

Random Process Background

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles, Jr. and B. Shi

Outline

- 1 Measurable Space
 - Measurable Space
 - Sigma Alebra
 - Topological Space
 - Open Set

- 2 Stochastic Process

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Measurable Space

Space (1)

- A **space** consists of selected **mathematical objects** that are treated as **points**, and selected **relationships** between these **points**.
 - the nature of the **points** can vary widely:
for example, the points can be
 - elements of a set
 - functions on another space
 - subspaces of another space
 - It is the **relationships** that define the nature of the space.

[https://en.wikipedia.org/wiki/Space_\(mathematics\)](https://en.wikipedia.org/wiki/Space_(mathematics))

Space (2)

- While modern mathematics uses many types of **spaces**, such as
 - Euclidean spaces
 - linear spaces
 - topological spaces
 - Hilbert spaces
 - probability spaces
- it does not define the notion of **space** itself.

[https://en.wikipedia.org/wiki/Space_\(mathematics\)](https://en.wikipedia.org/wiki/Space_(mathematics))

Space (3)

- a **space** is
 - a **set** (or a **universe**) with some added **structure**
- It is not always clear whether a given **mathematical object** should be considered as a geometric **space**, or an algebraic **structure**
- A general definition of **structure** embraces all common types of **space**

[https://en.wikipedia.org/wiki/Space_\(mathematics\)](https://en.wikipedia.org/wiki/Space_(mathematics))

Mathematical objects (1)

- A **mathematical object** is an **abstract concept** arising in mathematics.
- an **mathematical object** is anything that has been (or could be) **formally defined**, and with which one may do
 - **deductive reasoning**
 - **mathematical proofs**

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (2)

- Typically, a **mathematical object**
 - can be a **value** that can be assigned to a **variable**
 - therefore can be involved in **formulas**

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (3)

- Commonly encountered **mathematical objects** include
 - numbers
 - sets
 - functions
 - expressions
 - geometric objects
 - transformations of other mathematical objects
 - spaces

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (4)

- **Mathematical objects** can be very *complex*;
 - for example, the followings are considered as **mathematical objects** in **proof theory**.
 - theorems
 - proofs
 - theories

https://en.wikipedia.org/wiki/Mathematical_object

Structure (1)

- a **structure** is a **set** endowed with some *additional features* on the **set**
 - e.g. an *operation*
 - *relation*
 - *metric*
 - *topology*
- Often, the *additional features* are attached or related to the set, so as to provide it with some *additional meaning* or *significance*.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Structure (2)

- A partial list of possible **structures** are
 - measures
 - algebraic structures (groups, fields, etc.)
 - topologies
 - metric structures (geometries)
 - orders
 - events
 - equivalence relations
 - differential structures
 - categories.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Mathematical space (1)

- A **mathematical space** is, informally, a **collection** of **mathematical objects** under consideration.
- The **universe** of **mathematical objects** within a **space** are *precisely defined entities* whose **rules** of *interaction* come baked into the **rules** of the **space**.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Mathematical space (2)

- A **space** differs from a **mathematical set** in several important ways:
 - A **mathematical set** is also a **collection** of **objects**
 - but these **objects** are being pulled from a **space** (or **universe**) of **objects** where the **rules** and **definitions** have already been agreed upon

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Mathematical space (3)

- A **space** differs from a **mathematical set** in several important ways:
 - A **mathematical set** has no **internal structure**,
 - whereas a **space** usually has some **internal structure**.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Mathematical space (4)

- having some **internal structure** could mean a variety of things, but typically it involves
 - *interactions* and *relationships* between **elements** of the **space**
 - *rules* on how to *create* and *define* **new elements** of the **space**

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Measurable space (1)

- A **measurable space** is any **space** with a **sigma-algebra** which can then be equipped with a **measure**
 - collection of **subsets** of the **space** following certain **rules** with a way to assign **sizes** to those sets.

<https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have>

Measurable space (2)

- Intuitively, certain sets belonging to a **measurable space** can be given a **size** in a *consistent way*.

consistent way means that certain **axioms** are met:

- the **empty set** is given a **size** of zero
- if a measurable set is **contained** inside another one, then its **size** is **less than** or **equal to** the size of the **containing set**
- the size of a **disjoint union** of sets is the **sum** of the individual sets' **sizes**

<https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have>

Probability space

- A **probability space** is simply a **measurable space** equipped with a **probability measure**.
- A **probability measure** has the special property of giving the entire space a size of **1**.
 - this then implies that the **size** of any disjoint union of sets (the sum of the **sizes** of the sets) in the **probability space** is less than or equal to 1

<https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have>

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Sigma algebra

Sigma algebra (1)

- We term the **structures** which allow us to use **measure** to be **sigma algebras**
- the only requirements for **sigma algebras** (on a **set** X) are:
 - the $\{\}$ and X are in the **set**.
 - if A is in the **set**, $\text{complement}(A)$ is in the **set**.
 - for any **sets** E_i in the set,
 $\bigcup_i E_i$ is in the **set** (for countable i).

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

Sigma algebra (2)

- The most intuitive way to think about a **sigma algebra** is that it is the kind of **structure** we can do **probability** on.
 - for example, we can assign ratios of areas and length, so the **measure** on such a **set** X tells something about the **probability** of its **subsets**.
 - we can find the **probability** of **subsets** A and B because we know their ratios with respect to a **set** X ;
 - we also know that
 - (the measure of) their **complements** are defined, and
 - their **unions** and **intersections** are defined,
 - so we know how to find the **probability** of things in this set X .

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

Sigma algebra (3)

- The **sigma algebra** which contains the **standard topology** on \mathbb{R} (that is, *all open sets* on \mathbb{R}) is called the **Borel Sigma Algebra**, and the elements of this **set** are called **Borel sets**.
- What this gives us, is the set of **sets** on which outer measure gives our list of dreams. That is, if we take a **Borel set** and we check that length follows translation, additivity, and interval length, it will always hold.

[https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-](https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7)

f5cea0cc2e7

Sigma algebra (4)

- The **set** of Lebesgue measurable sets is the **set** of **Borel sets**, along with (union) all the sets which differ from a Borel set by a **set of measure 0**.
- More intuitively, it is all the sets we can normally measure, plus a bunch of stuff that doesn't affect our ideas of area or volume (think about the **border** of the circle above).

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

Borel Sets (1-1)

- a **Borel set** is any **set** in a **topological space** that can be formed from **open sets** (or, equivalently, from **closed sets**) through the operations of
 - countable union,
 - countable intersection, and
 - relative complement.

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (1-2)

- For a **topological space X** , the collection of all Borel sets on X forms a σ -algebra, known as the **Borel algebra** or **Borel σ -algebra**.
- The **Borel algebra on X** is the smallest **σ -algebra** containing all open sets (or, equivalently, all closed sets).

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (1-3)

- **Borel sets** are important in measure theory, since any measure defined on the open sets of a space, or on the closed sets of a space, must also be defined on all Borel sets of that space.
- Any measure defined on the Borel sets is called a **Borel measure**.
- **Borel sets** and the associated **Borel hierarchy** also play a fundamental role in descriptive set theory.

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (2)

- **Borel sets** are those obtained from intervals by means of the operations allowed in a **σ -algebra**. So we may construct them in a (transfinite) "sequence" of steps:
- ... And again and again.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Borel Sets (3-1)

1. Start with **finite unions** of **closed-open intervals**.
These sets are completely **elementary**, and they form an **algebra**.
2. **Adjoin countable unions** and **intersections** of elementary sets.
What you get already includes **open sets** and **closed sets**, **intersections** of an open set and a closed set, and so on.
Thus you obtain an **algebra**, that is still not a **σ -algebra**.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Borel Sets (3)

3. Again, **adjoin countable unions** and **intersections** to 2.
Observe that you get a strictly larger class, since a **countable intersection** of **countable unions** of intervals is not necessarily included in 2.
Explicit examples of sets in 3 but not in 2 include F_σ sets, like, say, the set of *rational numbers*.
4. And do the same again.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Borel Sets (4-1)

- And even after a sequence of steps we are not yet finished. Take, say, a countable union of a set constructed at step 1, a set constructed at step 2, and so on. This union may very well not have been constructed at any step yet. By axioms of σ -algebra, you should include it as well - if you want, as step ∞

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Borel Sets (4-2)

- (or, technically, the first infinite ordinal, if you know what that means).
- And then continue in the same way until you reach the first uncountable ordinal. And only then will you finally obtain the generated σ -algebra.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

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Topological Space

Topology

- **topology**
from the Greek words
τόπος, 'place, location',
and λόγος, 'study'
is concerned with the **properties** of a **geometric object**
 - that are *preserved* under continuous deformations,
such as stretching, twisting, crumpling, and bending;
 - that is, without closing holes, opening holes,
tearing, gluing, or passing through itself.

<https://en.wikipedia.org/wiki/Topology>

Topological space (1)

- a **topological space** is, roughly speaking, a **geometrical space** in which **closeness** is defined but cannot necessarily be **measured** by a **numeric distance**.

https://en.wikipedia.org/wiki/Borel_set

Topological space (2)

- More specifically, a **topological space** is
- a set whose elements are called points,
- along with an additional structure called a topology,
 - which can be defined as
 - a set of neighbourhoods for each point
 - that satisfy some axioms
 - formalizing the concept of closeness.

https://en.wikipedia.org/wiki/Borel_set

Topological space (3)

- There are several equivalent **definitions** of a topology, the most commonly used of which is the **definition** through **open sets**, which is easier than the others to manipulate.

https://en.wikipedia.org/wiki/Borel_set

Topological space (4)

- A **topological space** is the most general type of a **mathematical space** that allows for the definition of
 - **limits**,
 - **continuity**, and
 - **connectedness**.
- Common types of **topological spaces** include
 - **Euclidean spaces**,
 - **metric spaces** and
 - **manifolds**.

https://en.wikipedia.org/wiki/Borel_set

Topological space (5)

- Although very general, the concept of **topological spaces** is fundamental, and used in virtually every branch of modern mathematics.
- The study of **topological spaces** in their own right is called point-set topology or general topology.

https://en.wikipedia.org/wiki/Borel_set

Open set (1)

- an **open set** is a generalization of an **open interval** in the real line.
- a **metric space** is a **set** along with a **distance** defined between any two **points**
- in a **metric space**, an **open set** is a **set** that, along with every **point** P , contains all **points** that are **sufficiently near** to P
 - all **points** whose **distance** to P is less than some value depending on P

https://en.wikipedia.org/wiki/Open_set

Open set (2)

- More generally, an **open set** is a **member** of a given **collection** of **subsets** of a given **set**, a **collection** that has the property of **containing**
 - every union of its **members**
 - every finite intersection of its members
 - the **empty set**
 - the **whole set** itself

https://en.wikipedia.org/wiki/Open_set

Open set (2)

- A **set** in which such a **collection** is given is called a **topological space**, and the **collection** is called a **topology**.
- These conditions are very loose, and allow enormous flexibility in the choice of **open sets**.
- For example,
 - every **subset** can be **open** (the discrete topology), or
 - no **subset** can be **open** (the indiscrete topology) except
 - the space itself and
 - the empty set .

https://en.wikipedia.org/wiki/Open_set

Open set (3)

Example:

- The *circle* represents the **set** of **points** (x, y) satisfying $x^2 + y^2 = r^2$.
- The *disk* represents the **set** of **points** (x, y) satisfying $x^2 + y^2 < r^2$.
- The *circle* set is an **open set**,
- the *disk* set is its **boundary set**, and
- the **union** of the *circle* and *disk* sets is a **closed set**.

https://en.wikipedia.org/wiki/Open_set

Open set (4)

- A **set** is a **collection** of distinct **objects**.
- Given a **set** A , we say that a is an **element** of A if a is one of the distinct **objects** in A , and we write $a \in A$ to denote this
- Given two **sets** A and B , we say that A is a **subset** of B if every element of A is also an element of B write $A \subseteq B$ to denote this.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (5) Open Balls

- We give these definitions in general, for when one is working in \mathbb{R}^n since they are really not all that different to define in \mathbb{R}^n than in \mathbb{R}^2
- An **open ball** $B_r(\mathbf{a})$ in \mathbb{R}^n centered at $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ with radius r is the set of all points $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ such that the distance between \mathbf{x} and \mathbf{a} is less than r
- In \mathbb{R}^2 an **open ball** is often called an **open disk**

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (6) Interior points

- Suppose that $S \subseteq \mathbb{R}^n$.
- A point $\mathbf{p} \in S$ is an **interior point** of S if there exists an **open ball** $B_r(\mathbf{p}) \subseteq S$.
- Intuitively, \mathbf{p} is an **interior point** of S if we can *squeeze* an entire open ball centered at \mathbf{p} within S

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (7) Boundary points

- A point $p \in \mathbb{R}^n$ is a **boundary point** of S if all **open balls** centered at p contain both **points** in S and **points** not in S .
- The **boundary** of S is the **set** ∂S that consists of all of the **boundary points** of S .

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (8) Open and Closed Sets

- A set $O \subseteq \mathbb{R}^n$ is **open**
if every point in O is an **interior point**.
- A set $C \subseteq \mathbb{R}^n$ is **closed**
if it contains all of its **boundary points**.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (8) Bounded and Unbounded

- A set S is **bounded** if there is an **open ball** $B_M(0)$ such that

$$S \subseteq B.$$

- intuitively, this means that we can enclose all of the **set** S within a large enough ball centered at the origin.
- A **set** that is not **bounded** is called **unbounded**

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

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Open Set

Topologically distinguishable points

- Intuitively, an **open set** provides a *method* to *distinguish* two **points**.
- two **points** in a **topological space**, there exists an **open set**
 - containing one point but
 - not containing the other (distinct) point
 - the two **points** are **topologically distinguishable**.

https://en.wikipedia.org/wiki/Open_set

Metric spaces

- In this manner, one may speak of whether two **points**, or more generally two **subsets**, of a **topological space** are "**near**" without concretely defining a **distance**.
- Therefore, **topological spaces** may be seen as a generalization of **spaces** equipped with a notion of **distance**, which are called **metric spaces**.

https://en.wikipedia.org/wiki/Open_set

The set of all real numbers

- In the [set](#) of all [real numbers](#), one has the natural [Euclidean metric](#); that is, a function which *measures* the [distance](#) between two [real numbers](#): $d(x, y) = |x - y|$.

https://en.wikipedia.org/wiki/Open_set

All points close to a real number x

- Therefore, given a **real number** x , one can speak of the **set** of all **points** close to that **real number** x ; that is, **within** ε of x .
- In essence, **points** within ε of x approximate x to an **accuracy** of **degree** ε .
- Note that $\varepsilon > 0$ always, but as ε becomes *smaller* and *smaller*, one obtains **points** that approximate x to a *higher* and *higher* **degree** of **accuracy**.

https://en.wikipedia.org/wiki/Open_set

The points within ε of x

- For example, if $x = 0$ and $\varepsilon = 1$, the **points** within ε of x are precisely the **points** of the interval $(-1, 1)$;
- However, with $\varepsilon = 0.5$, the **points** within ε of x are precisely the **points** of $(-0.5, 0.5)$.
- Clearly, these **points** approximate x to a *greater degree* of **accuracy** than when $\varepsilon = 1$.

https://en.wikipedia.org/wiki/Open_set

without a concrete Euclidean metric

- The previous examples shows, for the case $x = 0$, that one may approximate x to *higher* and *higher* degrees of accuracy by defining ε to be *smaller* and *smaller*.
- In particular, sets of the form $(-\varepsilon, \varepsilon)$ give us a lot of information about points close to $x = 0$.
- Thus, rather than speaking of a concrete Euclidean metric, one may use sets to describe points close to x .

https://en.wikipedia.org/wiki/Open_set

Different collections of sets containing 0

- This innovative idea has far-reaching consequences; in particular, by defining

different collections of sets containing 0
(distinct from the sets $(-\varepsilon, \varepsilon)$),
one may find different results
regarding the distance
between 0 and other real numbers.

https://en.wikipedia.org/wiki/Open_set

A set for measuring distance

- For example, if we were to define R as the *only* such set for "*measuring distance*", all points are close to 0
- since there is only one possible degree of accuracy one may achieve in approximating 0: being a member of R .

https://en.wikipedia.org/wiki/Open_set

The measure as a binary condition

- Thus, we find that in some sense, every real number is **distance** 0 away from 0.
- It may help in this case to think of the **measure** as being a **binary condition**:
 - all things in \mathbf{R} are equally close to 0,
 - while any item that is not in \mathbf{R} is not close to 0.

https://en.wikipedia.org/wiki/Open_set

Family of sets (1)

- a **collection** F of **subsets** of a given **set** S is called a **family** of **subsets** of S , or a **family** of **sets** over S .
- More generally, a **collection** of any **sets** whatsoever is called a **family** of **sets**, **set family**, or a **set system**

https://en.wikipedia.org/wiki/Family_of_sets

Family of sets (2)

- The term "**collection**" is used here because,
 - in some contexts,
a **family** of **sets** may be allowed
to contain repeated copies of any given **member**, and
 - in other contexts
it may form a **proper class** rather than a **set**.

https://en.wikipedia.org/wiki/Family_of_sets

Family of sets – examples

- The **set** of all **subsets** of a given **set** S is called the **power set** of S and is denoted by $\wp(S)$.
The **power set** $\wp(S)$ of a given **set** S is a **family** of **sets** over S .
- A **subset** of S having k elements is called a **k -subset** of S .
The **k -subset** $S^{(k)}$ of a set S form a **family** of **sets**.
- Let $S = \{a, b, c, 1, 2\}$. An example of a **family** of **sets** over S (in the multiset sense) is given by $F = \{A_1, A_2, A_3, A_4\}$, where $A_1 = \{a, b, c\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2\}$, and $A_4 = \{a, b, 1\}$.

https://en.wikipedia.org/wiki/Family_of_sets

Class (1)

- a **class** is a **collection** of **sets**
(or sometimes other **mathematical objects**)
that can be unambiguously defined
by a **property** that all its members share.
- **Classes** act as a way to have **set-like collections**
while differing from **sets** so as to avoid **Russell's paradox**

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class (2)

- A **class** that is not a **set** is called a **proper class**, and
- a **class** that is a **set** is sometimes called a **small class**.
- the followings are **proper classes** in many formal systems
 - the **class** of all sets
 - the **class** of all ordinal numbers
 - the **class** of all cardinal numbers

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class (3)

- "the **set** of all **sets** with **property** X ."
- especially when dealing with **categories**, since the **objects** of a **concrete category** are all **sets** with certain additional **structure**.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

Class (4)

- However, if we're not careful about this we can get into serious trouble –
"let X be the set of all sets which do not contain themselves."
 - Since X is a set, we can ask whether X is an element of itself.
 - But then we run into a paradox – if X contains itself as an element, then it does not, and vice versa.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

Class (5)

- In order to avoid this paradox, we cannot consider the **collection** of all sets to be itself a **set**.
- This means we have to *throw out* the whole "the **set** of all **sets** with **property** X" construction. But we wanted that.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

Class (6)

- So the way we get around it is to say that there's something called a "class," which is like a set but not a set.
- Then we can talk about "the class X of all sets with property Y ."
- Since X isn't a set, it can't be an element of itself, and we're fine.
- Of course, if we need to talk about the collection of all classes, we need to create another name that goes another step back, and so forth.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

Class Examples (1)

- The **collection** of all **algebraic structures** of a given type will usually be a **proper class**.
 - the **class** of all **groups**
 - the **class** of all **vector spaces**
 - and many others.
- a **category**
 - whose **collection** of **objects** forms a **proper class** or
 - whose **collection** of **morphisms** forms a **proper class**is called a **large category**.

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class Examples (2)

- Within set theory, many **collections** of **sets** turn out to be **proper classes**.
- One way to *prove* that a **class** is **proper** is to place it in **bijection** with the **class** of all ordinal numbers.
- This method is used, for example, in the *proof* that there is no free complete lattice on three or more generators.

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class Paradoxes (1)

- The **paradoxes** of naive set theory can be explained in terms of the *inconsistent tacit assumption* that "all **classes** are **sets**".
- With a rigorous foundation, these **paradoxes** instead *suggest proofs* that certain **classes** are proper (i.e., that they are not **sets**).
- For example, **Russell's paradox** *suggests a proof* that the **class** of all **sets** which do not contain themselves is **proper**
- the **Burali-Forti paradox** *suggests* that the **class** of all ordinal numbers is **proper**.

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class Paradoxes (2)

- The paradoxes do not arise with classes because there is no notion of classes containing classes.
- Otherwise, one could, for example, define a class of all classes that do not contain themselves, which would lead to a Russell paradox for classes.
- A conglomerate, on the other hand, can have proper classes as members, although the theory of conglomerates is not yet well-established.

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Russell's Paradox (1-1)

- According to the unrestricted comprehension principle, for any sufficiently well-defined **property**, there is the **set** of **all** and only the **objects** that have that **property**.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (1-2)

- Let R be the **set of all sets** ($R = \{x \mid x \notin x\}$) that are not members of themselves ($R \notin R$).
 - if** R is not a **member** of itself ($R \notin R$), **then** its definition (the **set of all sets**) entails that it is a **member** of itself ($R \in R$);
 - yet, **if** it is a **member** of itself ($R \in R$), **then** it is not a **member** of itself ($R \notin R$), since it is the **set of all sets** that are not members of themselves ($R \notin R$)
- the resulting **contradiction** is **Russell's paradox**.
- Let $R = \{x \mid x \notin x\}$, then $R \in R \iff R \notin R$

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (2-1)

- Most **sets** commonly encountered are not **members** of themselves.
- For example, consider the **set** of all squares in a plane.
- This **set** is not itself a square in the plane, thus it is not a **member** of itself.
- Let us call a **set** "**normal**" if it is not a **member** of itself, and "**abnormal**" if it is a **member** of itself.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (2-2)

- Clearly every **set** must be either **normal** or **abnormal**.
- The **set** of squares in the plane is **normal**.
- In contrast, the **complementary set** that contains everything which is not a square in the plane is itself not a square in the plane, and so it is one of its own **members** and is therefore **abnormal**.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (3)

- Now we consider the **set** of all **normal sets**, R , and try to determine whether R is **normal** or **abnormal**.
 - **If** R were **normal**, it would be contained in the **set** of all **normal sets** (itself), and therefore be **abnormal**;
 - on the other hand **if** R were **abnormal**, it would not be contained in the **set** of all **normal sets** (itself), and therefore be **normal**.
- This leads to the conclusion that R is neither **normal** nor **abnormal**: **Russell's paradox**.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Filter

- a **filter** on a set X is a family \mathcal{B} of subsets such that:
- $X \in \mathcal{B}$ and $\emptyset \notin \mathcal{B}$ if $A \in \mathcal{B}$ and $B \in \mathcal{B}$,
then $A \cap B \in \mathcal{B}$
If $A, B \subset X, A \in \mathcal{B}$, and $A \subset B$,
then $B \in \mathcal{B}$
- A **filter** on a set may be thought of
as representing a "collection of large subsets",
one intuitive example being the **neighborhood filter**.
- **Filters** appear in order theory, model theory, and set theory,
but can also be found in topology, from which they originate.
The dual notion of a **filter** is an **ideal**.

[https://en.wikipedia.org/wiki/Filter_\(set_theory\)#filter_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

Neighbourhood basis (1)

- A **neighbourhood basis** or **local basis** (or **neighbourhood base** or **local base**) for a **point** x is a **filter base** of the **neighbourhood filter**;
- this means that it is a **subset** $\mathcal{B} \subseteq \mathcal{N}(x)$ such that for all $V \in \mathcal{N}(x)$, there exists some $B \in \mathcal{B}$ such that $B \subseteq V$. That is, for any **neighbourhood** V we can find a **neighbourhood** B in the **neighbourhood basis** that is contained in V .

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

Neighbourhood basis (2)

- Equivalently, \mathcal{B} is a local basis at x if and only if the neighbourhood filter \mathcal{N} can be recovered from \mathcal{B} in the sense that the following equality holds:

$$\mathcal{N}(x) = \{V \subseteq X : B \subseteq V \text{ for some } B \in \mathcal{B}\}$$

- A family $\mathcal{B} \subseteq \mathcal{N}(x)$ is a neighbourhood basis for x if and only if \mathcal{B} is a cofinal subset of $(\mathcal{N}(x), \supseteq)$ with respect to the partial order \supseteq (importantly, this partial order is the superset relation and not the subset relation).

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

A collection of sets around x

- In general, one refers to the family of sets containing 0, used to approximate 0, as a neighborhood basis;
- a member of this neighborhood basis is referred to as an **open set**.
- In fact, one may generalize these notions to an arbitrary set (X); rather than just the real numbers.
- In this case, given a point (x) of that set (X), one may define a **collection** of sets "around" (that is, containing) x , used to approximate x .

https://en.wikipedia.org/wiki/Open_set

Smaller sets containing x

- Of course, this **collection** would have to *satisfy* certain properties (known as **axioms**) for otherwise we may not have a *well-defined method* to measure **distance**.
- For example, every **point** in X should **approximate** x to some **degree** of **accuracy**.
- Thus X should be in this **family**.
- Once we begin to define "smaller" **sets** containing x , we tend to **approximate** x to a greater **degree** of **accuracy**.
- Bearing this in mind, one may define the remaining **axioms** that the **family** of **sets** about x is required to satisfy.

https://en.wikipedia.org/wiki/Open_set

Open ball (1)

- a **ball** is the solid figure bounded by a **sphere**;
it is also called a **solid sphere**.
 - a **closed ball**
includes the *boundary points* that constitute the sphere
 - an **open ball**
excludes them

[https://en.wikipedia.org/wiki/Ball_\(mathematics\)](https://en.wikipedia.org/wiki/Ball_(mathematics))

Open ball (2)

- A **ball** in n dimensions is called a **hyperball** or **n-ball** and is bounded by a **hypersphere** or $(n - 1)$ -sphere
- One may talk about **balls** in any **topological space** X , not necessarily induced by a **metric**.
- An **n -dimensional topological ball** of X is any **subset** of X which is **homeomorphic** to an **Euclidean n-ball**.

[https://en.wikipedia.org/wiki/Ball_\(mathematics\)](https://en.wikipedia.org/wiki/Ball_(mathematics))

Homeomorphism (1)

- a **homeomorphism**

(from Greek ὁμοιος (homoios) 'similar, same', and μορφή (morphē) 'shape, form', named by Henri Poincaré), **topological isomorphism**, or **bicontinuous function** is a **bijjective** and **continuous** function between topological spaces that has a **continuous inverse** function.

<https://en.wikipedia.org/wiki/Homeomorphism>

Homeomorphism (2)

- **Homeomorphisms** are the **isomorphisms** in the category of **topological spaces** – the **mappings** that **preserve** all the **topological properties** of a given space.
- Two **spaces** with a **homeomorphism** between them are called **homeomorphic**, and from a topological viewpoint they are the same.

<https://en.wikipedia.org/wiki/Homeomorphism>

Homeomorphism (3)

- Very roughly speaking,
a **topological space** is a **geometric object**,
and the **homeomorphism** is
a *continuous stretching* and *bending*
of the object into a *new shape*.

<https://en.wikipedia.org/wiki/Homeomorphism>

Homeomorphism (4)

- Thus, a *square* and a *circle* are **homeomorphic** to each other, but a *sphere* and a *torus* are not.
- However, this description can be misleading.
- Some *continuous deformations* are not **homeomorphisms**, such as the *deformation* of a *line* into a *point*.
- Some **homeomorphisms** are not *continuous deformations*, such as the homeomorphism between a *trefoil knot* and a *circle*.

<https://en.wikipedia.org/wiki/Homeomorphism>

Euclidean space definition (1)

- A **subset** U of the **Euclidean n-space** \mathbb{R}^n is **open** if, for every **point** x in U , there exists a positive **real number** ε (depending on x) **such that** any **point** in \mathbb{R}^n whose **Euclidean distance** from x is smaller than ε belongs to U

https://en.wikipedia.org/wiki/Open_set

Euclidean space definition (2)

- Equivalently, a subset U of \mathbb{R}^n is **open** if every point in U is the center of an **open ball** contained in U
- An example of a subset of \mathbb{R} that is not **open** is the **closed interval** $[0, 1]$, since neither $0 - \varepsilon$ nor $1 + \varepsilon$ belongs to $[0, 1]$ for any $\varepsilon > 0$, no matter how small.

https://en.wikipedia.org/wiki/Open_set

Metric space definition (1)

- A **subset** U of a **metric space** (M, d) is called **open** if, for any **point** x in U , there exists a **real number** $\varepsilon > 0$ such that any **point** $y \in M$ satisfying $d(x, y) < \varepsilon$ belongs to U .
- Equivalently, U is **open** if every **point** in U has a **neighborhood** contained in U .
- This generalizes the **Euclidean space** example, since **Euclidean space** with the **Euclidean distance** is a **metric space**.

https://en.wikipedia.org/wiki/Open_set

Metric space definition (2)

- Formally, a **metric space** is an **ordered pair** (M, d) where M is a **set** and d is a **metric** on M , i.e., a **function**

$$d : M \times M \rightarrow \mathbb{R}$$

satisfying the following **axioms** for all points $x, y, z \in M$:

- $d(x, x) = 0$.
- If $x \neq y$, then $d(x, y) > 0$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$.

https://en.wikipedia.org/wiki/Open_set

Metric space definition (3)

- satisfying the following **axioms** for all points $x, y, z \in M$:
 - The distance from a point *to itself* is zero:
 - (**Positivity**) The **distance** between two distinct points is always **positive**:
 - (**Symmetry**) The **distance** from x to y is always the same as the **distance** from y to x :
 - The **triangle inequality** holds: This is a natural property of both physical and metaphorical notions of distance: you can arrive at z from x by taking a detour through y , but this will not make your journey any faster than the shortest path.
- If the **metric** d is unambiguous, one often refers by abuse of notation to "the **metric space** M ".

https://en.wikipedia.org/wiki/Open_set

Topological space definition (1)

- A **topology** τ on a **set** X is a **set** of **subsets** of X with the *properties* below. Each **member** of τ is called an **open set**. [3]
 - $X \in \tau$ and $\emptyset \in \tau$
 - Any **union** of sets in τ belong to τ : if $\{U_i : i \in I\} \subseteq \tau$ then

$$\bigcup_{i \in I} U_i \in \tau$$

- Any finite **intersection** of sets in τ belong to τ : if $U_1, \dots, U_n \in \tau$ then

$$U_1 \cap \dots \cap U_n \in \tau$$

- X together with τ is called a **topological space**.

Topological space definition (2)

- Infinite intersections of open sets need not be open.
- For example, the intersection of all intervals of the form $(-1/n, 1/n)$, where n is a positive integer, is the set $\{0\}$ which is not open in the real line.
- A metric space is a topological space, whose topology consists of the collection of all subsets that are unions of open balls.
- There are, however, topological spaces that are not metric spaces.

https://en.wikipedia.org/wiki/Open_set

Topological space via neighborhoods (1)

- This axiomatization is due to Felix Hausdorff.
- Let X be a **set**;
- the **elements** of X are usually called **points**, though they can be any mathematical object.
- We allow X to be **empty**.

https://en.wikipedia.org/wiki/Topological_space

Topological space via neighborhoods (2)

- Let \mathcal{N} be a **function** assigning to each x (**point**) in X a non-empty **collection** $\mathcal{N}(x)$ of **subsets** of X .
- The **elements** of $\mathcal{N}(x)$ will be called **neighbourhoods** of x with respect to \mathcal{N} (or, simply, **neighbourhoods** of x).
- The **function** \mathcal{N} is called a neighbourhood topology if *the axioms* below are satisfied; and
- then X with \mathcal{N} is called a **topological space**.

https://en.wikipedia.org/wiki/Topological_space

Topological space via neighborhoods (3)

- If N is a neighbourhood of x (i.e., $N \in \mathcal{N}(x)$), then $x \in N$.
In other words, each point belongs to every one of its neighbourhoods.
- If N is a subset of X and includes a neighbourhood of x , then N is a neighbourhood of x . I.e., every superset of a neighbourhood of a point $x \in X$ is again a neighbourhood of x .
- The intersection of two neighbourhoods of x is a neighbourhood of x .
- Any neighbourhood \mathcal{N} of x includes a neighbourhood \mathcal{M} of x such that \mathcal{M} is a neighbourhood of each point of M .

https://en.wikipedia.org/wiki/Topological_space

Topological space via neighborhoods (4)

- The first three axioms for neighbourhoods have a clear meaning.
- The fourth axiom has a very important use in the structure of the theory, that of linking together the neighbourhoods of different points of X .
- A standard example of such a system of neighbourhoods is for the real line \mathbb{R} , where a subset N of \mathbb{R} is defined to be a neighbourhood of a real number x if it includes an open interval containing x .

https://en.wikipedia.org/wiki/Topological_space

Topological space via open sets (1)

- A **topology** on a set X may be defined as a **collection** τ of **subsets** of X , called **open sets** and satisfying the following **axioms**:
 - The **empty set** and X itself belong to τ .
 - Any arbitrary (**finite** or **infinite**) **union** of members of τ belongs to τ .
 - The **intersection** of any **finite** number of members of τ belongs to τ .

https://en.wikipedia.org/wiki/Topological_space

Topological space via open sets (2)

- As this definition of a topology is the most commonly used, the set τ of the **open sets** is commonly called a **topology** on X .
- A **subset** $C \subseteq X$ is said to be **closed** in (X, τ) if its complement $X \setminus C$ is an **open set**.

https://en.wikipedia.org/wiki/Topological_space

Topological space via neighborhoods (3)

- Given such a **structure**, a **subset** U of X is defined to be **open** if U is a **neighbourhood** of all **points** in U .
- The **open sets** then satisfy the **axioms** given below.
- Conversely, when given the **open sets** of a **topological space**, the **neighbourhoods** satisfying the above **axioms** can be recovered by defining N to be a **neighbourhood** of x if N includes an open set U such that $x \in U$.

https://en.wikipedia.org/wiki/Topological_space

Examples of topology (1)

- Given $X = \{1, 2, 3, 4\}$,
the trivial or indiscrete topology on X is
the family $\tau = \{\{\}, \{1, 2, 3, 4\}\} = \{\emptyset, X\}$
consisting of only the two subsets of X
required by the axioms
forms a topology of X .
- Given $X = \{1, 2, 3, 4\}$,
the family $\tau = \{\{\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$
 $= \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, X\}$
of six subsets of X forms another topology of X .

https://en.wikipedia.org/wiki/Topological_space

Examples of topology (2)

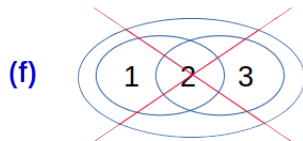
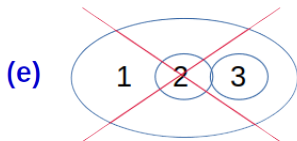
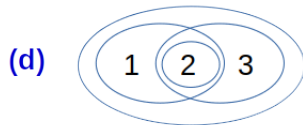
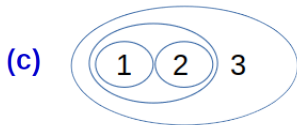
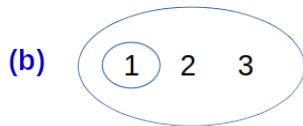
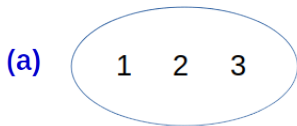
- Given $X = \{1, 2, 3, 4\}$,
the discrete topology on X is
the power set of X , which is the family $\tau = \mathcal{P}(X)$
consisting of all possible subsets of X .
In this case the topological space (X, τ)
is called a discrete space.
- Given $X = \mathbb{Z}$, the set of integers,
the family τ of all finite subsets
of the integers plus \mathbb{Z} itself
is not a topology,
because (for example) the union of all finite sets
not containing zero is not finite
but is also not all of \mathbb{Z} , and so it cannot be in τ .

Examples of topology (3)

- Let τ be denoted with the circles, here are four examples **(a)**, **(b)**, **(c)**, **(d)**, and two non-examples **(e)**, **(f)** of topologies on the three-point set $\{1,2,3\}$.
- **(e)** is not a topology because the union of $\{2\}$ and $\{3\}$ [i.e. $\{2,3\}$] is missing;
- **(f)** is not a topology because the intersection of $\{1,2\}$ and $\{2,3\}$ [i.e. $\{2\}$], is missing.

https://en.wikipedia.org/wiki/Topological_space

Examples of topology (4)



Definitions via closed sets

- Using **de Morgan's laws**, the above axioms defining **open sets** become axioms defining **closed sets**:
- The **empty set** and X are **closed**.
 - The **intersection** of any **collection** of **closed sets** is also **closed**.
 - The **union** of any finite number of **closed sets** is also **closed**.
- Using these **axioms**, another way to define a **topological space** is as a set X together with a **collection** τ of **closed subsets** of X . Thus the **sets** in the **topology** τ are the **closed sets**, and their complements in X are the **open sets**.

https://en.wikipedia.org/wiki/Open_set

Open)

- (Open and Closed Sets)

https://en.wikipedia.org/wiki/Topological_space

Stochastic Process (1)

In probability theory and related fields, a **stochastic** (/stou'kæstɪk/) or **random** process is a mathematical object usually defined as a family of **random variables**.

The word stochastic in English was originally used as an adjective with the definition "pertaining to **conjecturing**", and stemming from a Greek word meaning "to aim at a mark, guess", and the Oxford English Dictionary gives the year 1662 as its earliest occurrence.

From Ancient Greek στοχαστικός (stokhastikós), from στοχάζομαι (stokhá-zomai, "aim at a target, guess"), from στόχος (stókchos, "an aim, a guess").

<https://en.wikipedia.org/wiki/Stochastic>
<https://en.wiktionary.org/wiki/stochastic>

Stochastic Process (2)

The definition of a **stochastic process** varies, but a **stochastic process** is *traditionally* defined as a collection of **random variables** indexed by some set.

The terms **random process** and **stochastic process** are considered synonyms and are used interchangeably, without the **index set** being precisely specified.

Both "**collection**", or "**family**" are used while instead of "**index set**", sometimes the terms "**parameter set**" or "**parameter space**" are used.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (3)

The term **random function** is also used to refer to a **stochastic** or **random process**, though sometimes it is only used when the stochastic process takes real values.

This term is also used when the **index sets** are **mathematical spaces** other than the **real line**,

while the terms **stochastic process** and **random process** are usually used when the **index set** is interpreted as time,

and other terms are used such as **random field** when the **index set** is n -dimensional **Euclidean space** \mathbb{R}^n or a manifold

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (4)

A **stochastic process** can be denoted, by $\{X(t)\}_{t \in T}$, $\{X_t\}_{t \in T}$, $\{X(t)\}$, $\{X_t\}$ or simply as X or $X(t)$, although $X(t)$ is regarded as an abuse of function notation.

For example, $X(t)$ or X_t are used to refer to the **random variable** with the **index** t , and not the entire **stochastic process**.

If the **index set** is $T = [0, \infty)$, then one can write, for example, $(X_t, t \geq 0)$ to denote the **stochastic process**.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (1)

A **stochastic process** is defined as a collection of **random variables** defined on a common **probability space** (Ω, \mathcal{F}, P) ,

- Ω is a **sample space**,
- \mathcal{F} is a σ -**algebra**,
- P is a **probability measure**;
- the **random variables**, indexed by some set T ,
- all take values in the same **mathematical space** S , which must be **measurable** with respect to some σ -algebra Σ

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (2)

In other words, for a given **probability space** (Ω, \mathcal{F}, P) and a **measurable space** (S, Σ) , a **stochastic process** is a **collection** of S -valued **random variables**, which can be written as:

$$\{X(t) : t \in T\}.$$

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (3)

Historically, in many problems from the natural sciences a point $t \in T$ had the meaning of time, so $X(t)$ is a **random variable** representing a value observed at time t .

A **stochastic process** can also be written as $\{X(t, \omega) : t \in T\}$ to reflect that it is actually a function of two variables, $t \in T$ and $\omega \in \Omega$.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (4)

There are other ways to consider a stochastic process, with the above definition being considered the traditional one.

For example, a stochastic process can be interpreted or defined as a S^T -valued **random variable**, where S^T is the space of all the possible functions from the set T into the space S .

However this alternative definition as a "**function-valued random variable**" in general requires additional regularity assumptions to be **well-defined**.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (1)

The set T is called the **index set** or **parameter set** of the **stochastic process**.

Often this set is some subset of the real line, such as the natural numbers or an interval, giving the set T the interpretation of time.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (2)

In addition to these sets, the index set T can be another set with a **total order** or a more general set, such as the Cartesian plane R^2 or n -dimensional **Euclidean space**, where an element $t \in T$ can represent a point in space.

That said, many results and theorems are only possible for **stochastic processes** with a **totally ordered index set**.

https://en.wikipedia.org/wiki/Stochastic_process

State space

The **mathematical space** S of a **stochastic process** is called its **state space**.

This mathematical space can be defined using integers, real lines, n -dimensional Euclidean spaces, complex planes, or more abstract mathematical spaces.

The **state space** is defined using elements that reflect the different values that the **stochastic process** can take.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (1)

A **sample function** is a single outcome of a **stochastic process**, so it is formed by taking a single possible value of each **random variable** of the **stochastic process**.

More precisely, if $\{X(t, \omega) : t \in T\}$ is a **stochastic process**, then for any point $\omega \in \Omega$, the mapping $X(\cdot, \omega) : T \rightarrow S$, is called a **sample function**, a **realization**, or, particularly when T is interpreted as time, a **sample path** of the **stochastic process** $\{X(t, \omega) : t \in T\}$.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (2)

This means that for a fixed $\omega \in \Omega$,
there exists a **sample function**
that maps the **index set** T to the **state space** S .

Other names for a **sample function** of a **stochastic process**
include **trajectory**, **path function** or **path**

https://en.wikipedia.org/wiki/Stochastic_process

