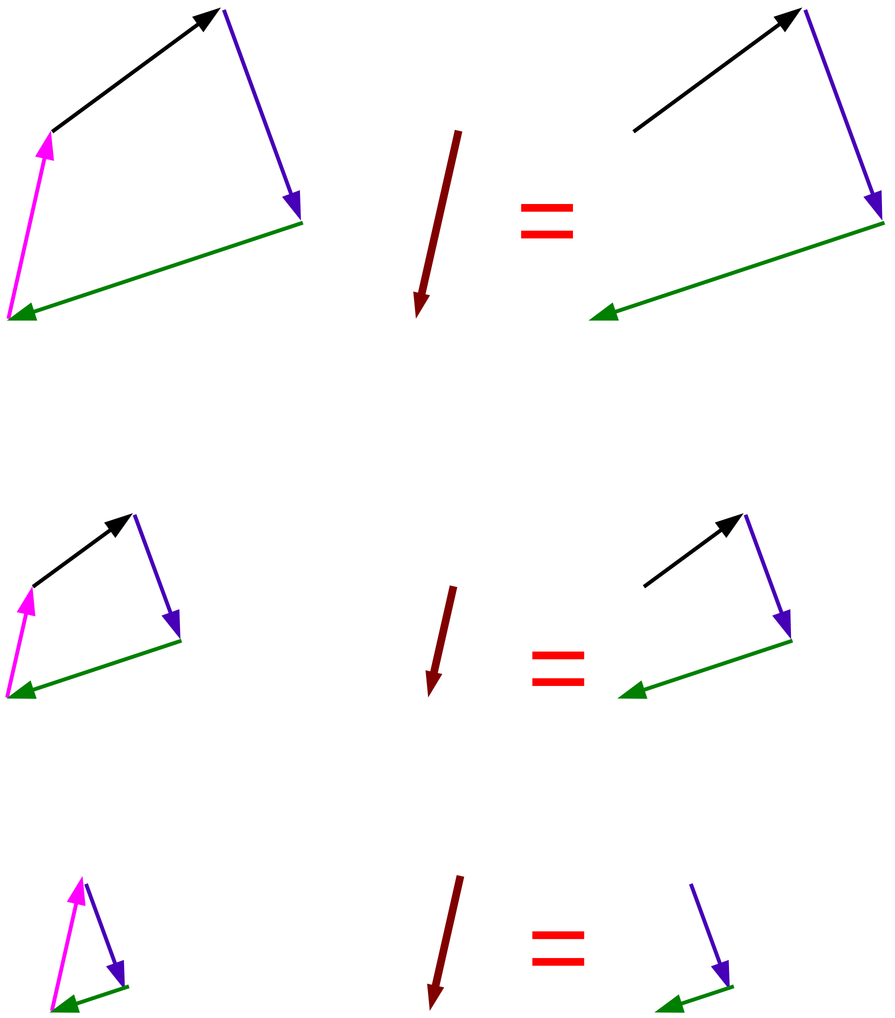
$$0\,\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 + m_3\boldsymbol{v}_3 + m_4\boldsymbol{v}_4 = \boldsymbol{0}$$

$$(m_1 = 0) \lor (m_2 \neq 0) \lor (m_3 \neq 0) \lor (m_4 \neq 0)$$

Linear Dependent (4)

$\{v_1, v_2, v_3, v_4\}$ linearly dependent





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12/21/15

Linear Independent (1)

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ non-empty set of vectors in V

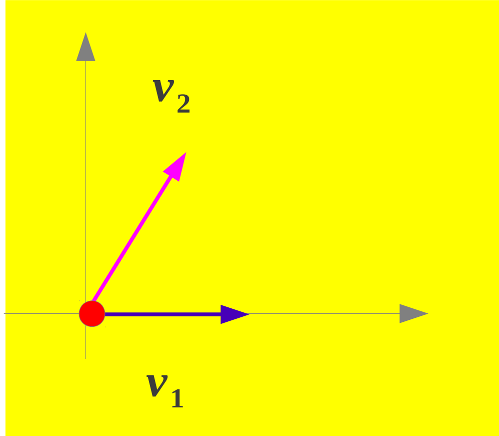
$k_1 \boldsymbol{v}_1 + k_2 \boldsymbol{v}_2 + \cdots + k_n \boldsymbol{v}_n = \boldsymbol{0}$

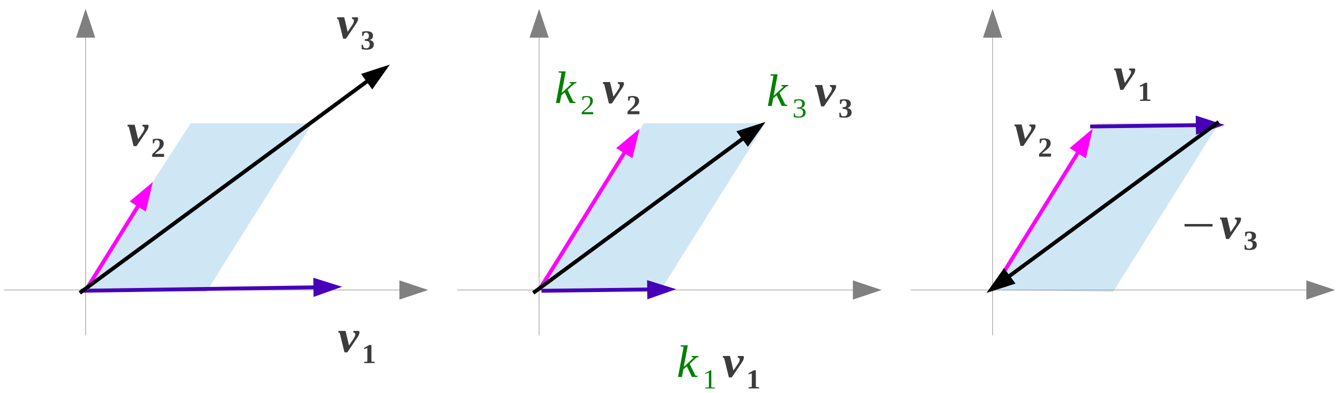
the solution of the above equation

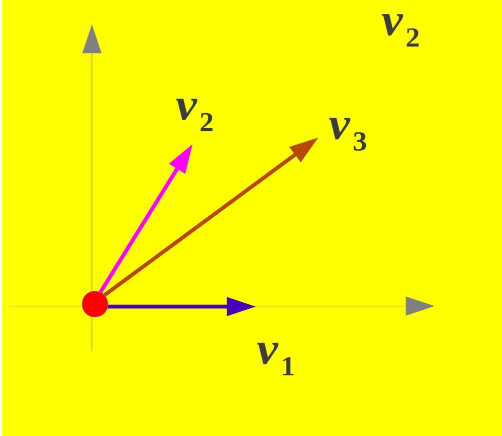
trivial solution: $k_1 = k_2 = \cdots = k_n = 0$

if other solution exists — S linearly dependent

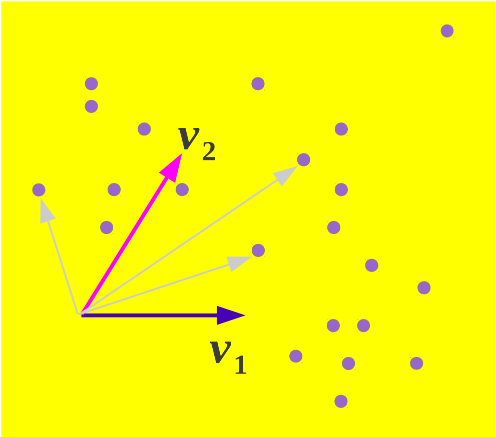
if no other solution exists — S linearly independent







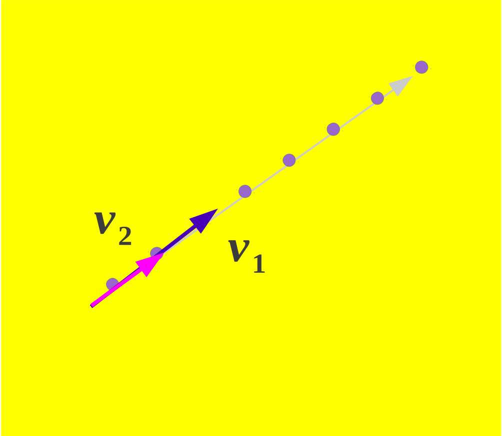
Linear Independent (2)



every point in R^2 can be represented by

$$k_1 \boldsymbol{v_1} + k_2 \boldsymbol{v_2}$$

linear combination of v1 and v2
which are one set of linear independent
two vectors

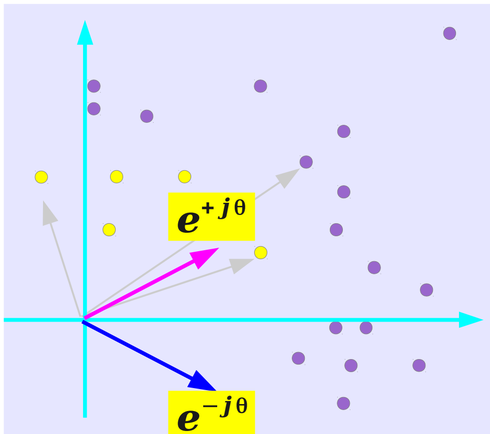


only points on a line in R^2

$$k_1 \boldsymbol{v_1} + k_2 \boldsymbol{v_2}$$

linear combination of v1 and v2
which are one set of linear dependent
two vectors

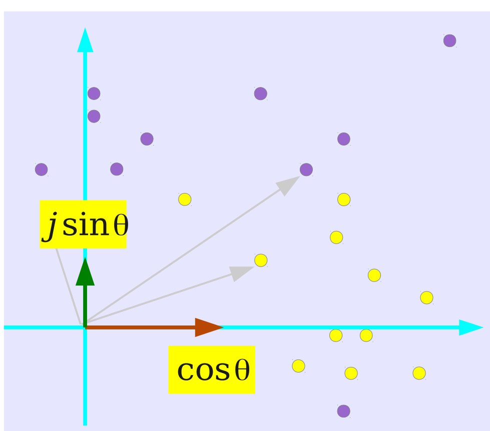
Basis : a set of linear independent spanning vectors



every complex number can be represented by

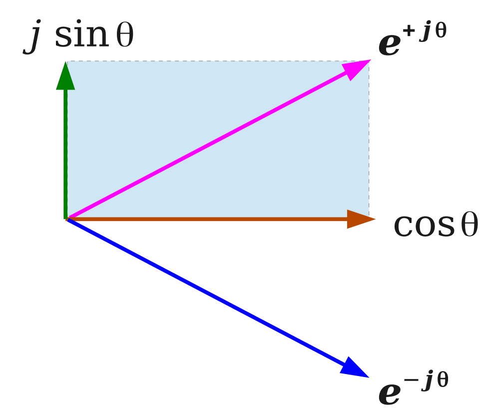
$$\boxed{k_1} e^{+j\theta} + \boxed{k_2} e^{+j\theta}$$

linear combination of $e^{+j\theta}$ and $e^{+j\theta}$
which are one set of linear independent
two vectors



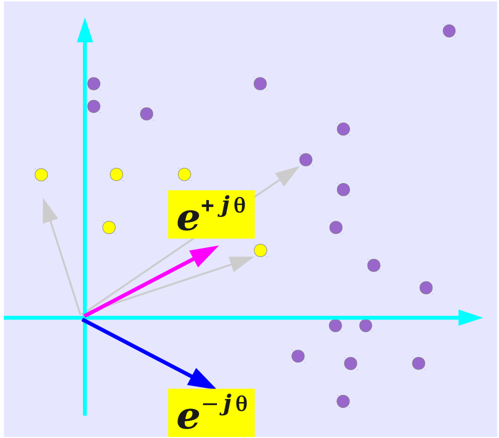
every complex number can also be represented by

$$\boxed{l_1} \cos\theta + \boxed{l_2\, j} \sin\theta$$

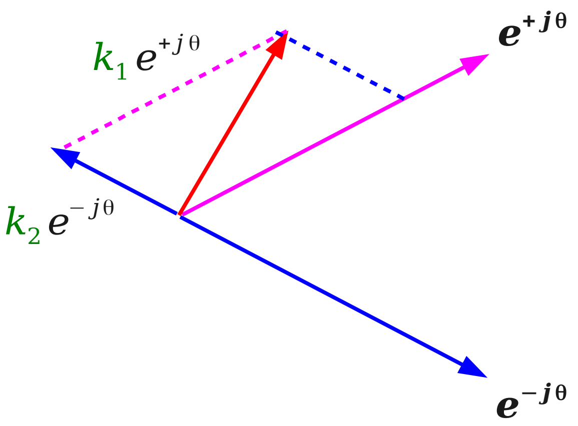


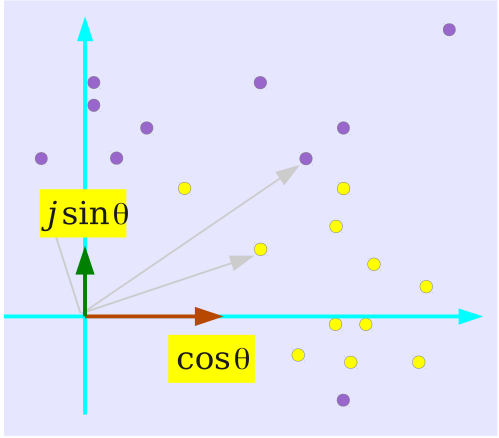
Basis (2)

Basis : a set of linear independent spanning vectors

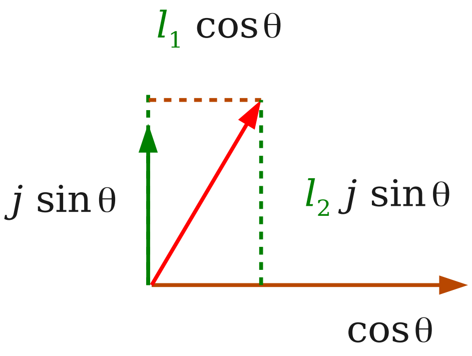


$$\boxed{k_1} e^{+j\theta} + \boxed{k_2} e^{+j\theta}$$





$$\boxed{l_1} \cos\theta + \boxed{l_2\, j} \sin\theta$$



Leibniz Formula

Anti-derivative Examples

differentiation | derivative of F(x)

$$F(x) \rightleftarrows f(x)$$

Anti-derivative of f(x) | Anti-differentiation

$$f(x) = x^2$$

$$\int_0^x f(x)\,dx = \left[\frac{1}{3}x^3\right]_0^x = \frac{1}{3}x^3$$

$$\int_a^x f(x)\,dx = \left[\frac{1}{3}x^3\right]_a^x = \frac{1}{3}x^3 - \frac{1}{3}a^3$$

$$\int_a^x f(t)\,dt = \left[\frac{1}{3}t^3\right]_a^x = \frac{1}{3}x^3 - \frac{1}{3}a^3$$

anti-derivative by the definite integral of f(x)

$$\int_a^x f(t)\,dt = \frac{1}{3}x^3 + C$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) = x^2$$

the Indefinite Integral of f(x)

$$\int x^2\,dx = \frac{1}{3}x^3 + C$$

Fundamental Theorem of Calculus

Derivative of an Antiderivative

$$F(x) = \int_a^x f(t)\,dt \qquad F'(x) = f(x)$$

$$F(g(x)) = \int_a^{g(x)} f(t)\,dt \qquad F'(g(x)) = f(g(x))\,g'(x)$$

$$F(x) = \int_a^x t^2\,dt = \frac{1}{3}x^3 - \frac{1}{3}a^3 \qquad f(x)=x^2$$

$$F(2x+1) = \int_a^{2x+1} t^2\,dt = \frac{1}{3}(2x+1)^3 - \frac{1}{3}a^3 \qquad F'(2x+1) = \frac{d}{dx}\left(\frac{1}{3}(2x+1)^3 - \frac{1}{3}a^3\right) = \frac{1}{3}(2x+1)^2\cdot 2$$

an Antiderivative and an Definite Integral

$$F'(x) = f(x) \qquad \int_a^b f(t)\,dt = F(a) - F(b)$$

Differentiation under the Integral Sign

$$F(x) = \int_a^x f(t)\,dt \qquad F'(x) = f(x)$$

$$F(g(x)) = \int_a^{g(x)} f(t)\,dt \qquad F'(g(x)) = f(g(x))g'(x)$$

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)\,dt\right) = f(b(x))b'(x) - f(a(x))a'(x)$$

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt$$

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
[5] www.chem.arizona.edu/~salzmanr/480a