Angle Recoding CORDIC
2. Wu

20180414

Extended EAS (EEAS) − Wu

more flexible way of decomposing the rotation angle

better
the number of iterations
the error performance

$$S_{EAS} = \{ (\sigma \cdot \tan^{-1}(2^{-r})) : \sigma \in \{+1, 0, -1\}, \ r \in \{1, 2, \cdots, n-1\} \}$$

$$S_{EEAS} = \{ (\sigma_1 \cdot \tan^{-1}(2^{-r_1}) + \sigma_2 \cdot \tan^{-1}(2^{-r_2})) : \sigma_1, \sigma_2 \in \{+1, 0, -1\}, \ r_1, r_2 \in \{1, 2, \cdots, n-1\} \}$$

The pseudo-rotation
for i-th micro rotations

$$x_{i+1} = x_i - [\sigma_1(i) \cdot 2^{-r_1(i)} + \sigma_2(i) \cdot 2^{-r_2(i)}] \, y_i$$
$$y_{i+1} = y_i + [\sigma_1(i) \cdot 2^{-r_1(i)} + \sigma_2(i) \cdot 2^{-r_2(i)}] \, x_i$$

The pseudo-rotated vector $[x_{R_m}, y_{R_m}]$
after R_m (the required number of micro-rotations)

Needs to be scaled by a factor $K = \prod K_i$

$$K_i = \left[1 + \left(\sigma_1(i) \cdot 2^{-r_1(i)} + \sigma_2(i) \cdot 2^{-r_2(i)} \right)^2 \right]^{-\frac{1}{2}}$$

$$\tilde{x}_{i+1} = \tilde{x}_i - [k_1(i) \cdot 2^{-s_1(i)} + k_2(i) \cdot 2^{-s_2(i)}] \, \tilde{y}_i$$
$$\tilde{y}_{i+1} = \tilde{y}_i + [k_1(i) \cdot 2^{-s_1(i)} + k_2(i) \cdot 2^{-s_2(i)}] \, \tilde{x}_i$$

$\tilde{x}_0 = x_{R_m}$ $\qquad k_1, k_2 \in \{-1, 0, 1\}$

$\tilde{y}_0 = y_{R_m}$ $\qquad s_1, s_2 \in \{1, 2, \dots, n-1\}$

[21] C.-S. Wu, A.-Y. Wu, and C.-H. Lin, "A high-performance/low-latency vector rotational CORDIC architecture based on extended elementary angle set and trellis-based searching schemes," *IEEE Trans. Circuits Syst. II: Anal. Digital Signal Process.*, vol. 50, no. 9, pp. 589–601, Sep. 2003.

IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS—I: FUNDAMENTAL THEORY AND APPLICATIONS, VOL. 49, NO. 10, OCTOBER 2002

A Unified View for Vector Rotational CORDIC Algorithms and Architectures Based on Angle Quantization Approach

An-Yeu Wu and Cheng-Shing Wu

AR : to approximate θ
with the combination
of selected angle elements
from a pre-defined **EAS**
(Elementary Angle Set)

EAS : all possible values of $\theta(j)$

$$\text{EAS } S_1 = \{ \tan^{-1}(\alpha^* \cdot 2^{-s^*}) : \alpha^* \in \{-1, 0, +1\}, \; s^* \in \{0, 1, \cdots, N-1\} \}$$

EAS S_1 consists of $\tan^{-1}$(Single Signed Power of Two)
$\tan^{-1}$(single SPT)
$\tan^{-1}(\alpha^* \cdot 2^{-s^*})$

SPT-based digital filter design

to increase the coefficient resolution

→ imploy more SPT terms to represent filter coefficients

[12] H. Samueli, "An improved search algorithm for the design of multiplierless FIR filters with power-of-two coefficients," *IEEE Trans. Circuits Syst.*, vol. 36, pp. 1044–1047, July 1989.
[13] Y. C. Lim, R. Yang, D. Li, and J. Song, "Signed power-of-two term allocation scheme for the design of digital filters," *IEEE Trans. Circuits Syst. II*, vol. 46, pp. 577–584, May 1999.

EAS $\hat{S}_1$ consists of $\tan^{-1}$(single signed power of two)

$\tan^{-1}$(single SPT)

$\tan^{-1}(\alpha^* \cdot 2^{-s^*})$

EAS $\hat{S}_2$ consists of $\tan^{-1}$(two signed power of two)

$\tan^{-1}$(two SPT)

$\tan^{-1}(\alpha_0^* \cdot 2^{-s_0^*} + \alpha_1^* \cdot 2^{-s_1^*})$

Two Signed-Power-of-Two terms

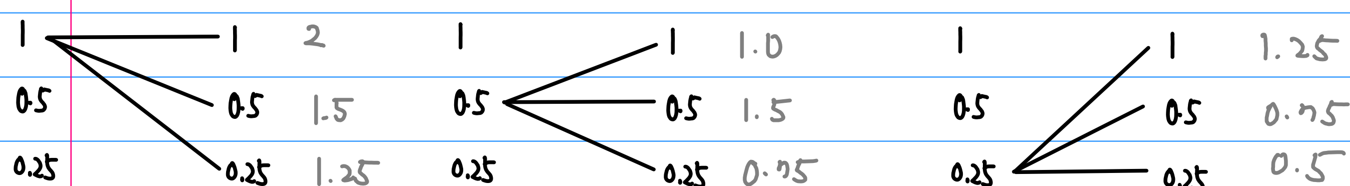
$$S_2 = \{ \tan^{-1}(\alpha_0^* \cdot 2^{-s_0^*} + \alpha_1^* \cdot 2^{-s_1^*}) : \alpha_0^*, \alpha_1^* \in \{-1, 0, +1\}, \; s_0^*, s_1^* \in \{0, 1, \cdots, w-1\} \}$$

S_1

1	$1 = 2^{-0}$	$\tan^{-1}(2^{-0})$
0.5	$\frac{1}{2} = 2^{-1}$	$\tan^{-1}(2^{-1})$
0.25	$\frac{1}{4} = 2^{-2}$	$\tan^{-1}(2^{-2})$

S_2

2	$1+1 = 2^{-0} + 2^{-0}$	$\pm\tan^{-1}(2^{-0} + 2^{-0})$
1.5	$1+\frac{1}{2} = 2^{-0} + 2^{-1}$	$\pm\tan^{-1}(2^{-0} + 2^{-1})$
1.25	$1+\frac{1}{4} = 2^{-0} + 2^{-2}$	$\pm\tan^{-1}(2^{-0} + 2^{-2})$
1.0	$1 = 2^{-0}$	$\pm\tan^{-1}(2^{-0})$
0.75	$\frac{1}{2}+\frac{1}{4} = 2^{-1} + 2^{-2}$	$\pm\tan^{-1}(2^{-1} + 2^{-2})$
0.5	$\frac{1}{2} = 2^{-1}$	$\pm\tan^{-1}(2^{-1})$
0.25	$\frac{1}{4} = 2^{-2}$	$\pm\tan^{-1}(2^{-2})$



$2^{-0}, \quad 2^{-1}, \quad 2^{-2}$

$\{0, 1, 2\} = \{0, 1, W-1\}$

$W=3$

$s_0^*, s_1^* \in \{0, 1, 2\}$

$2^{s_0^*}, 2^{s_1^*} \in \{2^{-0}, 2^{-1}, 2^{-2}\}$

as the word size W increases,

the size of the set S_2 increases exponentially

$$\theta_i = \tan^{-1}\left(\alpha_0(j)\cdot 2^{-S_0(j)} + \alpha_1(j)\cdot 2^{-S_1(j)}\right)$$

R_m: the number of the subangle N_A

$$S_2 = \{ \theta_i \mid i = 0, 1, \cdots R_m \}$$

$$S_2 = \{ \tan^{-1}\left(\alpha_0^*\cdot 2^{-S_0^*} + \alpha_1^*\cdot 2^{-S_1^*}\right) : \alpha_0^*, \alpha_1^* \in \{-1, 0, +1\}, \; S_0^*, S_1^* \in \{0, 1, \cdots, W-1\} \}$$

the optimization problem of the EEAS-based CORDIC algorithm

given θ and R_m

find $\alpha_0(j)$, $\alpha_1(j)$, $S_0(j)$, and $S_1(j)$

the combination of elementary angles

from EEAS $\hat{S}_2$

minimize the angle quantization error

$$|\xi_{m,\,EEAS}| \triangleq \theta - \sum_{j=0}^{R_m-1} \tan^{-1}\left(\alpha_0(j)\, 2^{-S_0(j)} + \alpha_1(j)\, 2^{-S_1(j)}\right)$$

given $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$

$$\begin{bmatrix} x(j+1) \\ y(j+1) \end{bmatrix} = \begin{bmatrix} 1 & \alpha_0(j)\,2^{-S_0(j)} + \alpha_1(j)\,2^{-S_1(j)} \\ \alpha_0(j)\,2^{-S_0(j)} + \alpha_1(j)\,2^{-S_1(j)} & 1 \end{bmatrix} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix}$$

$$\begin{bmatrix} x_f \\ y_f \end{bmatrix} = P \begin{bmatrix} x(R_m) \\ y(R_m) \end{bmatrix} = \frac{1}{\prod_{j=0}^{R_s-1} \sqrt{1 + [\alpha_0(j) \cdot 2^{-S_0(j)} + \alpha_1(j) \cdot 2^{-S_1(j)}]^2}} \begin{bmatrix} x(R_m) \\ y(R_m) \end{bmatrix}$$

micro rotation procedure

the scaling operation

4 additions

increased hardware

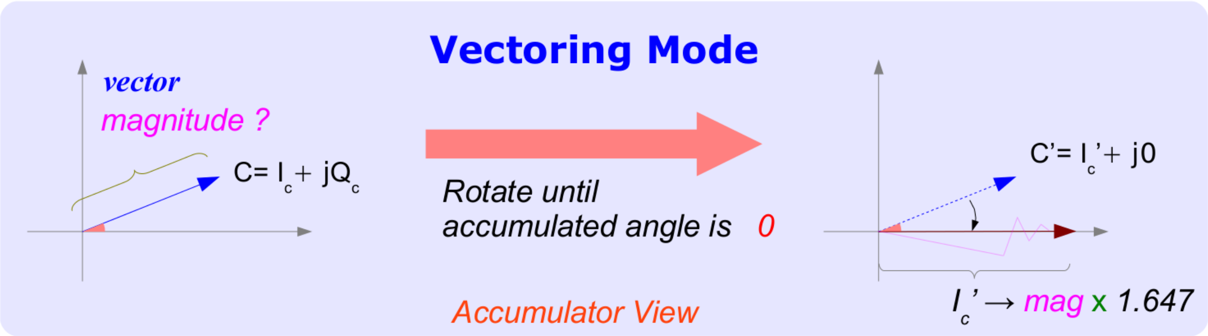
reduced iteration steps

MVR (Modified Vector Rotation)

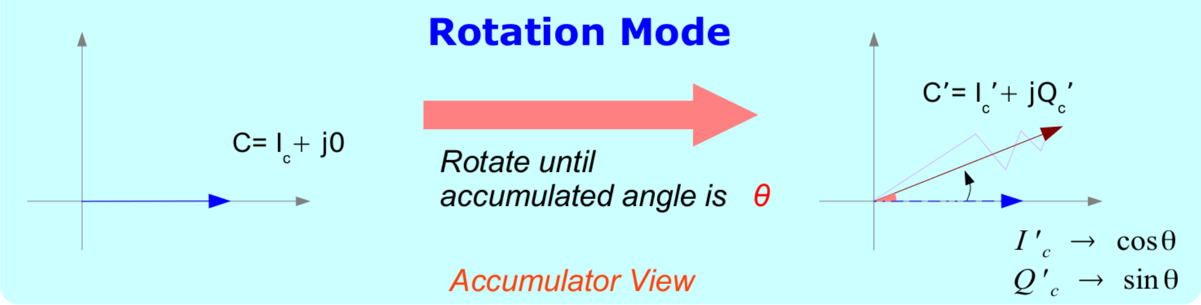
1) Repeat of Elementary Angles θ_i, σ_i

2) fixed total micro-rotation Number R_m

* Vector Rotation Mode
* and the rotation angles are known in advance



vector rotation mode



Modified Vector Rotational MVR CORDIC

- reduce the iteration number
- maintaining the SQNR performance
- modifying the basic microrotation procedure

Three Searching Algorithm

1. the selective prerotation
2. the selective scaling
3. iteration-tradeoff scheme

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Angle Quantization

Quantization process on the rotational angle θ

decompose θ into several subangles θ_i's

the angle quantization error

$$\xi_m \triangleq \theta - \sum_{i=0}^{N_A-1} \theta_i$$

N_A the number of subangles

$\theta_0, \theta_1, \cdots, \theta_{N_A-1}$

$$\theta = \theta_0 + \theta_1 + \cdots + \theta_{N_A-1} + \xi_m$$

data : W-bit word length

the iteration number : N $\qquad N \leqslant W$

the restricted iteration number : R_m $\qquad R_m \ll W$

AQ Process : 2 Design Issues

① need to determine the sub-angles θ_i

② select / combine subangles
to minimize the angle quantization angle ξ_m

Vector Rotation CORDIC Family

⓪ Conventional CORDIC

① AR

② MVR

③ EEAS

⓪ Conventional CORDIC

elementary angle $a(i) = \tan^{-1}(2^{-i})$

the number of elementary angles N

the rotation sequence $\mu(i) = \{-1, +1\}$

+1, -1, -1, +1, +1, ...

the i-the rotation angle $a(i)$

the W-bit word length

the iteration number $N \leq W$

the angle quantization error

$$\xi_{m,\,CORDIC} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\, a(i)$$

① AR [Hu]

skip certain micro rotations

the rotation sequence $\mu(i) = \{-1, 0, +1\}$

$\mu(i) = 0 \rightarrow$ skip

desire to minimize

$$\boxed{\sum_{i=0}^{N} |\mu(i)|}$$

so that the total number of CORDIC iterations can be minimized

Angle Recoding ← Multiplier Recoding

angle recoding method for efficient
implementation of the CORDIC algorithm
Hu & Naganathan, ISCAS 89

Greedy algorithm

$\theta(0) = \theta$, $\{\mu(i) = 0,\ 0 \leq i \leq N-1\}$, $k = 0$.

repeat until $|\theta(k)| < a(N-1)$ Do

Choose i_k, $0 \leq i_k \leq N-1$

$$\big|\,|\theta(k)| - a(i_k)\big| = \min_{0 \leq i \leq N-1} \big|\,|\theta(k)| - a(i_k)\big|$$

$$\theta(k+1) = \theta(k) - \mu(i_k)\, a(i_k)$$

$$\mu(i_k) = \operatorname{Sign}(\theta(k))$$

try to approach the target rotation angle θ
step by step

decisions are made in each step
by choosing the best combination of $\alpha(i)\, a(s(i))$

so as to minimize $|\xi_m|$

$\alpha(i)$, $a(i)$ are determined such that
the error function is minimized

$$J(i) = |\theta(i) - \alpha(i)\, a(s(i))|$$

$$\theta(i) = \theta - \sum_{m=0}^{i-1} \alpha(m)\, a(s(m))$$

terminated if no further improvement can be found

$$J(i) \geq J(i-1)$$

or $\alpha(R_m - 1)$ and $s(R_m - 1)$
are determined at the end

$$\xi_{m,\,CORDIC} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\, a(i) \qquad \mu(i) = \{-1, 0, +1\}$$

$$= \theta - \sum_{j=0}^{N'} \tilde{\theta}(j)$$

$$N' \equiv \sum_{i=0}^{N-1} |\mu(i)| \qquad \{ |+1|, |0|, |+1| \}$$

the effective iteration number N'

$S(j)$ the rotational sequence

determines the micro-rotation angle in the j-th iteration

$i \leftarrow S(j)$

$S(j)$: index $0, 1, \cdots, N'$ (reduced number)

i: index $0, 1, \cdots, N$ (original number)

$\mu(S(j)) \leftarrow \alpha(j)$

$\Downarrow \qquad \Uparrow$

$\{-1, +1\}$

$$\mu(i) = \begin{cases} \mu(S(j)) & i = S(j) \\ 0 & i \neq S(j) \end{cases}$$ --- reduced index

er

$i = 0,\ \boxed{1,\ 2,}\ 3, \cdots, N-1$

$s(j) = 0,\ \boxed{1,\ 2,}\ 3, \cdots, N-1$ rotational sequence

$\alpha(j) = -1,\ \boxed{0,\ 0,}\ +1, \cdots, -1$ directional sequence

$j = 0,\ -,\ -,\ 1, \cdots, N'-1$ effective iteration number

$N' = N-2$

the j-th micro-rotation of $a(s(j))$

elementary angle

$$a(i) = \tan^{-1}(2^{-i})$$

$$a(s(j)) = \tan^{-1}(2^{-s(j)})$$

$$\alpha(j)\,a(s(j)) = \alpha(j)\,\tan^{-1}(2^{-s(j)}) \qquad \alpha(j) \in \{-1, +1\}$$

$$\Leftrightarrow \mu(i)\,a(i) \qquad \mu(i) \in \{-1, 0, +1\}$$

$$\xi_{m,\,CORDIC} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\, a(i) \qquad \mu(i) \in \{-1, 0, +1\}$$

$$= \theta - \left[\sum_{j=0}^{N'} \tilde{\theta}(j) \right]$$

$$= \theta - \left[\sum_{j=0}^{N'} \tan^{-1}\left(\alpha(j) \cdot 2^{-s(j)} \right) \right] \qquad \alpha(j) \in \{-1, +1\}$$

$$\tilde{\theta}(j) = \alpha(j) \tan^{-1}\left(2^{-s(j)} \right)$$

$$= \tan^{-1}\left(\alpha(j) \cdot 2^{-s(j)} \right)$$

$$S_1 = \left\{ \tan^{-1}\left(\boxed{\alpha} \cdot 2^{-\boxed{s}} \right) \quad \boxed{\alpha} \in \{-1, 0, +1\},\ \boxed{s} \in \{0, 1, 2, \cdots N-1\} \right\}$$

② MVR (Modified Vector Rotational)

two modifications

① repeat of elementary angles

each micro-rotation of elementary angle can be performed repeatedly

- more possible combinations
- smaller ξ_m

② confines of total micro-rotation number

confine the iteration number in the micro-rotation phase to R_m ($R_m \ll W$)

The role of R_m is quite similar to the number of non-zero digit N_D in CSD recoding scheme

$$\xi_{m,MVR} \triangleq \theta - \sum_{i=0}^{R_m-1} \alpha(i)\, a(s(i))$$

the rotational sequence

$$s(i) \in \{0, 1, \cdots, W-1\}$$

determines the micro-rotation angle in the i-th iteration

the directional sequence

$$\alpha(i) \in \{-1, 0, +1\}$$

controls the direction of the i-th micro-rotation of $a(s(i))$

$$\alpha(i)\, a(s(i)) = \tilde{\theta}(j)$$

$$\xi_{m,MVR} \triangleq \theta - \sum_{j=0}^{R_m-1} \alpha(j)\, a(s(j))$$

the rotational sequence

$$s(j) \in \{0, 1, \cdots, W-1\}$$

determines the micro-rotation angle in the j-th iteration

the directional sequence

$$\alpha(j) \in \{-1, 0, +1\}$$

controls the direction of the j-th micro-rotation of $a(s(j))$

$$\alpha(j)\, a(s(j)) = \tilde{\theta}(j)$$

$i =$	$0,$	$1,$	$2,$	$3,$	$\cdots,$	$W-1$		
$s(j) =$	$0,$	$1,$	$2,$	$3,$	$\cdots,$	$W-1$	rotational sequence	
$\alpha(j) =$	$-1,$	$0,$	$0,$	$+1,$	$\cdots,$	-1	directional sequence	
$j =$	$0,$	$-,$	$-,$	$1,$	$\cdots,$	R_m-1	effective iteration number	

$R_m \ll W$