CTFS of Periodic Pulse Functions (2B)

- CTFS of a Periodic Pulse Train
- CTFS of a Shifted Periodic Pulse Train
- Spectrum Plots of the CTFS of a Periodic Pulse Train
- Spectrum Plots of the CTFS of a Shifted Periodic Pulse Train

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7/11/13

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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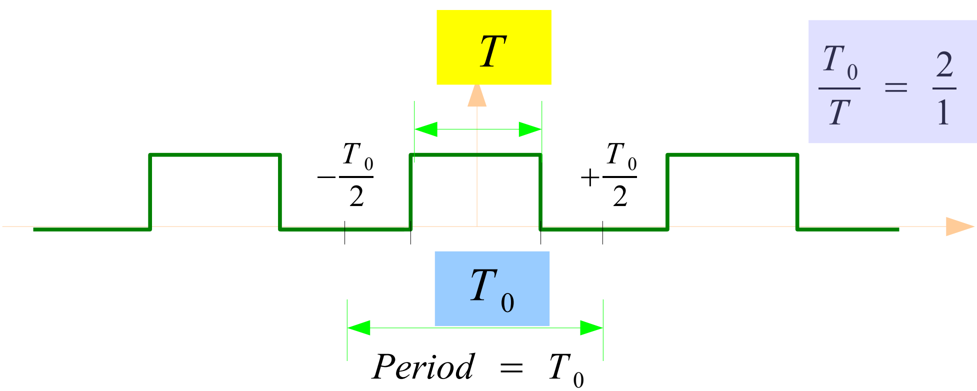
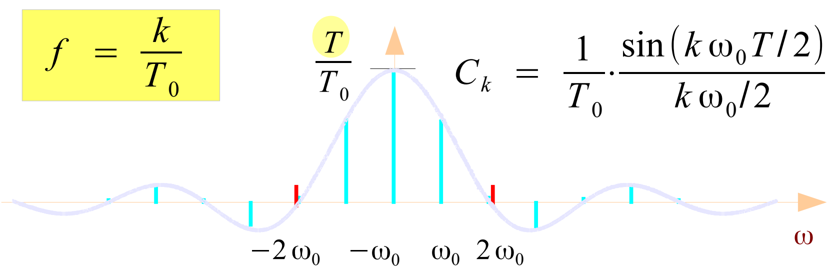
Continuous Time Fourier Series

CTFS

$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x_{T_0}(t) = \sum_{k=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$



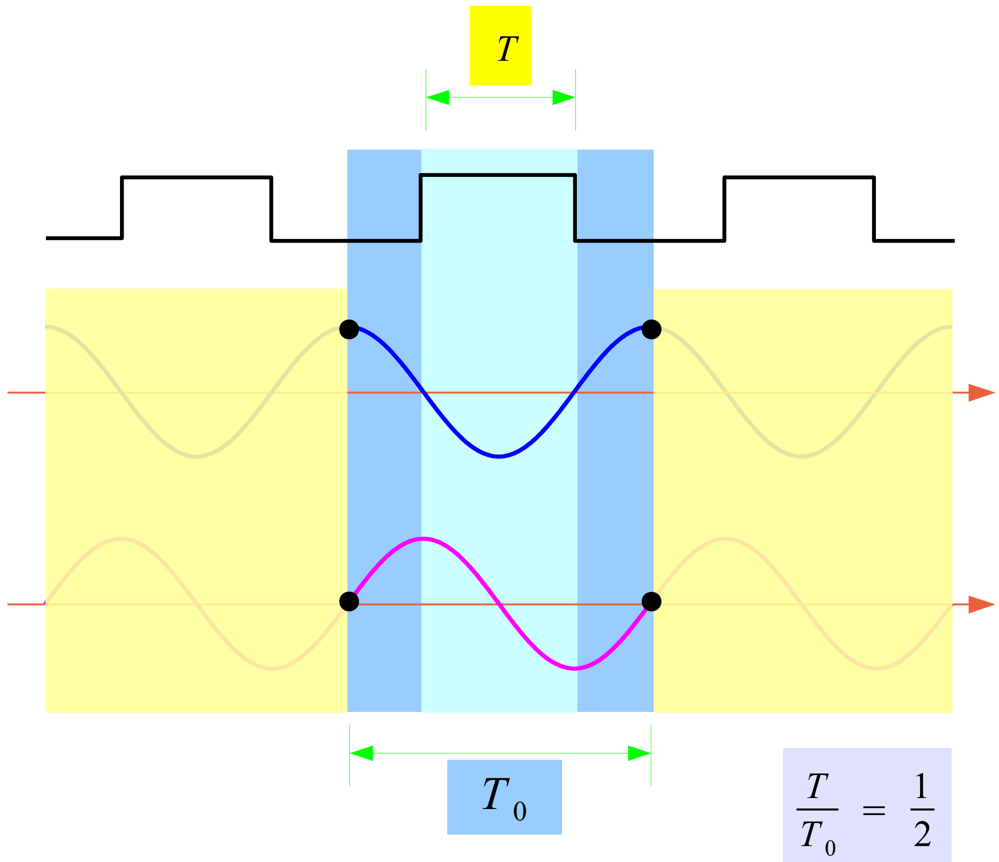


CTFS (Continuous Time Fourier Series)

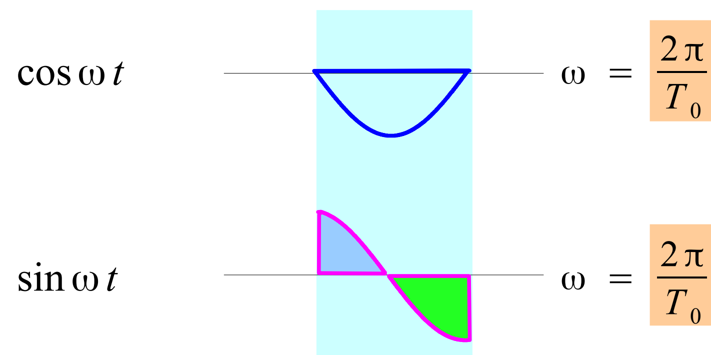
$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{T}{T_0}\cdot \mathbf{sinc}\left(k\frac{T}{T_0}\right)$$

$$C_0 = \frac{T}{T_0}$$

Zero Crossings of $T/T_0 \, sinc\left(f\,T/T_0\right)$ (1)



$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

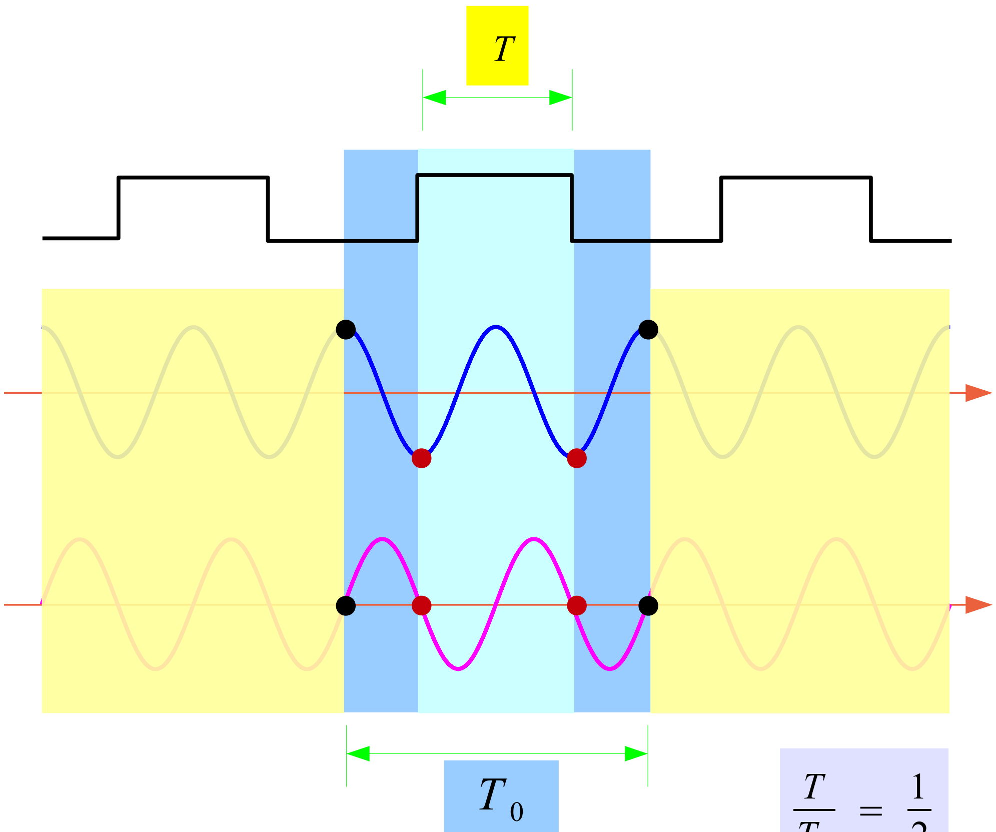


$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{T}{T_0} \cdot sinc\left(k\frac{T}{T_0}\right)$$

Non Zeros at odd k's $\quad \omega = \pi k$

$$C_1,\ C_3,\ C_5,\ \cdots \neq 0 \quad \Leftarrow \quad \omega = 2\pi k \frac{T}{T_0}$$

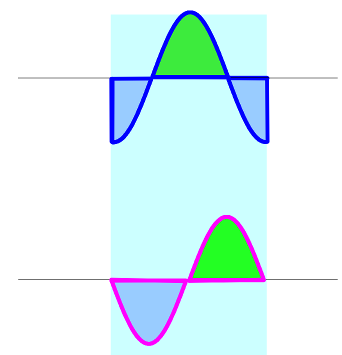
Zero Crossings of $T/T_0\, sinc\left(f\, T/T_0\right)$ (2)



$$\frac{T}{T_0} = \frac{1}{2}$$

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$\cos \omega t$ $\quad \omega = \dfrac{4\pi}{T_0}$



$\sin \omega t$ $\quad \omega = \dfrac{4\pi}{T_0}$

Zeros at <u>even</u> *k*'s $\quad \omega = \pi k$

$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{T}{T_0} \cdot sinc\left(k \frac{T}{T_0}\right)$$

$$C_2,\ C_4,\ C_6,\ \cdots = 0 \quad \Leftarrow \quad \omega = 2\pi k \frac{T}{T_0}$$

Periodic Pulse Train CTFS (1)

Continuous Time Fourier **Series**

$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x_{T_0}(t) = \sum_{k=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

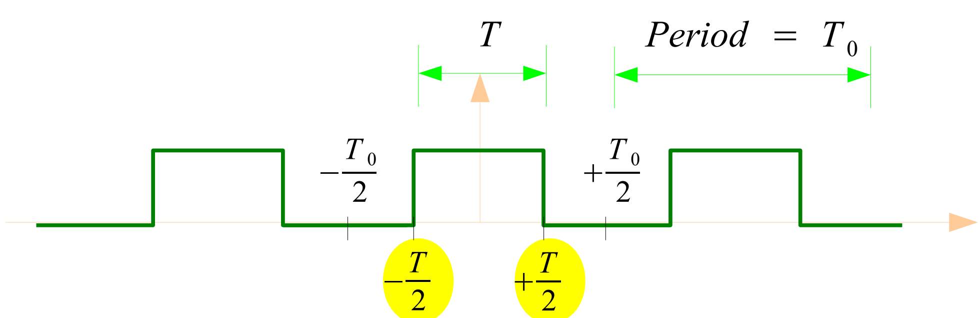
$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

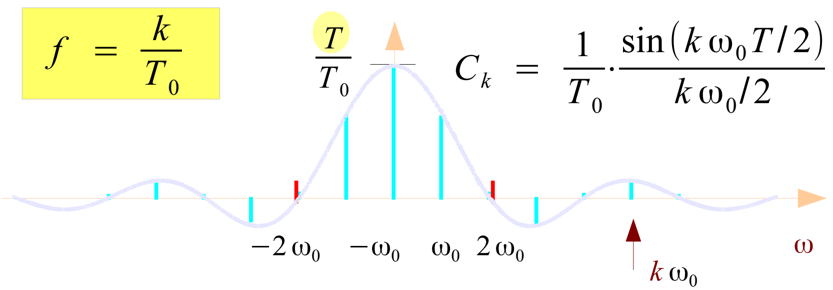
$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} e^{-jk\omega_0 t}\, dt = \left[\frac{-1}{jk\omega_0} e^{-jk\omega_0 t}\right]_{-T/2}^{+T/2}$$

$$= -\frac{e^{-jk\omega_0 T/2} - e^{+jk\omega_0 T/2}}{jk\omega_0}$$

$$C_0 = \frac{1}{T_0}\frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$



$$f = \frac{k}{T_0}$$



Periodic Pulse Train CTFS (2)

$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{T}{T_0}\cdot\frac{\sin(k\pi T/T_0)}{k\pi T/T_0}$$

Normalized Sinc function

$$sinc(t) = \frac{\sin(\pi t)}{\pi t}$$

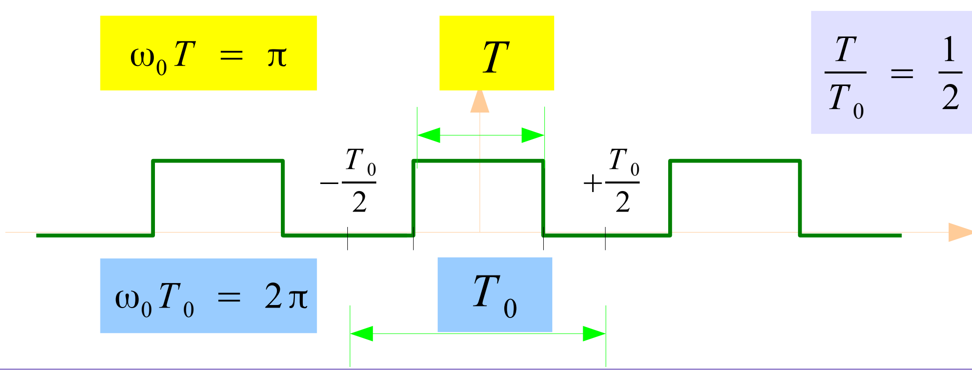
$$C_k = \frac{T}{T_0}\cdot sinc\left(k\frac{T}{T_0}\right)$$

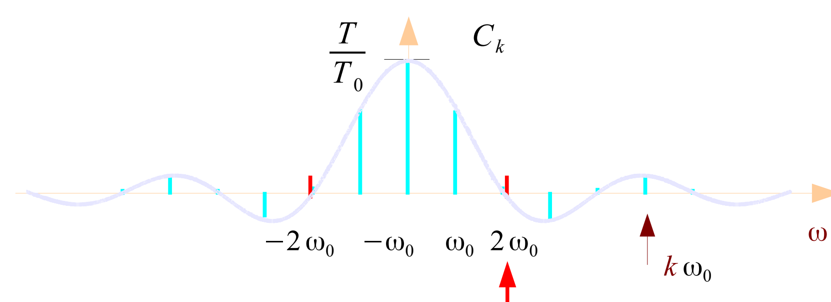
Zeros at $t = \pm 1,\ \pm 2,\ \cdots$

Zeros at $k\left(\frac{T}{T_0}\right) = \pm 1,\ \pm 2,\ \cdots$

$k = \pm 2,\ \pm 4,\ \cdots \quad \Leftarrow \quad \left(\frac{T}{T_0}\right)^{-1} = \left(\frac{1}{2}\right)^{-1}$

$\omega = \pm 2\omega_0,\ \pm 4\omega_0,\ \cdots \quad \Leftarrow \quad \omega_0 = \frac{2\pi}{T_0}$





Periodic Pulse Train CTFS (3)

Fundamental Frequency

$$\omega_0 = \frac{2\pi}{T_0} \qquad \omega_0 T_0 = 2\pi$$

Harmonic Frequencies

$$\omega = k\omega_0 \qquad \omega = 2\pi f = 2\pi \frac{k}{T_0}$$

$$f = \frac{k}{T_0}$$

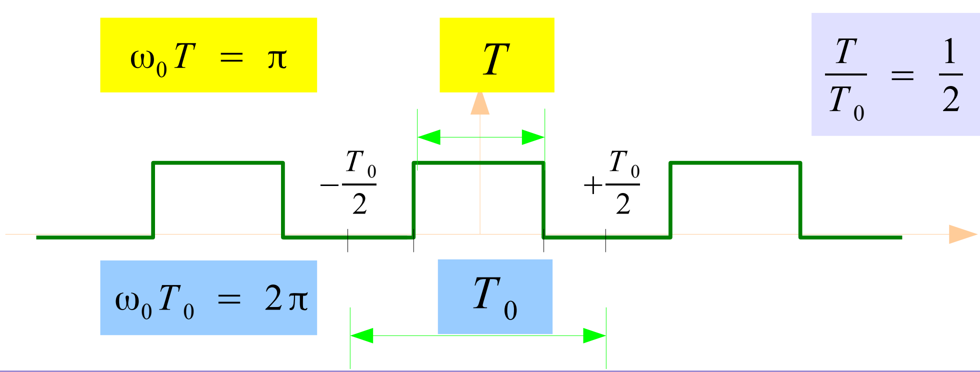
$$0 < T < T_0$$

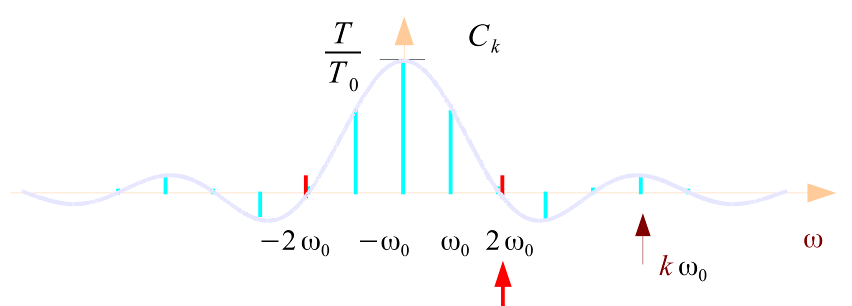
$$\omega_0 T = 2\pi \frac{T}{T_0} \qquad \omega_0 T = \pi \qquad \frac{T}{T_0} = \frac{1}{2}$$

$$k\frac{\omega_0 T}{2} = k\pi \frac{T}{T_0} \qquad \sin\left(k\frac{\omega_0 T}{2}\right) = \sin\left(k\pi \frac{T}{T_0}\right)$$

Zeros at $k = \pm 2, \pm 4, \cdots$

$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{1}{T_0} \cdot \frac{\sin(T k\omega_0/2)}{k\omega_0/2}$$





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Periodic Pulse Train CTFS (4)

$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{1}{T_0}\cdot\frac{\sin(T k\omega_0/2)}{k\omega_0/2}$$

$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\pi T/T_0)}{k\pi/T_0} = \frac{T}{T_0}\cdot\frac{\sin(k\pi T/T_0)}{k\pi T/T_0}$$

$$= \frac{T}{T_0}\cdot \textbf{sinc}\left(k\frac{T}{T_0}\right)$$

Zeros $\sin(k\omega_0 T/2) = 0$

$$k\omega_0 T/2 = \pm n\pi \quad \Rightarrow \quad k\frac{2\pi}{T_0}\frac{T}{2} = \pm n\pi$$

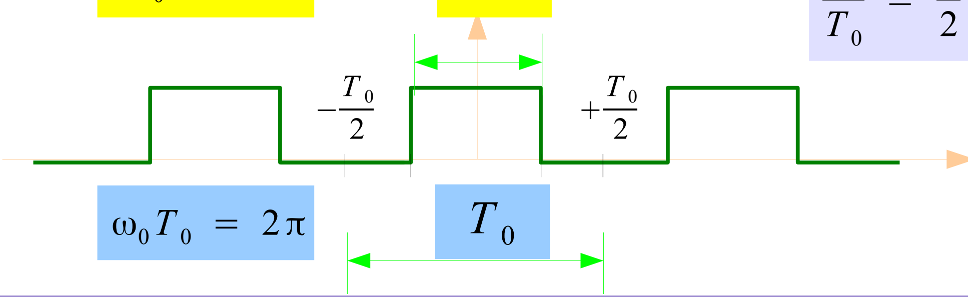
Zeros at $k = \pm n\left(\frac{T}{T_0}\right)^{-1}$

Zeros at $\omega = k\omega_0 = \pm n\left(\frac{T}{T_0}\right)^{-1}\omega_0 = \pm n\left(\frac{2\pi}{T}\right)$

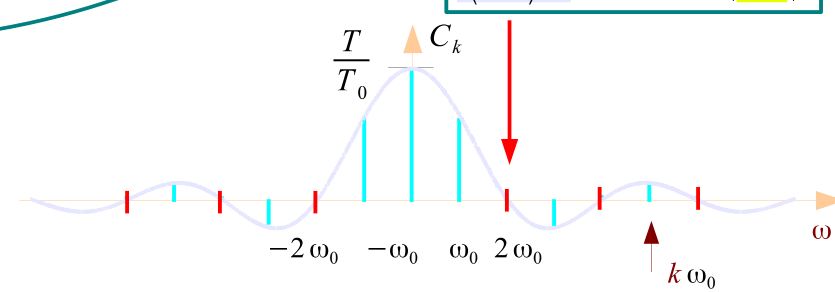
the first zero

$$\left(\frac{T}{T_0}\right)^{-1}\omega_0 = \pm n\left(\frac{2\pi}{T}\right)$$

$\omega_0 T = \pi$ T $\frac{T}{T_0} = \frac{1}{2}$



$\omega_0 T_0 = 2\pi$ T_0



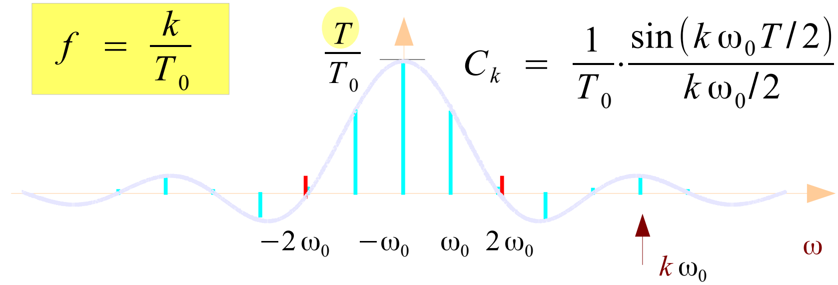
Periodic Pulse Train CTFS (4)

$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{T}{T_0}\cdot \textbf{sinc}\left(k\frac{T}{T_0}\right)$$

$$C_0 = \frac{T}{T_0}$$

$$f = \frac{k}{T_0}$$

$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

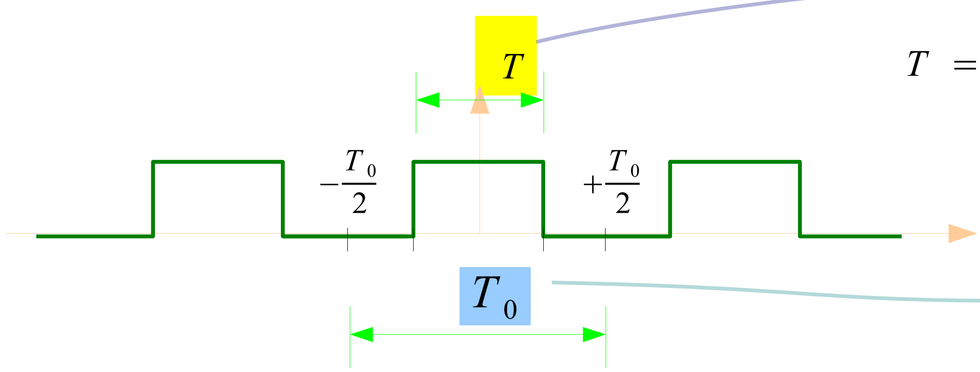


$$\omega_0 = \frac{2\pi}{T_0}$$

$$k\frac{\omega_0 T}{2} = k\pi\frac{T}{T_0}$$

Zeros at

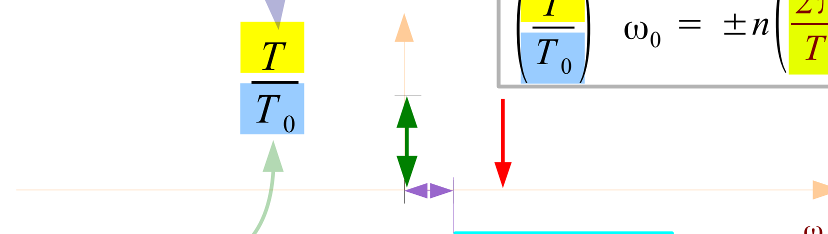
$$k = \pm n\left(\frac{T}{T_0}\right)^{-1} \Leftarrow k\left(\frac{T}{T_0}\right) = \pm 1,\ \pm 2,\ \cdots$$



$$T = T_0/2$$

the first zero

$$\left(\frac{T}{T_0}\right)^{-1}\omega_0 = \pm n\left(\frac{2\pi}{T}\right)$$



$$\omega_0 = \frac{2\pi}{T_0}$$

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Shifted Square Wave CTFS (1)

Continuous Time Fourier **Series**

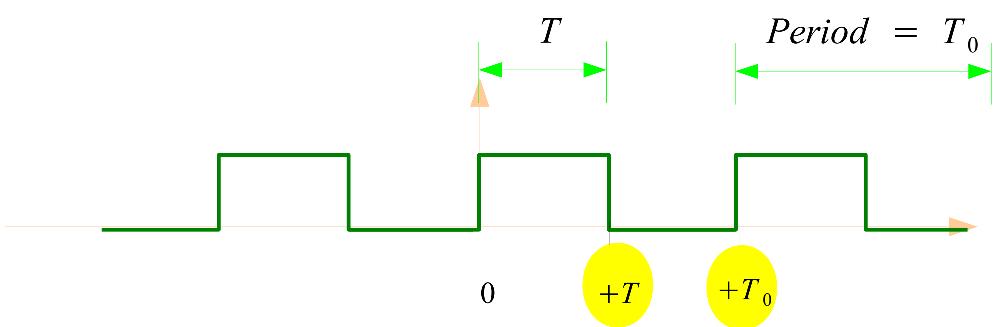
$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x_{T_0}(t) = \sum_{k=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_{0}^{+T_0} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k T_0 = \int_{0}^{+T_0} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \int_{0}^{+T} e^{-jk\omega_0 t}\, dt = \left[\frac{-1}{jk\omega_0} e^{-jk\omega_0 t}\right]_0^{+T}$$

$$= -\frac{e^{-jk\omega_0 T} - e^{0}}{jk\omega_0} = \frac{1 - e^{-jk\omega_0 T}}{jk\omega_0} = \frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi/T_0}$$



Fundamental Frequency

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0 T_0 = 2\pi$$
$$\omega_0 T = \pi$$

$$\frac{T_0}{T} = \frac{2}{1}$$

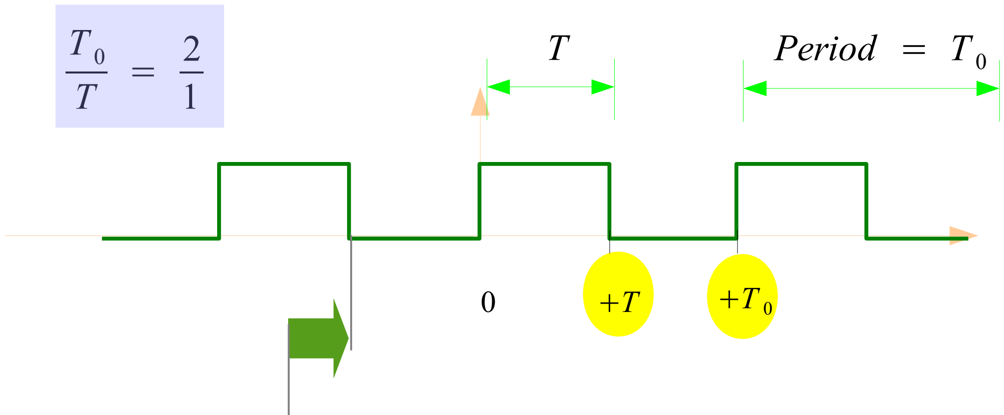
$$\omega_0 T = 2\pi \frac{T}{T_0}$$

$$C_k T_0 = \frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi/T_0} \Rightarrow C_k = \frac{1-(-1)^k}{j2\pi k}$$

$$C_0 T_0 = \int_0^{+T} e^{-j0\omega_0 t}\, dt = T \Rightarrow C_0 = \frac{1}{2}$$

C_{-4}	C_{-3}	C_{-2}	C_{-1}	C_0	C_1	C_2	C_3	C_4
0	$\frac{-1}{j3\pi}$	0	$\frac{-1}{j\pi}$	$\frac{1}{2}$	$\frac{1}{j\pi}$	0	$\frac{1}{j3\pi}$	0

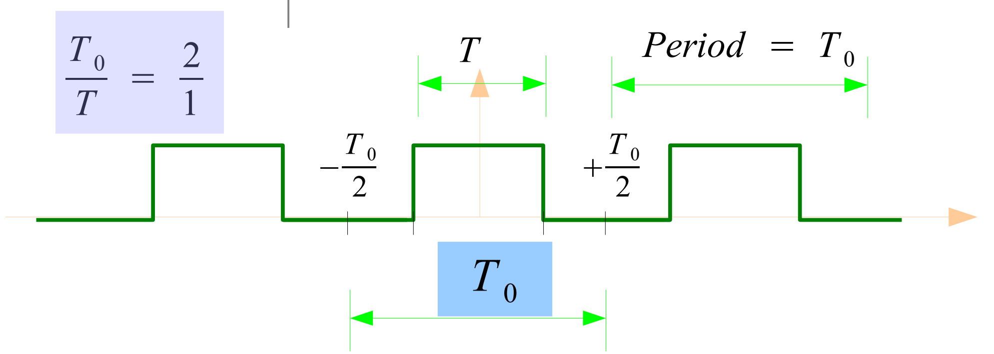
Shifted Square Wave CTFS (2)



$$C_k = \frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi}$$

$$C_0 = \frac{T}{T_0}$$

$e^{+jk\omega_0 T/2}$

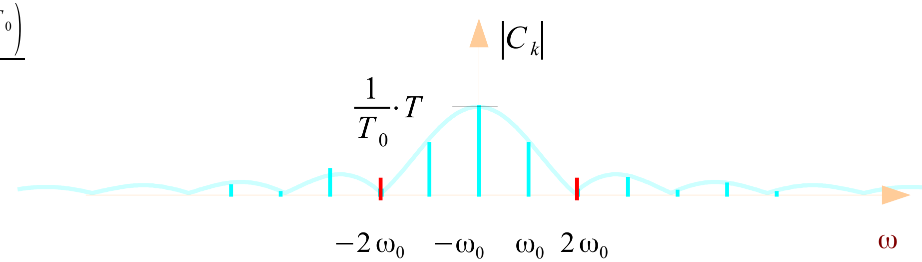


$$C_k = \frac{1}{T_0} \cdot \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$C_0 = \frac{T}{T_0}$$

$$\frac{1}{T_0} \cdot \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{\sin(k\pi T/T_0)}{k\pi} = \frac{\left(e^{+jk\pi T/T_0} - e^{-jk\pi T/T_0}\right)}{jk2\pi}$$

$$= e^{+jk\pi T/T_0}\left(\frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi}\right) = e^{+jk\omega_0 T/2}\left(\frac{1 - e^{-jk2\pi T/T_0}}{jk2\pi}\right)$$



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Spectrum of the CTFS of a Signal

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

Spectrum $C_k = |C_k| \, arg(C_k)$

Magnitude Spectrum $|C_k|$

Phase Spectrum $arg(C_k)$

Spectrum of the CTFS of a Real Signal

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_{-k} = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-j(-k)\omega_0 t}\, dt$$

$$C_k^* = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}^*(t)\, e^{+jk\omega_0 t}\, dt$$

a real signal $x_{T_0}(t) = x_{T_0}^*(t)$ ➡

- **magnitude: an even function** $|C_{-k}| = |C_k|$
- **phase: an even function** $arg(C_{-k}) = -arg(C_k)$

a real signal ➡ $x_{T_0}(t) = x_{T_0}^*(t)$

➡ $C_{-k} = C_k^*$

$$C_{-k} = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-j(-k)\omega_0 t}\, dt = C_k^*$$

Spectrum of the CTFS of a Real Even Signal

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_{-k} = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-j(-k)\omega_0 t}\, dt$$

$$C_k^* = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}^*(t)\, e^{+jk\omega_0 t}\, dt$$

a real even signal $x_{T_0}(t) = x_{T_0}^*(t) = x_{T_0}(-t)$

➡ **a real even spectrum** $C_k = C_k^* = C_{-k}$

a real even signal $x_{T_0}(t) = x_{T_0}^*(t) = x_{T_0}(-t)$

$$C_k^* = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}^*(t)\, e^{+jk\omega_0 t}\, dt = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(-t)\, e^{-jk\omega_0(-t)}\, dt = C_k$$

$$C_{-k} = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{+jk\omega_0 t}\, dt = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(-t)\, e^{-jk\omega_0(-t)}\, dt = C_k$$

Spectrum of a Periodic Pulse Train

Magnitude Spectrum of a Periodic Pulse Train

Phase Spectrum of a Periodic Pulse Train

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Spectrum Plots of a Shifted Periodic Pulse Train

Magnitude Spectrum Plots of a Shifted Periodic Pulse Train

Phase Spectrum Plots of a Shifted Periodic Pulse Train

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] G. Beale, http://teal.gmu.edu/~gbeale/ece_220/fourier_series_02.html
[4] C. Langton, http://www.complextoreal.com/chapters/fft1.pdf