Ergodic Random Processes - Examples

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

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Example: $X(t) = \alpha \cos(\omega t + \Theta)$

Consider the output of a sinusoidal oscillator
that has a **random phase** and **amplitude** of the form:

$$X(t) = cos(\omega t + \Theta)$$

where $\Theta \sim U([0, 2\pi])$
the explicit dependence on the underlying **sample space** S
the oscillator output can be written as

$$x(t, \Theta) = cos(\omega t + \Theta)$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Example: $X(t) = \alpha \cos(\omega t + \Theta)$

$$\begin{aligned}\mu_X &= E_\Theta(x_t(\Theta)) = E_\Theta[\cos(\omega t + \Theta)] \\ &= E_\Theta[\cos(\omega t)\cos(\Theta) - \sin(\omega t)\sin(\Theta)] \\ &= E_\Theta[\cos(\Theta)]\cos(\omega t) - E_\Theta[\sin(\Theta)]\sin(\omega t)\end{aligned}$$

$$\mu_X = \cos(\omega t)\left(\frac{1}{2\pi}\right)\int_0^{2\pi}\cos(\theta)d\theta - \sin(\omega t)\left(\frac{1}{2\pi}\right)\int_0^{2\pi}\sin(\theta)d\theta = 0$$

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Example: $X(t) = \alpha \cos(\omega t + \Theta)$

the correlation between the R.Vs $x(t_1)$ and $x(t_2)$
denoted as $R_{XX}(t_1, t_2)$

$$
\begin{aligned}
R_{XX}(t_1, t_2) = E_{\Theta}[x(t_1)x(t_2)] &= \int_0^{2\pi} \cos[\omega t_1 + \theta] \cos[\omega t_2 + \theta] d\theta \\
&= \left(\frac{1}{4\pi}\right) \int_0^{2\pi} \cos[\omega(t_1 + t_2) + 2\theta] d\theta \\
&+ \left(\frac{1}{4\pi}\right) \int_0^{2\pi} \cos[\omega(t_1 - t_2)] d\theta \\
&= \left(\frac{1}{2}\right) cos[\omega(t_1 - t_2)]
\end{aligned}
$$

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Example: $X(t) = \alpha \cos(\omega t + \Theta)$

time averages of a single sample function
or realization of the random process $X(t)$.

the **sample mean** of the random process
irrespective of the sample realization that we choose is:

$$\langle \mu_X \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} \cos[\omega t + \Theta] dt$$

As $T \to \infty$ we have:

$$\lim_{T \to \infty} \langle \mu_X \rangle_T = 0$$

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Example: $X(t) = \alpha \cos(\omega t + \Theta)$

The sample mean of the process is therefore independent of the particular ensemble waveform used to calculate the **time-average**, i.e., independent of the value of Θ for the realization.

$$\lim_{T \to \infty} E\langle \mu_X \rangle_T = \mu_X(t) = 0$$

$$\lim_{T \to \infty} Var\langle \mu_X \rangle_T = 0$$

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Example: $X(t) = \alpha \cos(\omega t + \Theta)$

The random process $X(t)$ is therefore
ergodic in the **mean** (**first-order ergodic**)

the sample ACF (Auto-Correlation Function)
of the random process $X(t)$.

The sample ACF is again independent
of the particular realization of the process

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Example: $X(t) = \alpha \cos(\omega t + \Theta)$

$$
\begin{aligned}
\lim_{T\to\infty} \langle R_{XX}(\tau) \rangle_T &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \cos[\omega t + \Theta] \cos[\omega(t-\tau) + \Theta] dt \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T/2}^{T/2} \cos[2\omega t - \omega\tau + +2\Theta] dt \\
&+ \lim_{T\to\infty} \frac{1}{2T} \int_{-T/2}^{T/2} \cos(\omega\tau) dt \\
&= \frac{1}{2} \cos(\omega\tau) = R_{XX}(\tau)
\end{aligned}
$$

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Example: $X(t) = \alpha \cos(\omega t + \Theta)$

The random process X(t) is therefore
ergodic in the **ACF (second-order ergodic)**

$$\lim_{T\to\infty} \langle R_{XX}(\tau) \rangle_T = \frac{1}{2}\cos(\omega\tau) = R_{XX}(\tau)$$

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Example: $X(t) = \alpha\cos(\omega t + \Theta)$

The **power-spectrum** of this random signal, i.e., the **Fourier transform** of the **ensemble ACF** can then be computed as :

$$P_{XX}(\Omega) = \frac{\pi}{2}[\delta(\omega + \omega_c) + [\delta(\omega - \omega_c)]$$

Note that this expression for the **power spectrum** is identical to the expression for the **spectrum** of a **deterministic** sinusoidal signal.

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