Sequence (4A)

- Closed form expression
- Recurrence expression
- Mathematical Induction

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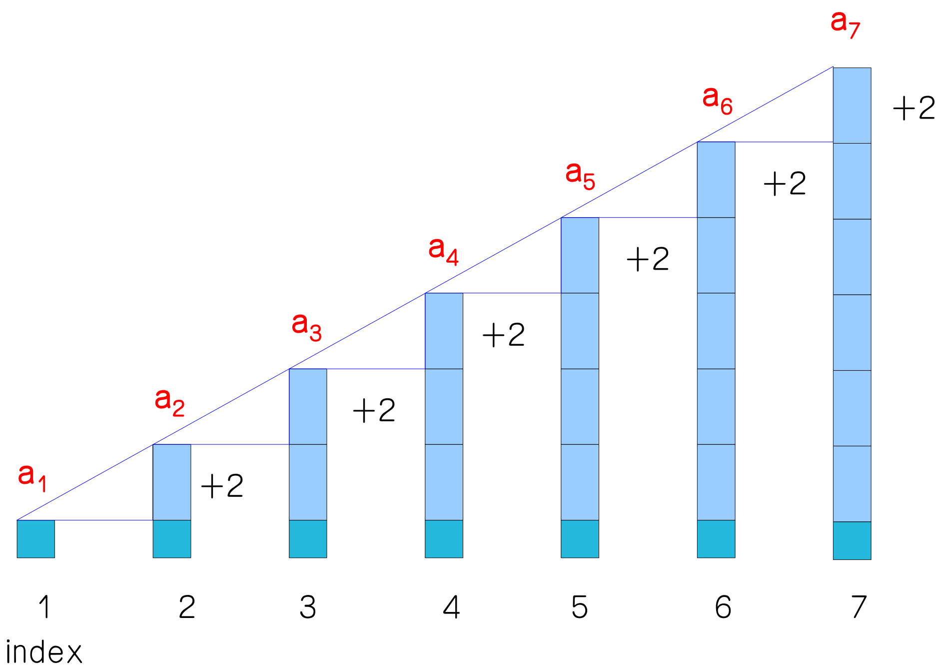
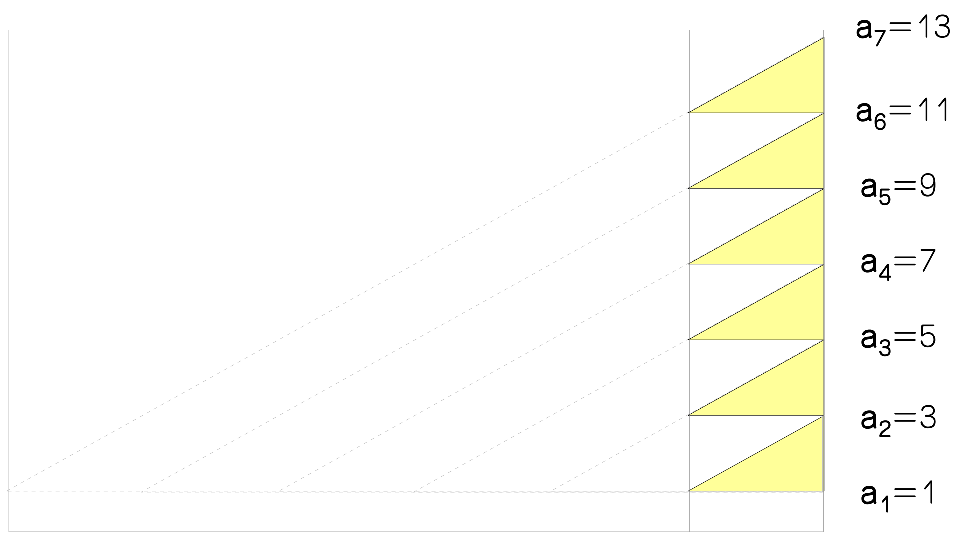
Arithmetic Progression – Closed Form Expression

- common difference: $d = 2$
- first term: $a = 1$

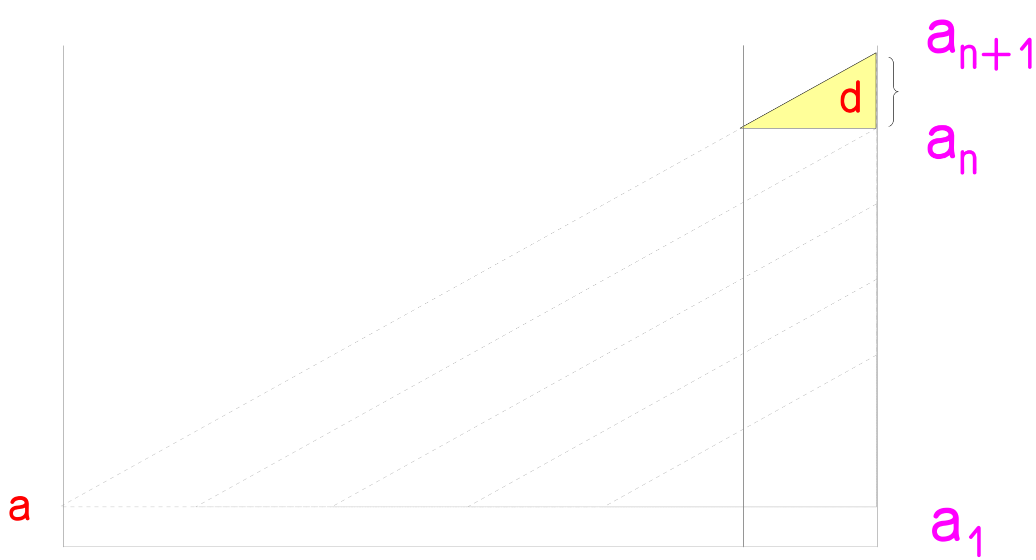




$$a_n = a + (n-1) \cdot d$$ ◀ closed form expression

$(n = 1, 2, 3, \cdots)$

- common difference: d
- first term: a



◀ recurrence expression

$$\begin{cases} a_{n+1} = a_n + d \\ a_1 = a \end{cases}$$

$(n = 1, 2, 3, \cdots)$

(+) operation

◀ closed form expression

$$a_n = a + (n-1)\cdot d$$

$(n = 1, 2, 3, \cdots)$

(X, +) operations

◀ recurrence expression

$$\begin{cases} a_{n+1} = a_n + d \\ a_1 = a \end{cases} \quad (n = 1, 2, 3, \cdots)$$

- common difference: d
- first term: a

$$\begin{aligned} a_5 &= 2 + a_4 \\ &= 2 + (2 + a_3) \\ &= 2 + (2 + (2 + a_2)) \\ &= 2 + (2 + (2 + (2 + a_1))) \\ &= 2 + (2 + (2 + (2 + 1))) \\ &= 2 + (2 + (2 + 3)) \\ &= 2 + (2 + 5) \\ &= 2 + 7 \\ &= 9 \end{aligned}$$

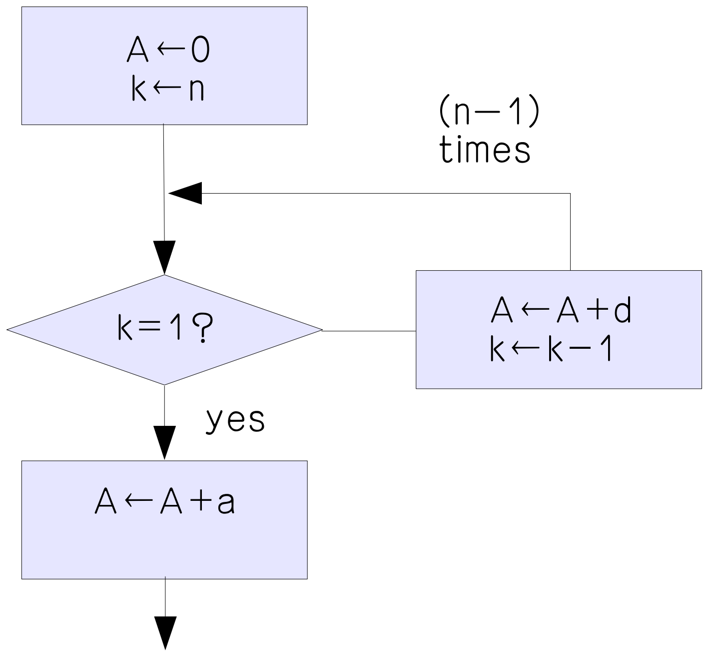
$$\begin{aligned} a_4 &= 2 + a_3 \\ a_3 &= 2 + a_2 \\ a_2 &= 2 + a_1 \\ a_1 &= 1 \\ a_2 &= 3 \\ a_3 &= 5 \\ a_4 &= 7 \\ a_5 &= 9 \end{aligned}$$

◀ closed form expression

$$a_n = a + (n-1) \cdot d$$

$(n = 1, 2, 3, \cdots)$

- common difference: d
- first term: a

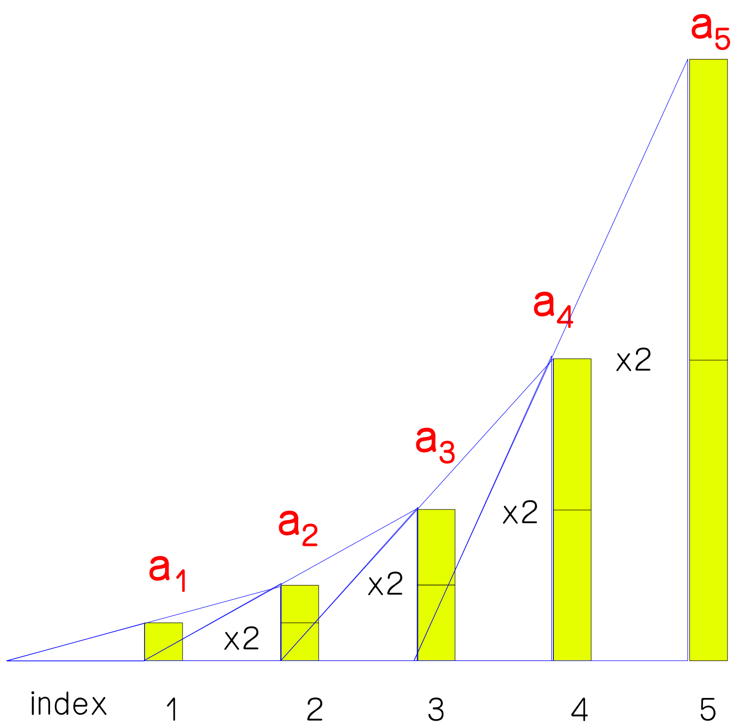


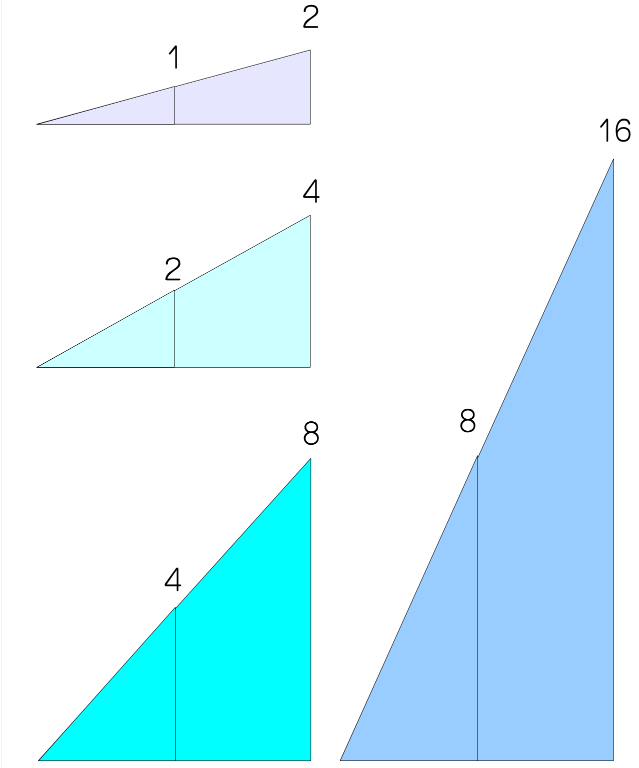
$$a_5 = \underbrace{2 + 2 + 2 + 2}_{(n-1)\ \text{times}} + 1$$

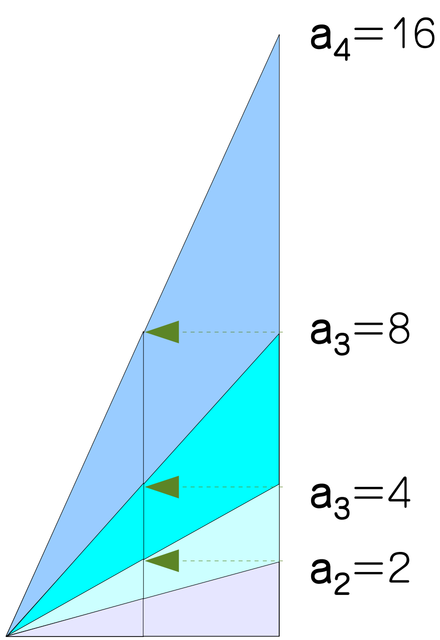
Geometric Progression – Closed Form Expression

- common ratio: $r = 2$
- first term: $a = 1$

$$a_{n+1} : a_n = r : 1$$



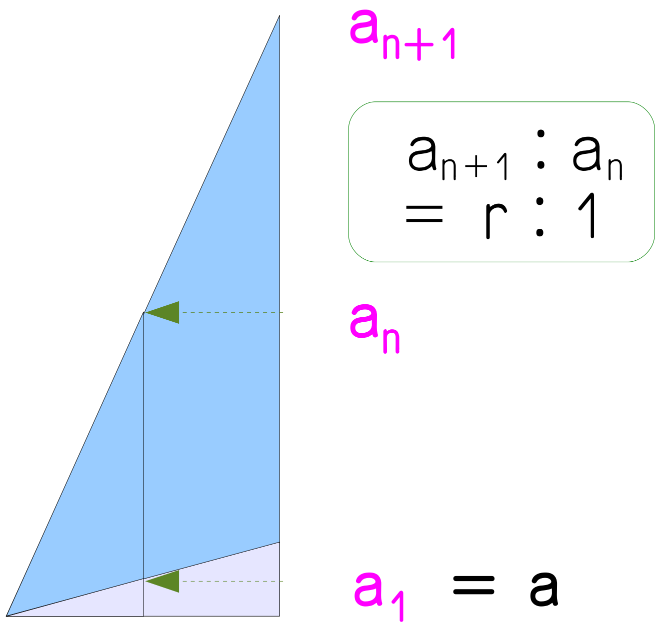




$$a_n = a \cdot r^{n-1}$$

◀ closed form expression

- common ratio: r
- first term: a



◀ recurrence expression

$$\begin{cases} a_{n+1} = r \cdot a_n \\ a_1 = a \end{cases}$$

$(n = 1, 2, 3, \cdots)$

(X) operation

◀ closed form expression

$$a_n = a \cdot r^{n-1}$$

$(n = 1, 2, 3, \cdots)$

(X, Exp) operations

Geometric Progression – Recursive Computation

◀ recurrence expression

$$\begin{cases} a_{n+1} = r \cdot a_n \\ a_1 = a \end{cases}$$

$(n = 1, 2, 3, \cdots)$

- common ratio: r
- first term: a

$$\begin{aligned} a_5 &= 2 \cdot a_4 \\ &= 2 \cdot (2 \cdot a_3) \\ &= 2 \cdot (2 \cdot (2 \cdot a_2)) \\ &= 2 \cdot (2 \cdot (2 \cdot (2 \cdot a_1))) \\ &= 2 \cdot (2 \cdot (2 \cdot (2 \cdot 1))) \\ &= 2 \cdot (2 \cdot (2 \cdot 2)) \\ &= 2 \cdot (2 \cdot 4) \\ &= 2 \cdot 8 \\ &= 16 \end{aligned}$$

$$\begin{aligned} a_4 &= 2 \cdot a_3 \\ a_3 &= 2 \cdot a_2 \\ a_2 &= 2 \cdot a_1 \\ a_1 &= 1 \\ a_2 &= 2 \\ a_3 &= 4 \\ a_4 &= 8 \\ a_5 &= 16 \end{aligned}$$

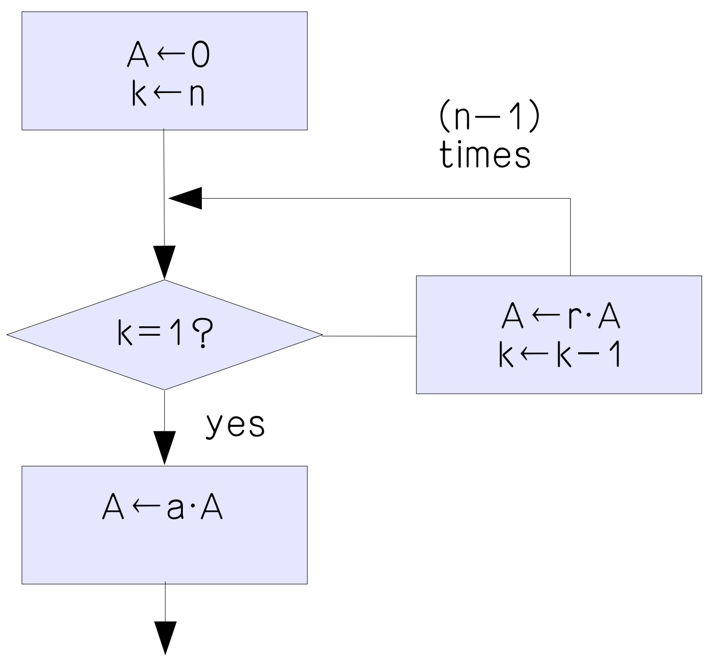
Geometric Progression – Iterative Computation

◀ closed form expression

$$a_n = a \cdot r^{n-1}$$

$(n = 1, 2, 3, \cdots)$

- common ratio: r
- first term: a



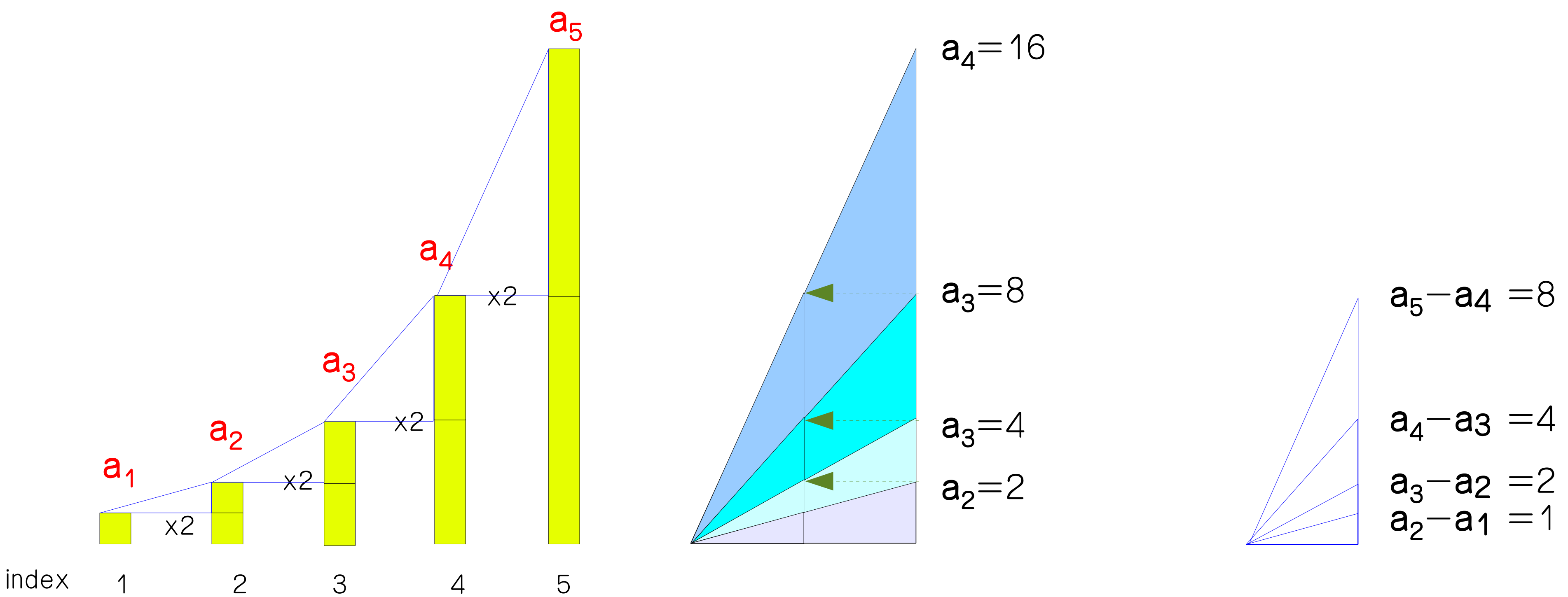
$$a_5 = \underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{(n-1)\ \text{times}} \cdot 1$$

Geometric Progression – Difference (1)

- common ratio: $r = 2$
- first term: $a = 1$

$$\frac{a_{n+1}}{a_n} = r$$

$$\frac{(a_{n+2} - a_{n+1})}{(a_{n+1} - a_n)} = r$$

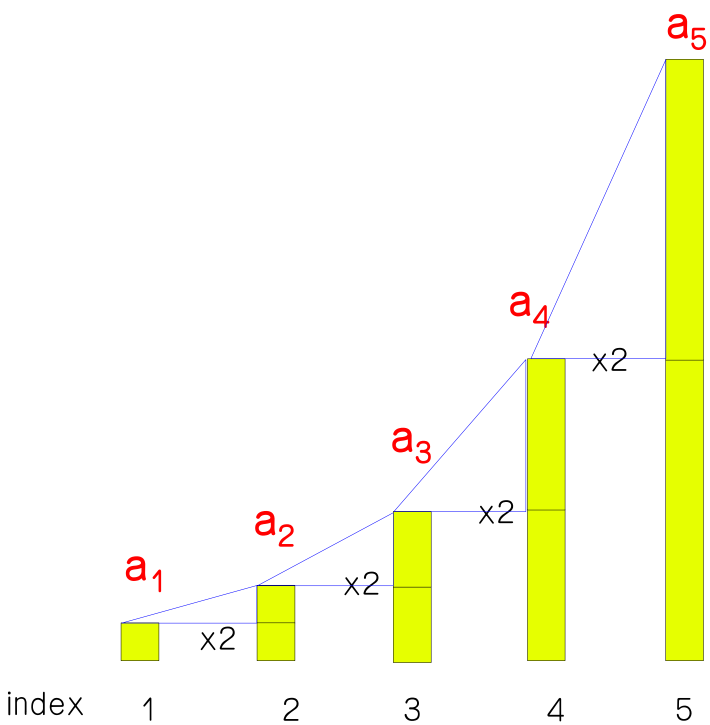


$$a_n = a \cdot r^{n-1}$$

◀ closed form expression

Geometric Progression – Difference (2)

- common ratio: $r = 2$
- first term: $a = 1$



$$a_1 = a$$
$$a_2 = a \cdot r$$
$$a_3 = a \cdot r^2$$
$$a_4 = a \cdot r^3$$
$$a_5 = a \cdot r^4$$

$$a_n = a \cdot r^{n-1}$$

$$b_1 = a_2 - a_1 = a \cdot (r-1)$$
$$b_2 = a_3 - a_2 = a \cdot (r-1) \cdot r$$
$$b_3 = a_4 - a_3 = a \cdot (r-1) \cdot r^2$$
$$b_4 = a_5 - a_4 = a \cdot (r-1) \cdot r^3$$
$$b_5 = a_6 - a_5 = a \cdot (r-1) \cdot r^4$$

$$b_n = a_{n+1} - a_n = a \cdot (r-1) \cdot r^{n-1}$$

Logical Reasoning (1)

Deduction: means determining the conclusion. (P ⇒ Q)
It is using the rule and its precondition to make a conclusion.

Induction: means determining the rule. (P ⇒ Q)
It is learning the rule after numerous examples of the conclusion following the precondition.

Abduction: means determining the precondition. (P ⇒ Q)
It is using the conclusion and the rule to support that the precondition could explain the conclusion.

Logical Reasoning (2)

Deduction: *Mathematicians* commonly use this style of reasoning

(P ⇒ Q)

- When it rains, the grass gets wet.
- It rains.
- Thus, the grass is wet.

Induction: *Scientists* commonly use this style of reasoning

(P ⇒ Q)

- The grass has been wet every time it has rained.

  Observation 1: The grass has been wet *when it has rained heavily.*

  Observation 2: The grass has been wet *when it has rained lightly.*

  Observation 3: The grass has been wet *when it has rained moderately.*

  …
- Thus, when it rains, the grass gets wet.

Abduction: *Diagnosticians* and *detectives* commonly use this style of reasoning

(P ⇒ Q)

- When it rains, the grass gets wet.
- The grass is wet
- Thus, it must have rained.

The Principle of Mathematical Induction:

Let P_n be a statement involving the positive integer n.

If

- P_1 is true, and
- the truth of the statement P_k implies the truth of the statement P_{k+1}, for every positive integer k,

then

the statement P_n is true for all positive integers n.

Proof by Mathematical Induction

To prove that P_n is true:

Show that

- P_1 is true.
- if P_k is assumed to be true,

  then P_{k+1} is also true,

  for every positive integer k.

Mathematical Induction – Example (1)

To prove that P_n is true:

$$f(n) = g(n) \qquad n = 1, 2, 3, \cdots$$

- Show P_1 is true.

$$f(1) = g(1)$$

- If P_k is assumed to be true, then P_{k+1} is also true, for any arbitrary k.

$$f(k) = g(k)$$

$$\Downarrow$$

$$f(k+1) = g(k+1)$$

- Consider f(n) and g(n) which have the following properties.

$$f(n+1) = f(n) + a(n+1)$$
$$g(n+1) = g(n) + a(n+1)$$

- then we can show $f(k) = g(k) \Rightarrow f(k+1) = g(k+1)$

$$f(k+1) \downarrow \quad g(k+1) \uparrow$$

$$f(k) + a(k+1) = g(k) + a(k+1)$$

Mathematical Induction – Example (3)

- An example class of such functions are:

$$S_n = \sum_{i=1}^{n} a_i$$

f(n+1)	$S_{n+1} = S_n + a_{n+1}$
g(n+1)	$\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n} a_i + a_{n+1}$

Ex1) $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Ex2) $$\sum_{i=1}^{n} 5\cdot 6^i = 6(6^n - 1)$$

Mathematical Induction – Example (4)

- To prove $$S_n = \sum_{i=1}^{n} a_i$$

- Show $S_1 = a_1$

- Show $S_{k+1} = \sum_{i=1}^{k+1} a_i$

  $\Downarrow \qquad \Uparrow$

  $S_k + a_{k+1} = \sum_{i=1}^{k} a_i + a_{k+1}$

Mathematical Induction – Ex 1)

- Prove $$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)} = \frac{n}{n+1}$$
- n = 1 $$\frac{1}{1\cdot 2} = \frac{1}{1+1}$$
- n = k $$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)} = \frac{k}{k+1}$$
- n = k+1 $$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

Mathematical Induction – Ex 2)

- Prove $$\sum_{i=1}^{n} 5\cdot6^{i} = 6(6^{n} - 1)$$
- n = 1 $$5\cdot6 = 6(6-1)$$
- n = k $$\sum_{i=1}^{k} 5\cdot6^{i} = 6(6^{k} - 1)$$
- n = k+1 $$\sum_{i=1}^{k+1} 5\cdot6^{i} = \sum_{i=1}^{k} 5\cdot6^{i} + 5\cdot6^{(k+1)}$$

$$= 6(6^{k} - 1) + 5\cdot6^{(k+1)} = 6^{(k+1)} - 6 + 5\cdot6^{(k+1)}$$

$$= 6(6^{(k+1)} - 1)$$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] Blitzer, R. "Algebra & Trigonometry." 3rd ed, Prentice Hall
[4] Smith, R. T., Minton, R. B. "Calculus: Concepts & Connections," Mc Graw Hill
[5] 홍성대, "기본/실력 수학의 정석," 성지출판