Phasor

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Phase Lags and Leads

$\frac{d}{dx} f(x) = \cos(x)$ **leads** $f(x) = \sin(x)$

$\frac{d}{dx} f(x) = -\sin(x)$ **leads** $f(x) = \cos(x)$

$\int f(x)\,dx = -\cos(x) + C$ **lags** $f(x) = \sin(x)$

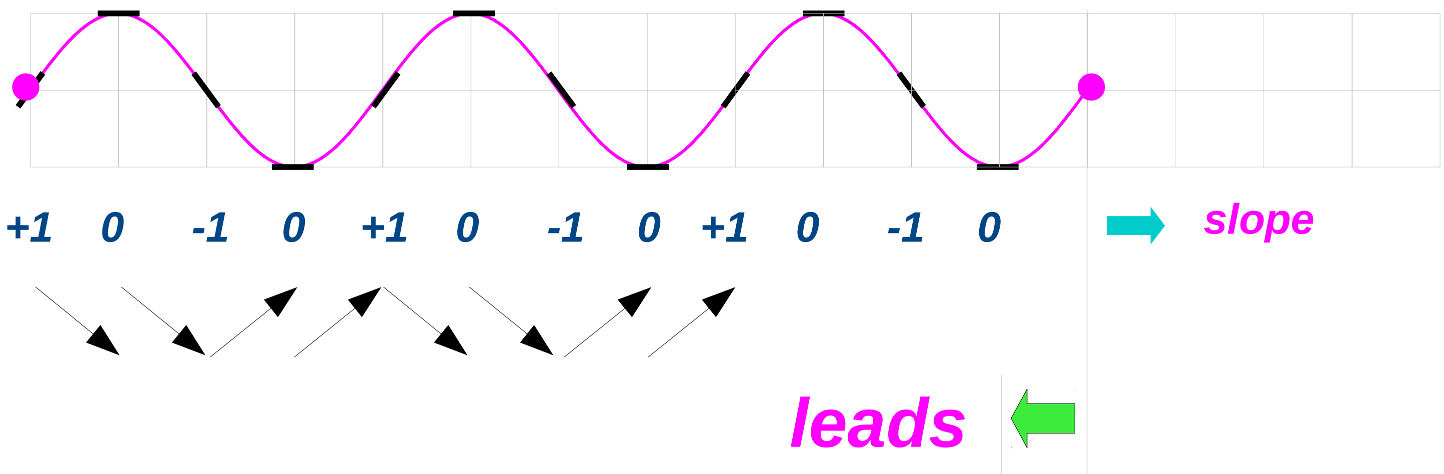
$\int f(x)\,dx = \sin(x) + C$ **lags** $f(x) = \cos(x)$

$\frac{d}{dx} f(x)$ **leads** $f(x)$ **by** $\frac{\pi}{2}$

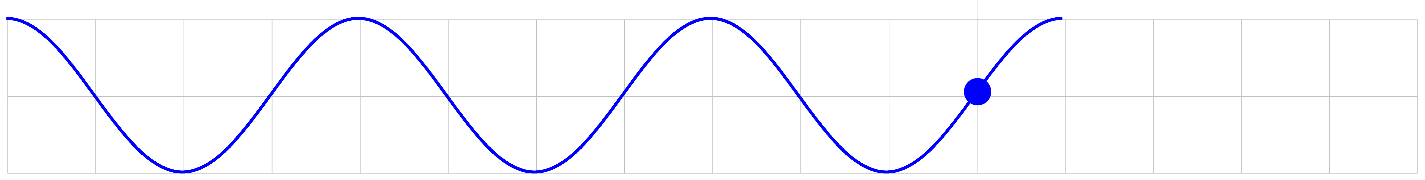
$\int f(x)\,dx$ **lags** $f(x)$ **by** $\frac{\pi}{2}$

Derivative of sin(x)

$$f(x) = \sin(x)$$

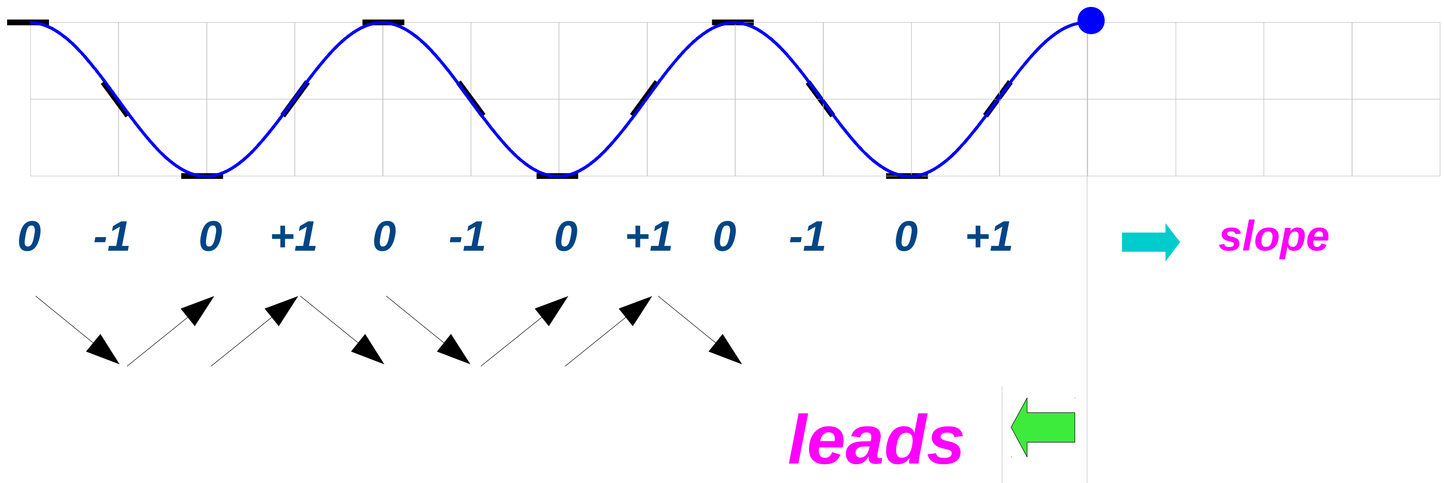


$$\frac{d}{dx} f(x) = \cos(x)$$

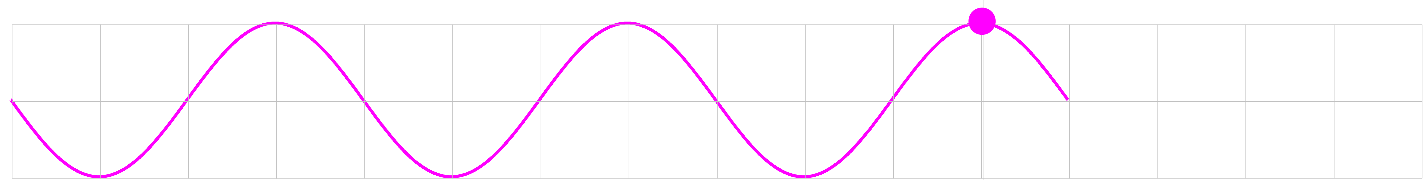


Derivative of cos(x)

$$f(x) = \cos(x)$$

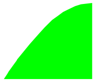


$$\frac{d}{dx} f(x) = -\sin(x)$$

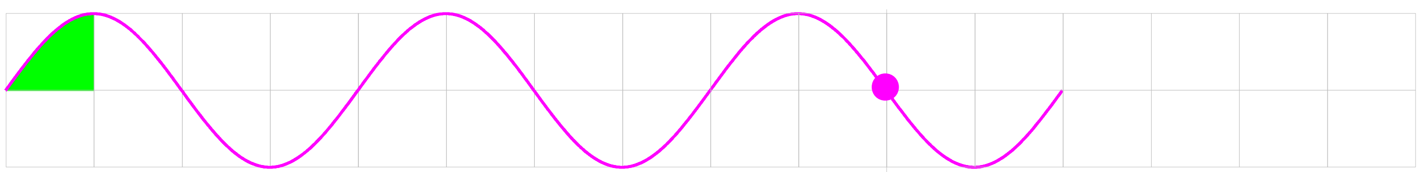


Integral of sin(x)

$$f(x) = \sin(x)$$



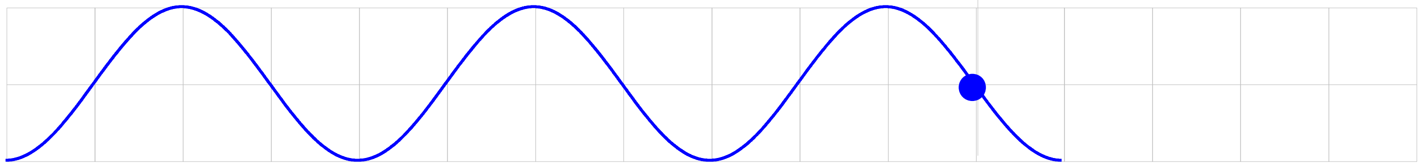
$$\int_0^{\pi/2} \sin(t)\,dt = 1$$



$C = 1$	0	1	2	1	0	1	2	1	0	1	2	1	➡	area	$\int_0^x \sin(t)\,dt$
$C = 0$	-1	0	+1	0	-1	0	+1	0	-1	0	+1	0	➡	area - 1	$\int_0^x \sin(t)\,dt - 1$

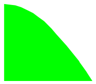
➡ **lags** $= -\cos(x)$

$$\int f(x)\,dx = -\cos(x) + C$$

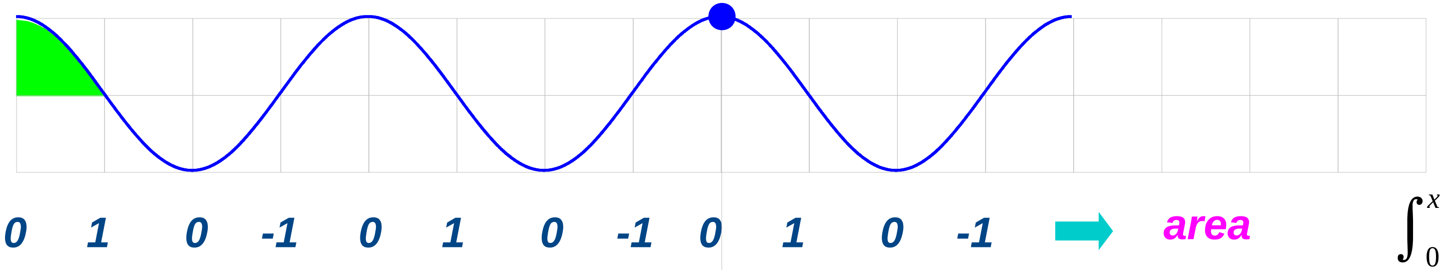


Integral of cos(x)

$$f(x) = \cos(x)$$



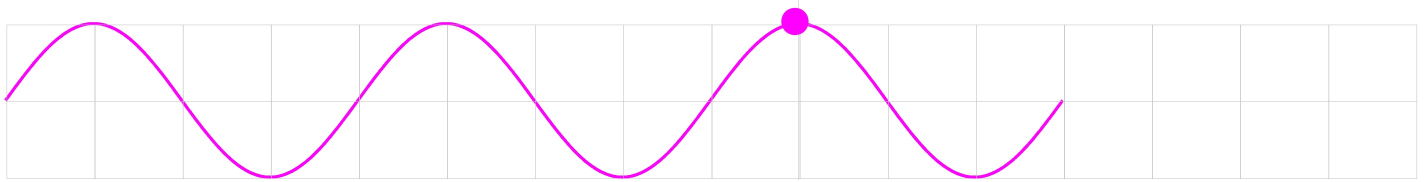
$$\int_0^{\pi/2} \cos(x)\, dx = 1$$



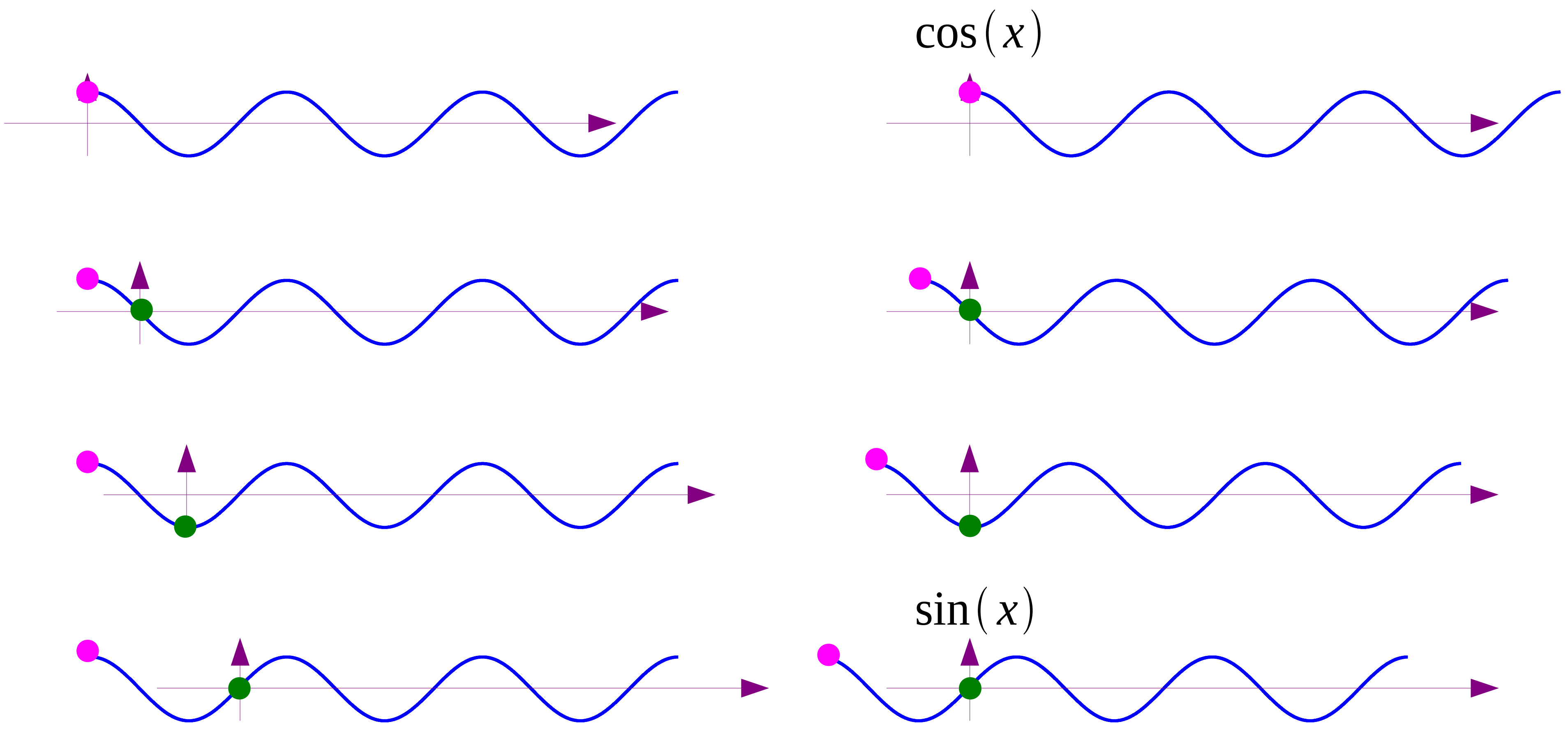
$$\int_0^{x} \cos(t)\, dt$$

$$= - \sin(x)$$

$$\int f(x)\, dx = \sin(x) + C$$



Sinusoid



Same Amplitude

Same Angular Frequency

$$\left.\begin{array}{l} \cos(x) \Rightarrow 1 \cdot \cos(1 \cdot t) \\ \sin(x) \Rightarrow 1 \cdot \sin(1 \cdot t) \end{array}\right\} \quad A\cos(\omega t + \theta)$$

Phasor

Sinusoid (Sine Waves)

$$A\cos(\omega t + \theta)$$

- *Amplitude* A
- *Angular Frequency* ω
- *Angle at t = 0* θ

1. Representation using Euler's Formula

$$A\cos(\omega t + \theta) = \frac{A}{2}\cdot e^{+i(\omega t + \theta)} + \frac{A}{2}\cdot e^{-i(\omega t + \theta)}$$

2. Representation using Real Part

$$A\cos(\omega t + \theta) = Re\{A e^{i(\omega t + \theta)}\} = Re\{A e^{i\theta}\cdot e^{i\omega t}\}$$

➡ $A e^{i\theta}\cdot e^{i\omega t}$

➡ $A e^{i\theta}$

➡ $A \angle \theta$

Phasor

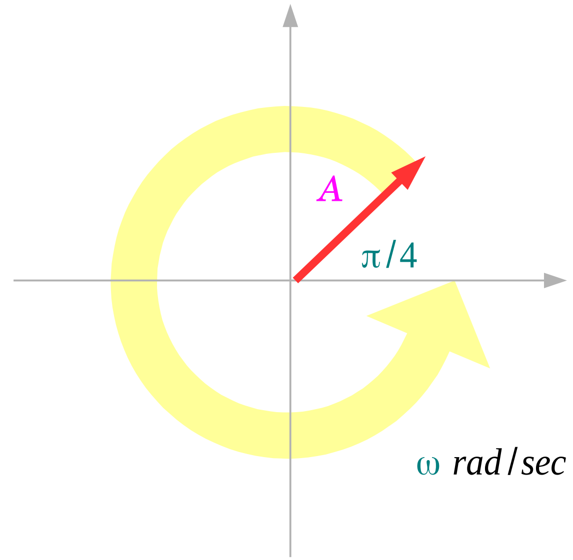
$$A\cos(\omega t + \theta)$$

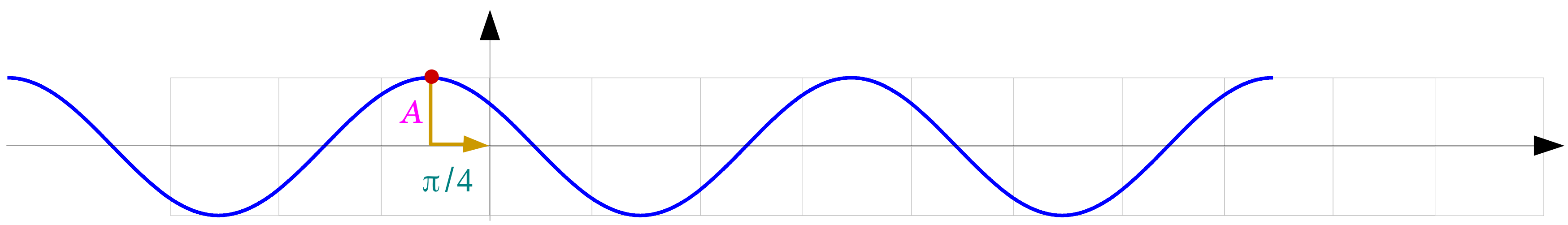
$$A\cos(\omega t + \theta) = \Re\{A e^{i(\omega t + \theta)}\}$$

$$= \Re\{e^{i\omega t} \cdot A e^{i\theta}\}$$

➡ $A e^{i\theta} = A\cos\theta + j A\sin\theta$

➡ $A \angle \theta$





A

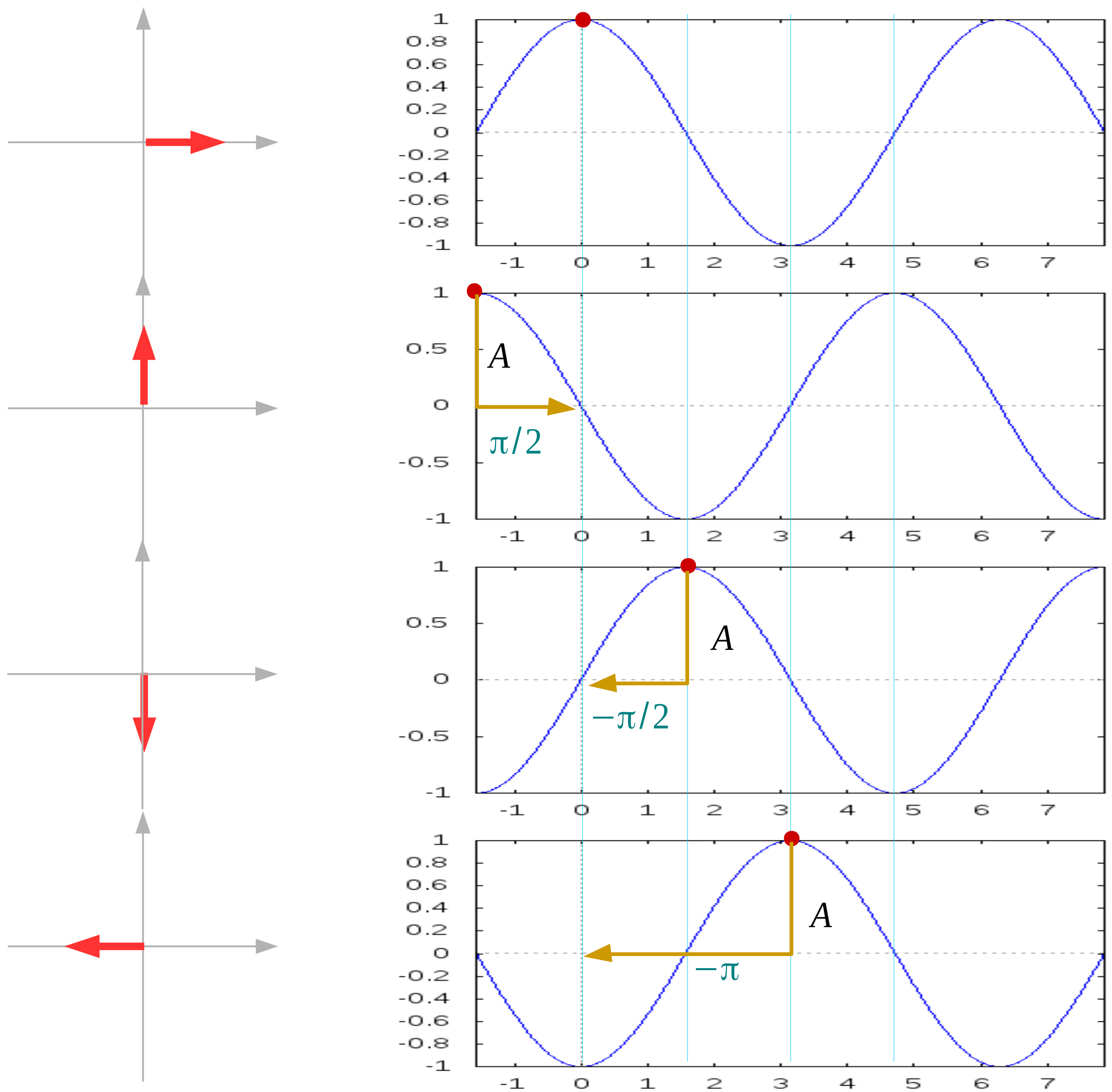
Phasor Example (1)

$A\angle 0$ $\quad$ $A\cos(\omega t + 0)$

$A\angle +\pi/2$ $\quad$ $A\cos(\omega t + \pi/2)$

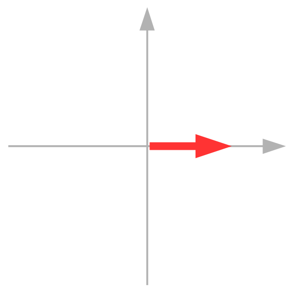
$A\angle -\pi/2$ $\quad$ $A\cos(\omega t - \pi/2)$

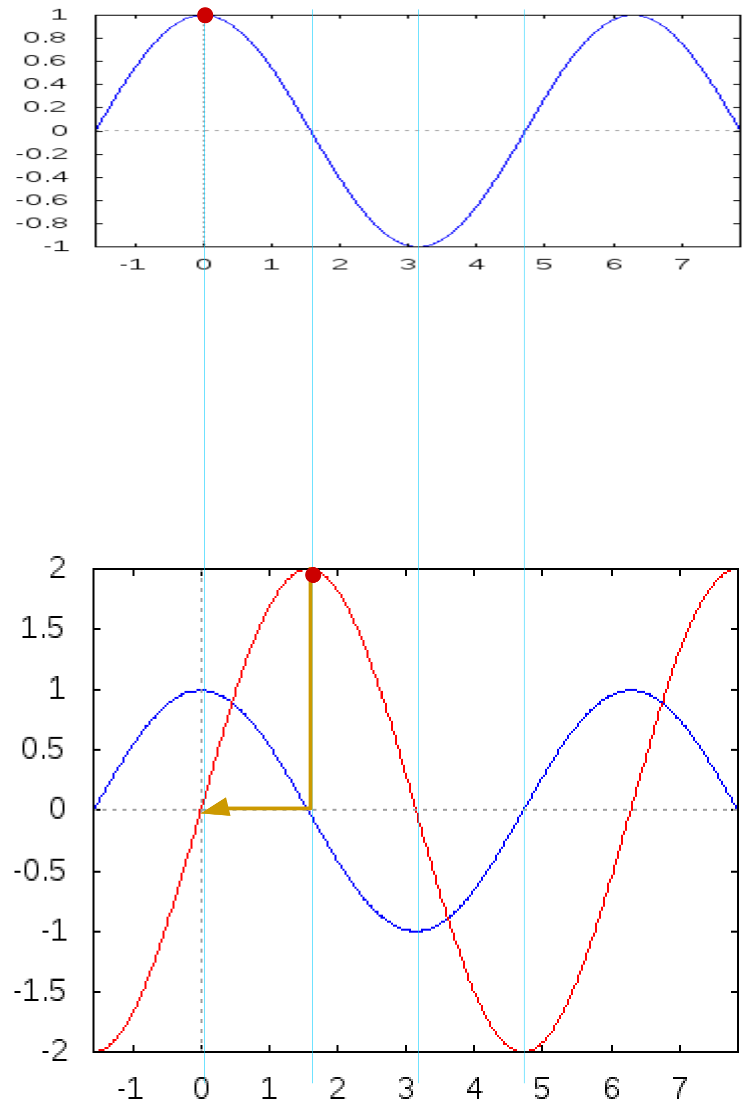
$A\angle -\pi$ $\quad$ $A\cos(\omega t - \pi)$



Phasor Example (2)

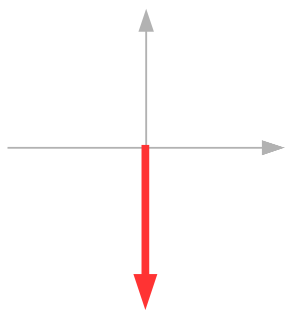
$A\angle 0$





$A\cos(\omega t + 0)$

$2A\angle -\pi/2$



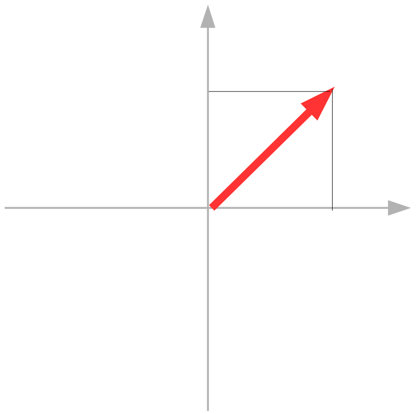
$2A\cos(\omega t - \pi/2)$

Phasor

$$A\cos(\omega t + \theta)$$

$$A\angle\theta$$

$$= A\cos(\omega t)\cos(\theta) - A\sin(\omega t)\sin(\theta)$$
$$= A\cos(\theta)\cos(\omega t) - A\sin(\theta)\sin(\omega t)$$
$$= X\cos(\omega t) - Y\sin(\omega t)$$



$$(X, Y) = (A\cos\theta, A\sin\theta)$$

$$X + jY = A\cos\theta + jA\sin\theta$$

$$A\cos(\theta) = X$$
$$A\sin(\theta) = Y$$

$$A = \sqrt{X^2 + Y^2}$$
$$\tan\theta = \frac{Y}{X}$$

$\theta > 0$ ***leading*** $\theta < 0$ ***lagging***

Linear Combination of cos(ωt) & sin(ωt)

$X\cos(\omega t) + Y\sin(\omega t)$

$\sqrt{X^2+Y^2}\cos(\omega t - \theta)$

$$X\cos(\omega t) + Y\sin(\omega t)$$

$$= \sqrt{X^2+Y^2}\left[\frac{X}{\sqrt{X^2+Y^2}}\cos(\omega t) + \frac{Y}{\sqrt{X^2+Y^2}}\sin(\omega t)\right]$$

$$= \sqrt{X^2+Y^2}\left[\cos(\theta)\cos(\omega t) + \sin(\theta)\sin(\omega t)\right]$$

$$= \sqrt{X^2+Y^2}\cos(\theta - \omega t)$$

$$= \sqrt{X^2+Y^2}\cos(\omega t - \theta)$$

$$X\cos(\omega t) + Y\sin(\omega t)$$

$$= \sqrt{X^2+Y^2}\cos(\omega t - \theta)$$

$$\cos(\theta) = \frac{X}{\sqrt{X^2+Y^2}}$$

$$\sin(\theta) = \frac{Y}{\sqrt{X^2+Y^2}}$$

$X\cos(\omega t) - Y\sin(\omega t)$

$\sqrt{X^2+Y^2}\cos(\omega t + \theta)$

Phasor as a starting point

$$(+1)\cos(\omega t) - (0)\sin(\omega t)$$

$\cos(0)$ $\sin(0)$

$$(0)\cos(\omega t) - (+1)\sin(\omega t)$$

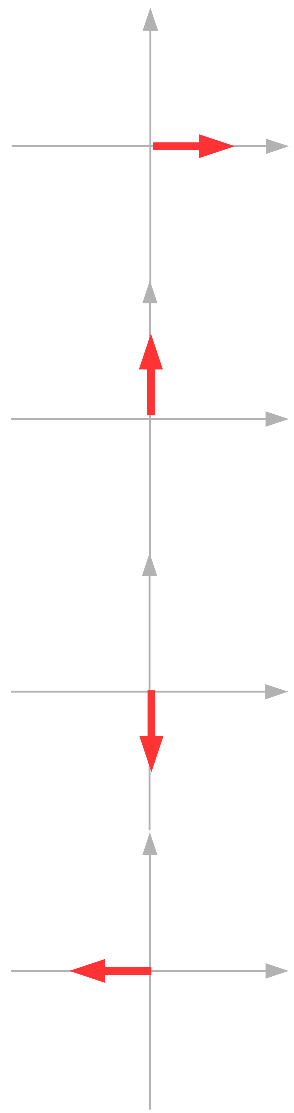
$\cos(\pi/2)$ $\sin(\pi/2)$

$$(0)\cos(\omega t) - (-1)\sin(\omega t)$$

$\cos(-\pi/2)$ $\sin(-\pi/2)$

$$(-1)\cos(\omega t) - (0)\sin(\omega t)$$

$\cos(-\pi)$ $\sin(-\pi)$



$$A\cos(\omega t + 0)$$

$$A\cos(\omega t + \pi/2)$$

$$A\cos(\omega t - \pi/2)$$

$$A\cos(\omega t - \pi)$$

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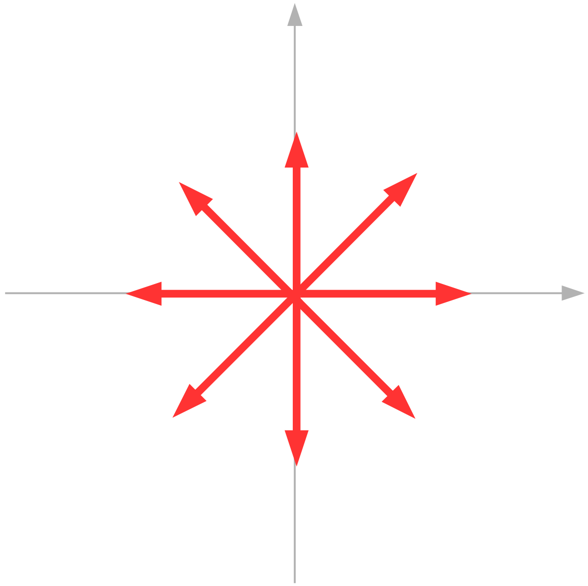
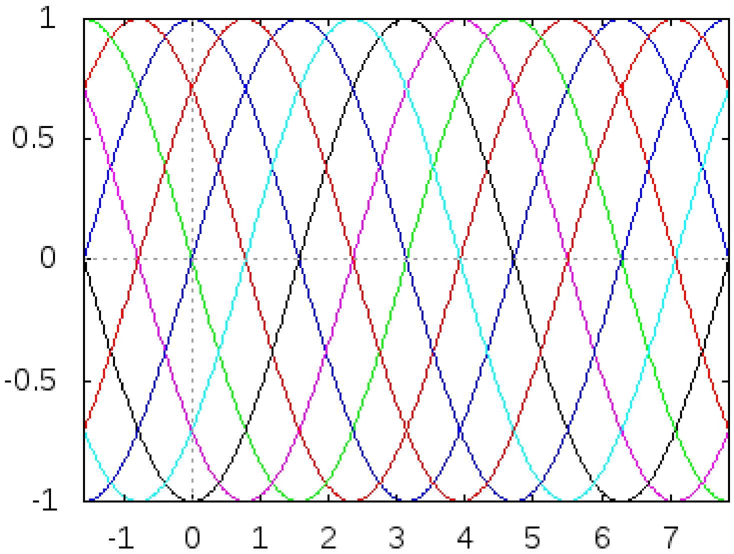
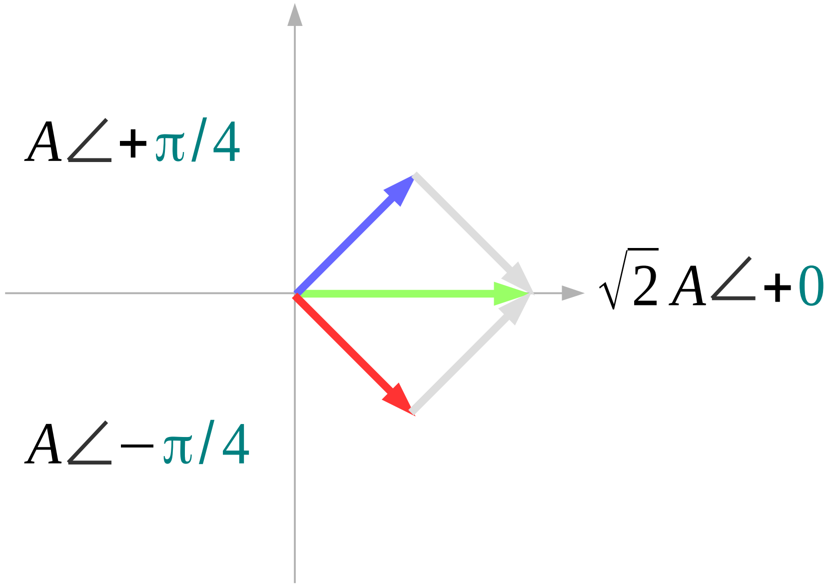
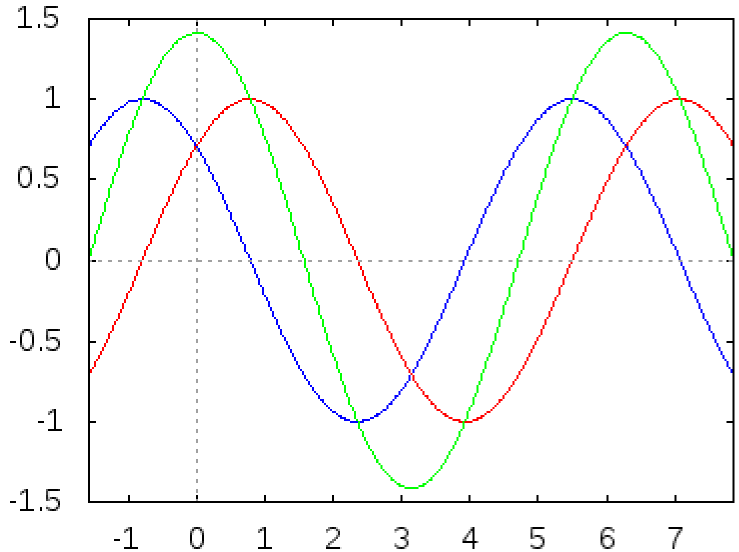
Phase Angles



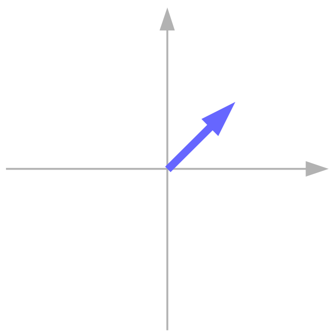


Phasor Arithmetic

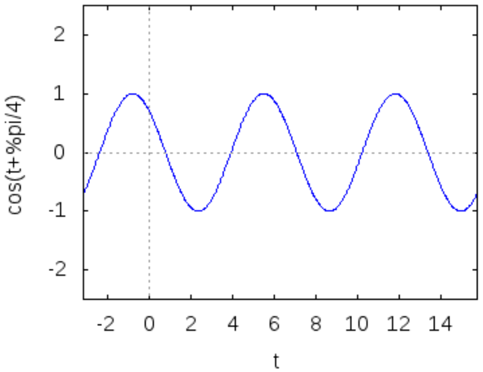




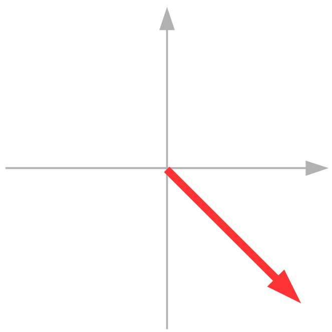
Phasor Addition



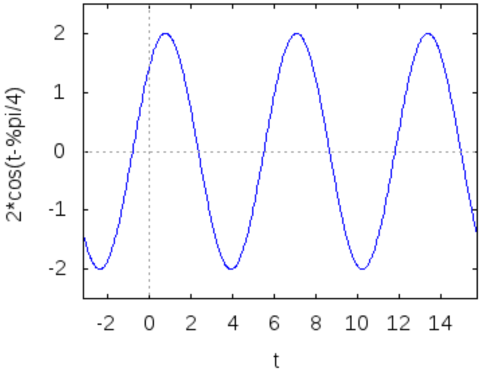
$1\angle +\pi/4 \iff \cos(\omega t + \pi/4)$



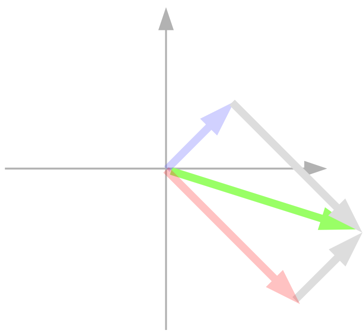
$+ \qquad +$



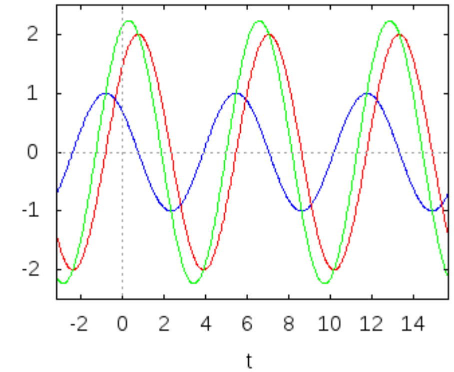
$2\angle -\pi/4 \iff 2\cos(\omega t - \pi/4)$



$= \qquad =$



$\sqrt{5}\angle +0 \iff \sqrt{5}\cos(\omega t + 0)$



Phasor Addition Rule

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega t + \theta_k)$$

$$= A\cos(\omega t + \theta)$$

$$= \sum_{k=1}^{N} \Re\{A_k e^{j(\omega t + \theta_k)}\}$$

$$= \Re\{\sum_{k=1}^{N} A_k e^{j(\omega t)} e^{j(\theta_k)}\}$$

$$= \Re\{\sum_{k=1}^{N} A_k e^{j\theta_k} e^{j(\omega t)}\}$$

$$= \Re\{\sum_{k=1}^{N} A_k e^{j\theta_k} e^{j(\omega t)}\}$$

$$= \Re\{\sum_{k=1}^{N} A e^{j\theta} e^{j(\omega t)}\}$$

$$= \Re\{\sum_{k=1}^{N} A e^{j(\omega t + \theta)}\}$$

$$= A\cos(\omega t + \theta)$$

adding complex numbers

$$\sum_{k=1}^{N} A_k e^{j(\theta_k)} = A e^{j\theta}$$

a complex number

Phasor Multiplication & Division

$$x(t) = A_1 \cos(\omega t + \theta_1) = \Re\{A_1 e^{j(\omega t + \theta_1)}\}$$

$$y(t) = A_2 \cos(\omega t + \theta_2) = \Re\{A_2 e^{j(\omega t + \theta_2)}\}$$

$$x(t) * y(t) = A_1 A_2 \cos(\omega t + \theta_1) \cos(\omega t + \theta_2)$$

$$= \Re\{A_1 A_2 e^{j(2\omega t + \theta_1 + \theta_2)}\}$$

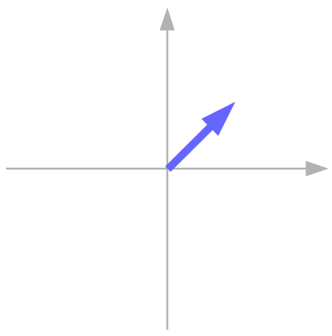
different angular frequency !

$$\frac{x(t)}{y(t)} = \frac{A_1}{A_2} \frac{\cos(\omega t + \theta_1)}{\cos(\omega t + \theta_2)}$$

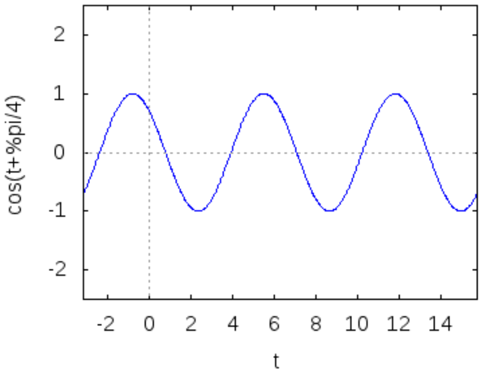
$$= \Re\{\frac{A_1}{A_2} e^{j(\theta_1 - \theta_2)}\}$$

no more rotating !

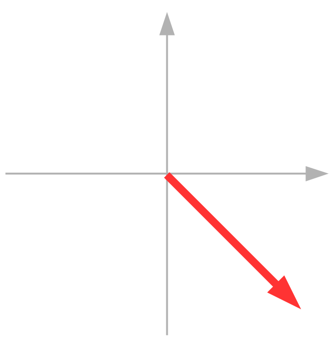
Phasor Multiplication



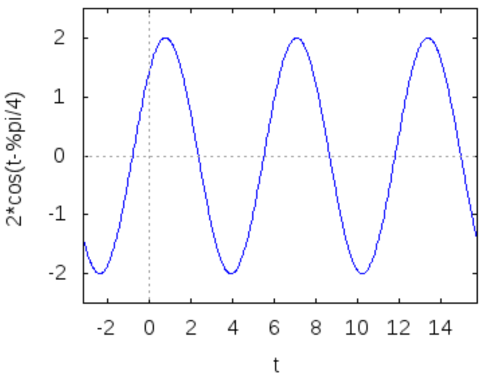
$1\angle+\pi/4 \Longleftrightarrow \cos(\omega t + \pi/4)$



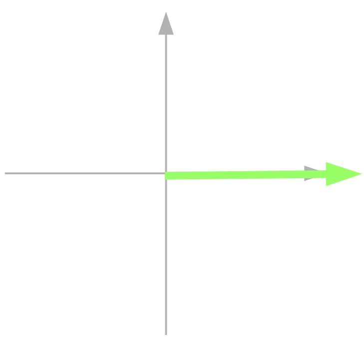
$*$ $\qquad$ $*$



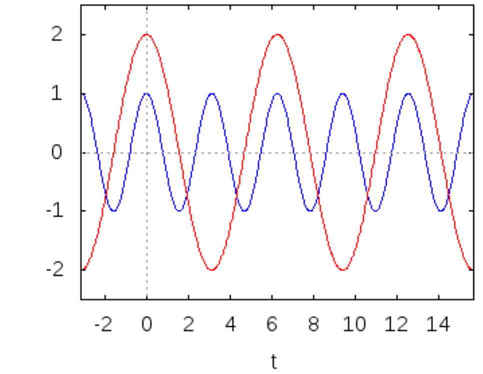
$2\angle-\pi/4 \Longleftrightarrow 2\cos(\omega t - \pi/4)$



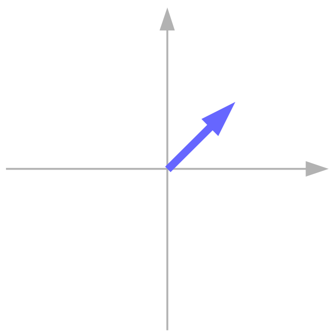
$\|$ $\qquad$ $\|$



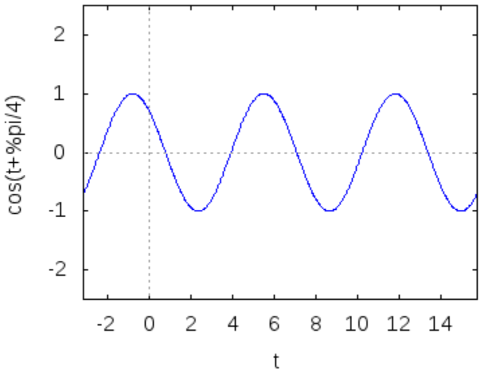
$2\angle+0 \Longleftrightarrow 2\cos(\omega t + 0)$



Phasor Scaling



$1\angle{+\pi/4} \iff \cos(\omega t + \pi/4)$



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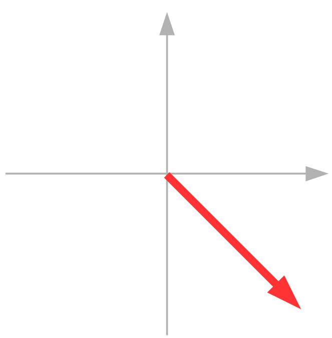


$2\angle{-\pi/4}$

$\Re\{A_1 e^{j(\omega t + \theta_1)}\}$

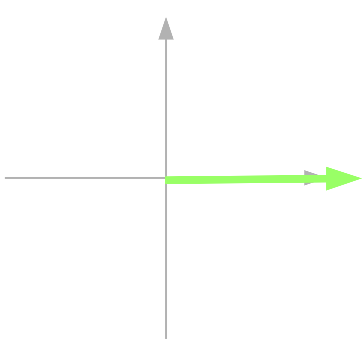
$\Re\{A_2 e^{j(\theta_2)}\}$

$\Re\{A_1 A_2 e^{j(\omega t + \theta_1 + \theta_2)}\}$

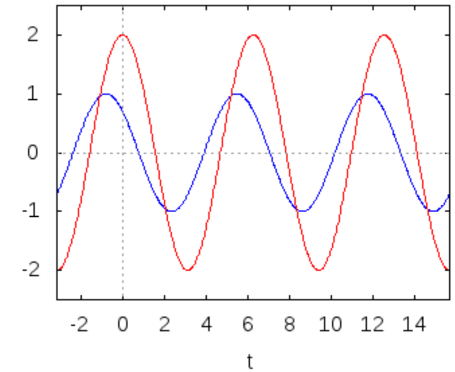
$2\angle{-\pi/4}$

not a phasor
just a scaling
complex number

$=$



$2\angle{+0} \iff 2\cos(\omega t + 0)$



Vector Space

V: non-empty set of objects

defined operations: addition $\mathbf{u} + \mathbf{v}$

scalar multiplication $k\,\mathbf{u}$

if the following axioms are satisfied

for all object **u**, **v**, **w** and all scalar *k*, *m*

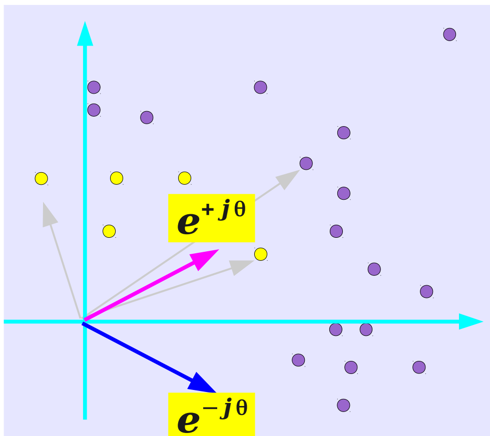
→ V: vector space

objects in V: vectors

1. if **u** and **v** are objects in V, then **u** + **v** is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if *k* is any scalar and **u** is objects in V, then *k***u** is in V
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

Basis

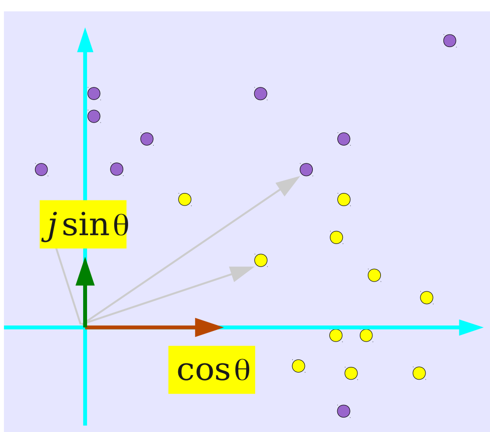
Basis : a set of linear independent spanning vectors



every complex number can be represented by

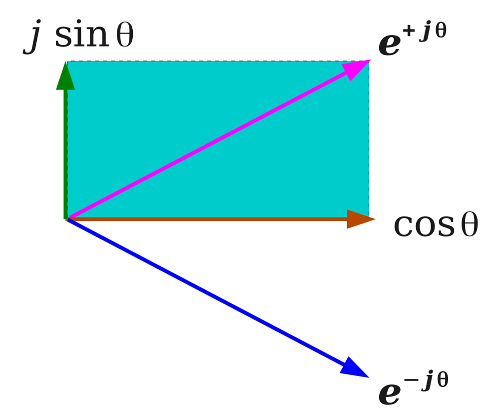
$$k_1 e^{+j\theta} + k_2 e^{+j\theta}$$

linear combination of $e^{+j\theta}$ and $e^{+j\theta}$
which are one set of linear independent
two vectors



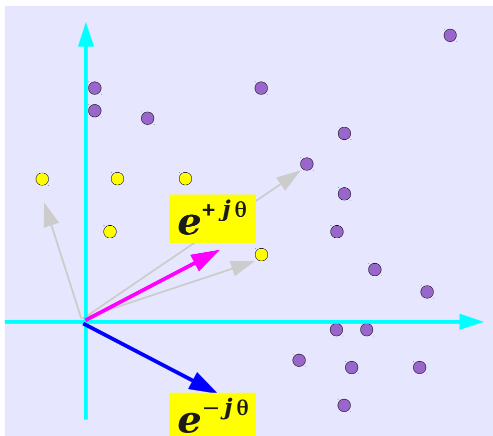
every complex number can also be represented by

$$l_1 \cos\theta + l_2\, j \sin\theta$$

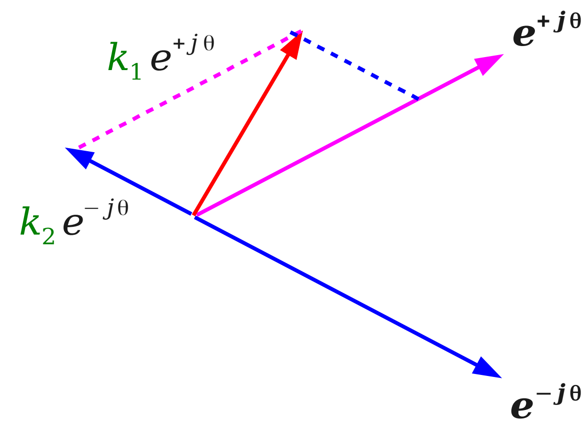


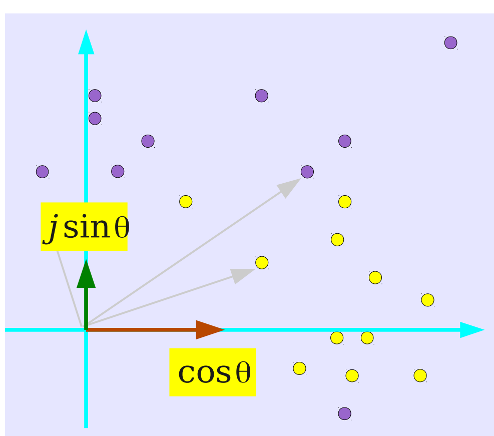
Basis (2)

Basis : a set of linear independent spanning vectors

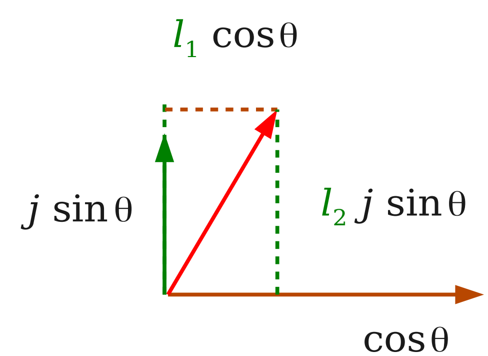


$$k_1 e^{+j\theta} + k_2 e^{+j\theta}$$





$$l_1 \cos\theta + l_2 j \sin\theta$$



C^1 and R^2 Spaces

$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟷ $c_3 \cos(\omega t) + c_4 i \sin(\omega t)$

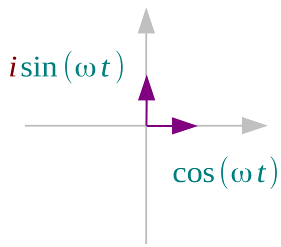
$c_1 = (c_3 + c_4)/2$ *real number*

$c_2 = (c_3 - c_4)/2$ *real number*

$c_3 = (c_1 + c_2)$ *real number*

$c_4 = (c_1 - c_2)$ *real number*

C^1



$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟷ $c_3 \cos(\omega t) + c_4 \sin(\omega t)$

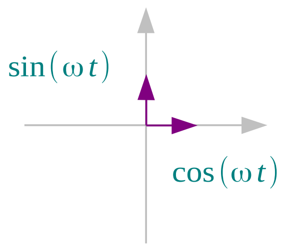
$c_1 = (c_3 - c_4 i)/2$ *conjugate*

$c_2 = (c_3 + c_4 i)/2$ *complex number*

$+2 * real\ part$ $c_3 = (c_1 + c_2)$ *real number*

$-2 * imag\ part$ $c_4 = i(c_1 - c_2)$ *real number*

R^2



Linear Combination of cos(ωt) & sin(ωt)

$X\cos(\omega t) + Y\sin(\omega t)$

$\sqrt{X^2+Y^2}\cos(\omega t - \theta)$

$X\cos(\omega t) - Y\sin(\omega t)$

$\sqrt{X^2+Y^2}\cos(\omega t + \theta)$

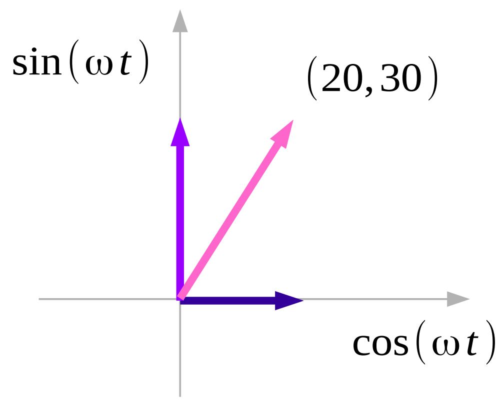
$20\cos(\omega t) + 30\sin(\omega t)$

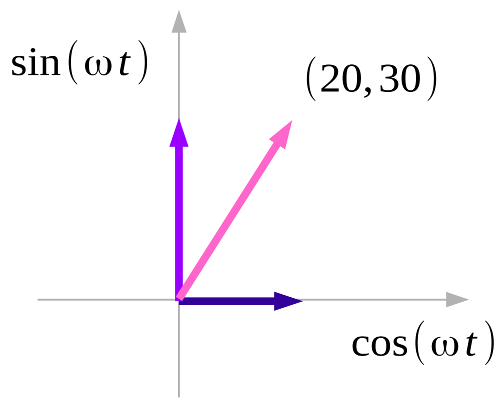
$36.06\cos(\omega t - 0.588)$

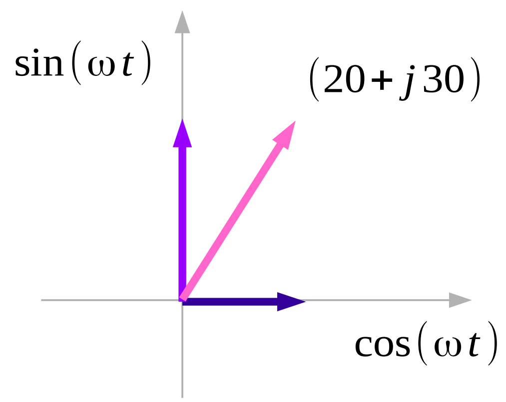
$20\cos(\omega t) - 30\sin(\omega t)$

$36.06\cos(\omega t + 0.588)$

$36.06 \angle 0.588$







References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003