# Z Transform (H.1)
# Definition

20170105

Based on
Complex Analysis for Mathematics and Engineering
J. Mathews

# z - Transform

$$X(z) = \sum_{k=-\infty}^{+\infty} x[k]\, z^{-k}$$

$$z = |r| e^{j2\pi F} = |r| e^{j\Omega}$$

$$x[n] \longleftrightarrow X(z)$$

One Sided z-transform

$$X(z) = \sum_{k=0}^{+\infty} x[k]\, z^{-k}$$

# Inverse z-Transform

$$X(z) = \mathcal{Z}\left[\{x_n\}_{n=0}^{\infty}\right] = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x_n = x[n] = \mathcal{Z}^{-1}\left[X(z)\right] = \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz$$

# Admissible Form of z-transform

$$X(z) = \sum_{k=0}^{\infty} x[n] \, z^{-n}$$

admissible z-transform

if $X(z)$ is a rational function

$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z^1 + b_2 z^2 + \cdots + b_{p-1} z^{p-1} + b_p z^p}{a_0 + a_1 z^1 + a_2 z^2 + \cdots + a_{q-1} z^{q-1} + a_q z^q}$$

$P(z)$ : a polynomial of degree $p$
$Q(z)$ : a polynomial of degree $q$

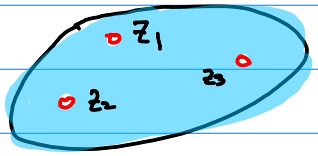
D: simply connected domain
C: simple closed contour (ccw) in D

if $f(z)$ is analytic inside C and on C
except at the points $z_1, z_2, \cdots, z_k$ in C
then

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \sum_{j=1}^{k} \mathrm{Res}(f(z), z_j)$$

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k)$$

finite number $k$ of
singular points $z_k$



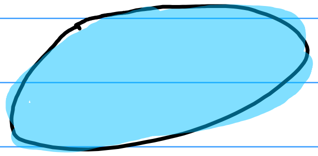
$$\oint_C f(z)dz = 0$$

if $f(z)$ is continuous in D and
$f(z) = F'(z)$ : $F(z)$ is an antiderivative of $f(z)$
fundamental theorem of calculus

$$\oint_C f(z)dz = 0$$

if $f(z)$ is analytic within and on C

no singularity



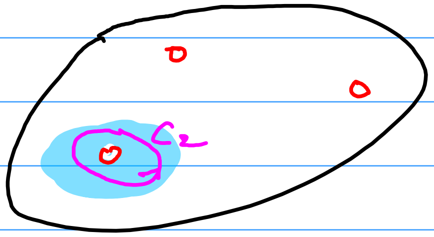
$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

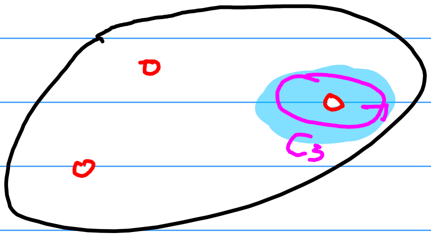
$$= \sum_k \mathrm{Res}\left( \frac{f(z)}{(z-z_m)^{n+1}} , z_k \right) \qquad z_k \text{ within } C$$

$$= \frac{1}{n!} f^{(n)}(z_m) \qquad n \geq 0$$

$a_n^{\{0\}}$ expansion at $z_0$



$a_n^{\{1\}}$ expansion at $z_1$



$a_n^{\{2\}}$ expansion at $z_2$

$$a_n^{\{m\}} = \mathrm{Res}(f(z), z_m)$$

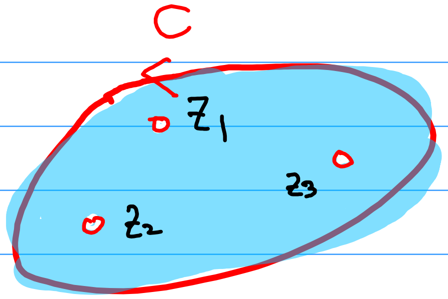
the residue of $f(z)$ at $z_m$ using $C_m$

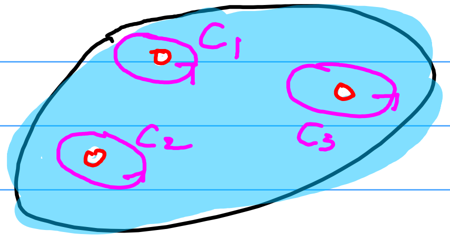
assumed that

there are several (m) singularities (poles) of $f(z)$ in a region

but that

C is taken to enclose only the pole $z_m$ : $C_m$





# Laurent's Theorem

$f$ : analytic within the annular domain $D$

$$r < |z - z_0| < R$$

then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k \quad , \text{ valid for } \quad r < |z - z_0| < R$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} \, ds , \qquad k = 0, \pm 1, \pm 2, \cdots$$

$C$ : a simple closed curve
that lies entirely within $D$
that encloses $z_0$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) \, ds \Longrightarrow \oint_C f(s) \, ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) \, ds = Res(f(z), z_0)$$

$$= \begin{cases} \lim\limits_{z \to z_0} (z - z_0) f(z) & (\text{simple}) \\ \dfrac{1}{(n-1)!} \lim\limits_{z \to z_0} \dfrac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & (\text{order } n) \end{cases}$$

# Cauchy's Residue Theorem

$f(z)$ : analytic on and within C

except a finite number of singular points

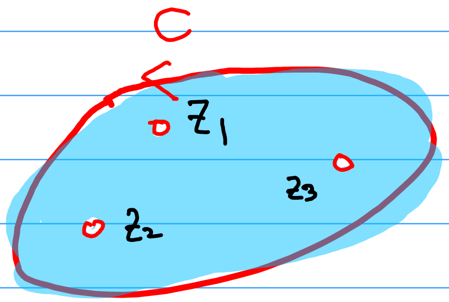
$z_1, z_2, \cdots, z_n$ within C

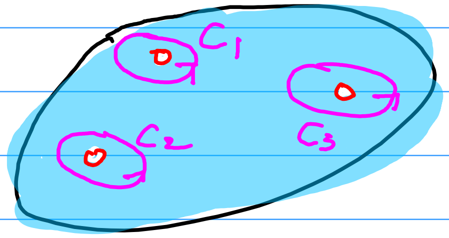
then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

D : a simply connected domain

C : a simple closed contour in D





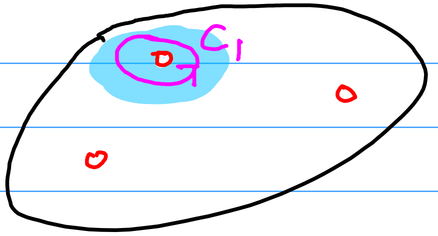
$\boxed{z_1}$ $f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_1)^k$ $a_{-1}^{\{1\}} = \frac{1}{2\pi i} \oint_{C_1} f(s)\,ds = \mathrm{Res}(f(z), z_1)$

$\boxed{z_2}$ $f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_2)^k$ $a_{-1}^{\{2\}} = \frac{1}{2\pi i} \oint_{C_2} f(s)\,ds = \mathrm{Res}(f(z), z_2)$

$\boxed{z_3}$ $f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_3)^k$ $a_{-1}^{\{3\}} = \frac{1}{2\pi i} \oint_{C_3} f(s)\,ds = \mathrm{Res}(f(z), z_3)$

$\boxed{z_1}$

## Laurent series expansion at $z_1$

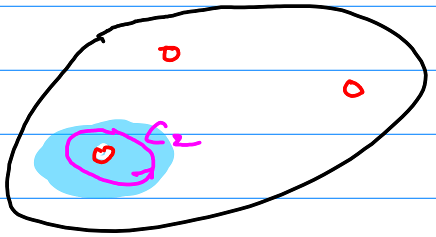


$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_1)^k$$

$$a_{-1}^{\{1\}} = \frac{1}{2\pi i}\oint_{C_1} f(s)\, ds = \text{Res}(f(z), z_1)$$

$\boxed{z_2}$

## Laurent series expansion at $z_2$

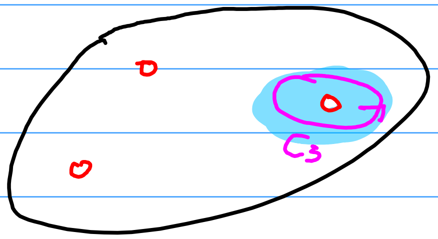


$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_2)^k$$

$$a_{-1}^{\{2\}} = \frac{1}{2\pi i}\oint_{C_2} f(s)\, ds = \text{Res}(f(z), z_2)$$

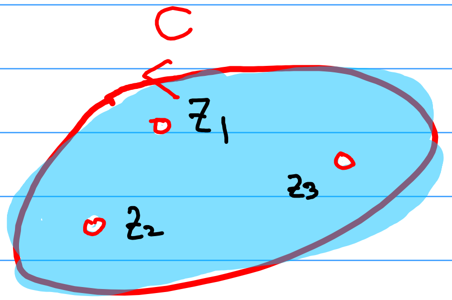
$\boxed{z_3}$

## Laurent series expansion at $z_3$

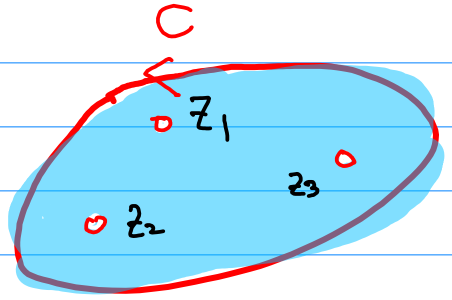


$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_3)^k$$

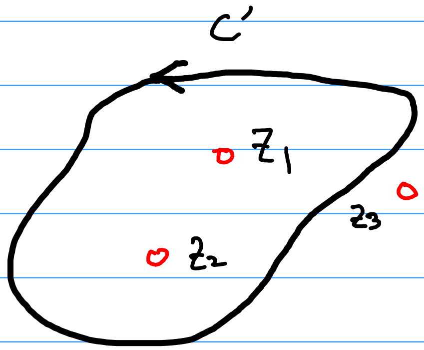
$$a_{-1}^{\{3\}} = \frac{1}{2\pi i}\oint_{C_3} f(s)\, ds = \text{Res}(f(z), z_3)$$



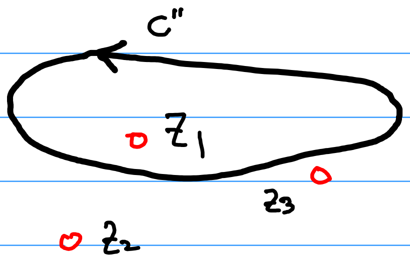
$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k)$$



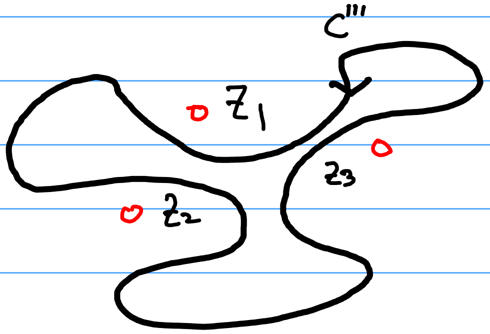
$$\int_c f(z)\,dz = 2\pi i\ \mathrm{Res}(f(z), z_1) + 2\pi i\ \mathrm{Res}(f(z), z_2) + 2\pi i\ \mathrm{Res}(f(z), z_3)$$



$$\int_{c'} f(z)\,dz = 2\pi i\ \mathrm{Res}(f(z), z_1) + 2\pi i\ \mathrm{Res}(f(z), z_2)$$

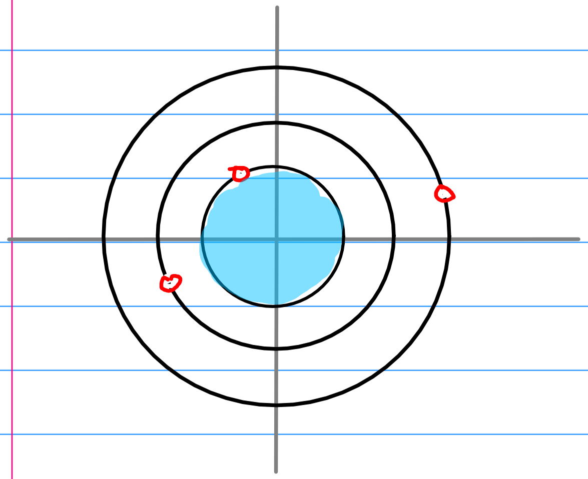


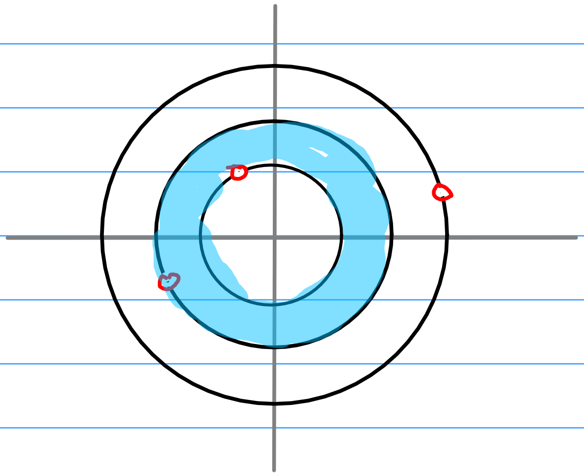
$$\int_{c''} f(z)\,dz = 2\pi i\ \mathrm{Res}(f(z), z_1)$$

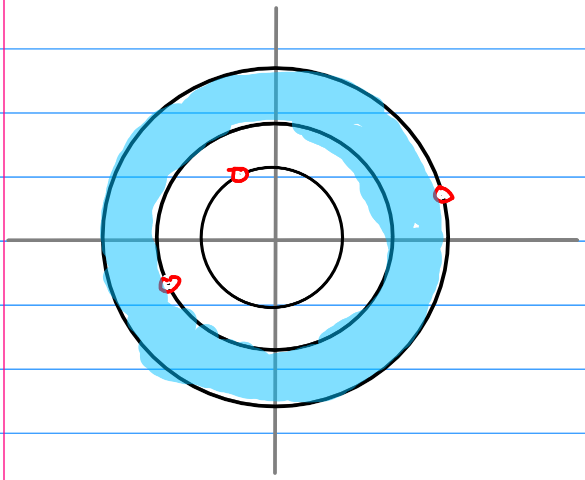


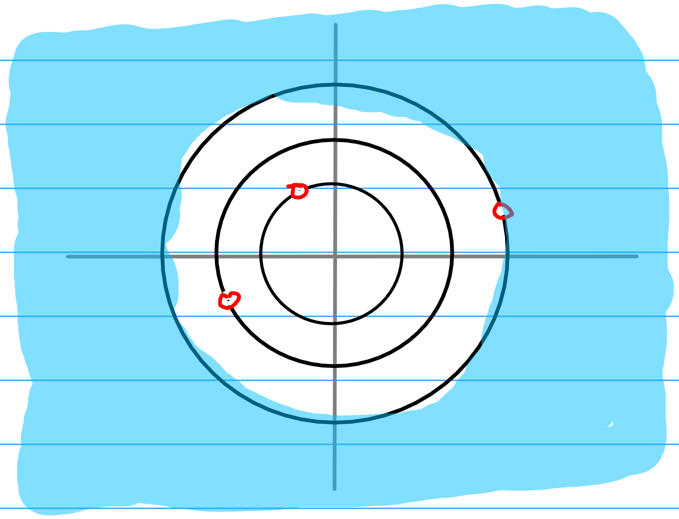
$$\int_{c''} f(z)\,dz = 0$$

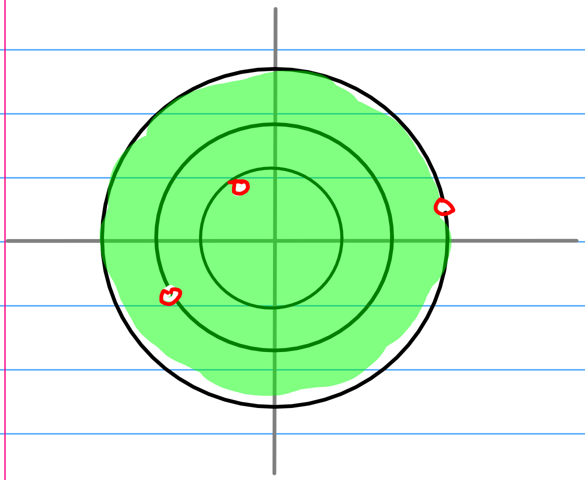
# Different D, Different Laurent Series

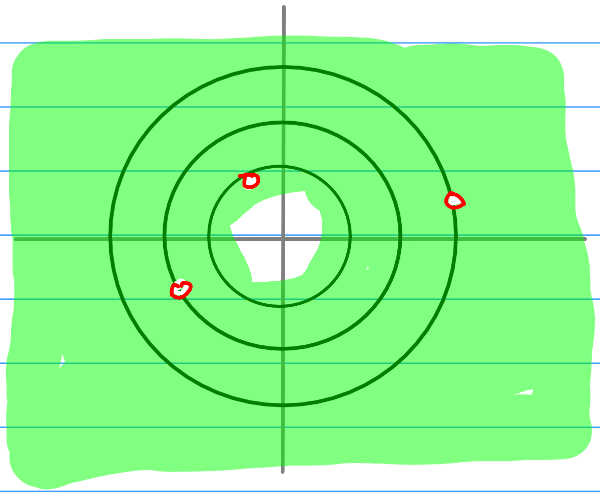










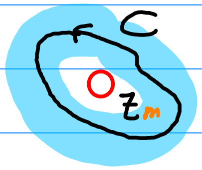


z-transform

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}}\, dz'$$

$$= \sum_k \mathrm{Res}\left( \frac{f(z)}{(z-z_m)^{n+1}} ,\ z_k \right)$$



C is in the same region of analyticity of $f(z)$
typically a circle centered on $z_m$

$z_k$ within C : singularities of $\frac{f(z)}{(z-z_m)^{n+1}}$

$n_1 = n_{f,m}$ depends on $f(z)$, $z_m$

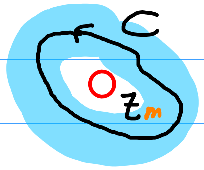
$a_n^{\{m\}}$ depends on $f(z)$, $z_m$, region of analyticity

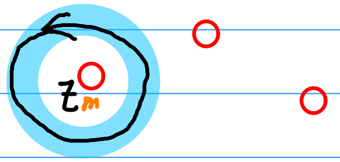
Whether $f(z)$ is singular at $z = z_m$ or not
or at other points between $z$ and $z_m$
We can expand $f(z)$ about any point $z_m$
over powers of $(z - z_m)$.

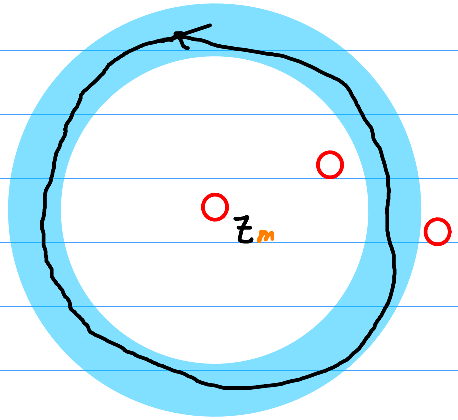
$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

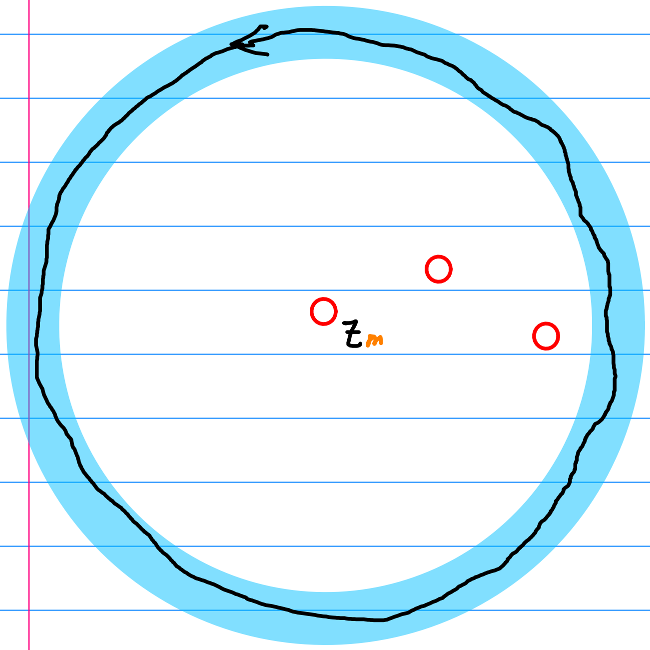
$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$









$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z'-z_m)^{n+1}} dz'$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

analytic at $z_m$

| | | |
|---|---|---|
| $n_1 \geq 0$ | | Taylor Series |
| general $n_1$ | $z_m = 0$ | MacLaurin Series |

singular at $z_m$

| | | |
|---|---|---|
| general $n_1$ | | Laurent Series |
| general $n_1$ | $z_m = 0$ | z - Transform |

$$\boxed{\begin{aligned} f(z) &= \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n \\ a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}}\, dz' \\ &= \sum_k \mathrm{Res}\left( \frac{f(z)}{(z - z_m)^{n+1}} , z_k \right) \end{aligned}}$$

$$\boxed{z_m = 0} \qquad \boxed{a_{-n}^{\{0\}} = h(n)} \qquad \boxed{n \to -n}$$

$$\boxed{\begin{aligned} H(z) &= \sum_{n=n_1}^{\infty} h(-n)\, z^n \\ h(n) &= \frac{1}{2\pi i} \oint_C \frac{H(z')}{z'^{\,n+1}}\, dz' \\ &= \sum_k \mathrm{Res}\left( \frac{H(z)}{z^{n+1}} , z_k \right) \end{aligned}}$$

$$\boxed{\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} \\ h(n) &= \frac{1}{2\pi i} \oint_C H(z')\, z'^{\,n-1}\, dz' \\ &= \sum_k \mathrm{Res}\left( H(z)\, z^{n-1} , z_k \right) \end{aligned}}$$

$C$ is in the same region of analyticity of $f(z)$
typically a circle centered on $z_m$

$z_k$ within $C$ : singularities of $\frac{f(z)}{(z-z_m)^{n+1}}$

$C$ is in the same region of analyticity of $H(z)$
typically a circle centered on $z_m$

generally a circle centered on the origin
may enclose any or all singularities of $H(z)$
often the unit circle

$z_k$ within $C$ : singularities of $H(z)\, z^{n-1}$

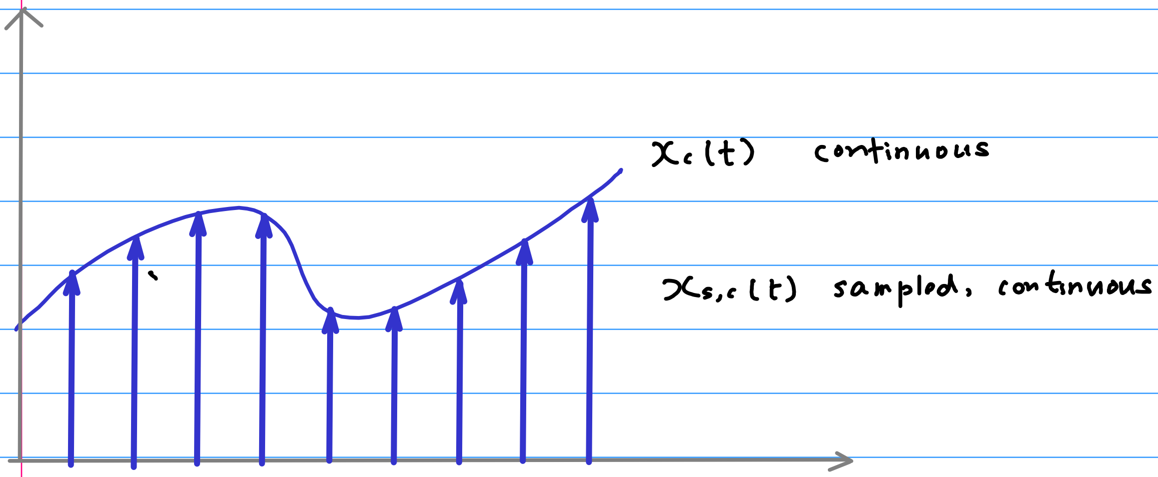
$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} \qquad z \in \text{R.O.C}$$

$$h(n) = \frac{1}{2\pi i} \oint_C H(z')\, z'^{\,n-1}\, dz' \qquad C \text{ in R.O.C.}$$

$$= \sum_k \text{Res}\left(H(z)\, z^{n-1},\ z_k\right)$$

① a power series representation of a function $f(z)$ of a complex variable $z$

② a transform $H(z)$ of a sequence of 1



$$x_{s,c}(t) = \sum_{n=-\infty}^{+\infty} x(n)\,\delta_c(t-n\Delta t)$$

$$X_{s,c}(s) = \mathcal{L}\{x_{s,c}(t)\} = \int_{-\infty}^{\infty} \boxed{\sum_{n=-\infty}^{+\infty} x(n)\,\delta_c(t-n\Delta t)}\, e^{-st}\,dt$$

$$= \sum_{n=-\infty}^{+\infty} x(n) \int_{-\infty}^{\infty} \delta_c(t-n\Delta t)\, e^{-st}\,dt$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-sn\Delta t} \qquad e^{s\Delta t} \triangleq z$$

$$X_{s,c}(s) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \Big|_{z=e^{s\Delta t}}$$

$$X_{s,c}(s) = X(z)\Big|_{z=e^{s\Delta t}}$$

$$X_{s.c}(s) = \mathcal{L}\{x_{s.c}(t)\} = X(z)\Big|_{z=e^{s\Delta t}}$$

$x_{s.c}(t)$ an impulse train
whose coefficients are given by $x[n] = x_c(n\Delta t)$