Laurent Series and z-Transform
- Geometric Series Combinations

A

20200702 Thr

Combinations of a and z
-- common ratio in a geometric series

$\boxed{a \quad z}$ a^{n}	$\boxed{a^{\boxed{-1}} \quad z}$ a^{-n}
$\Big\downarrow \boxed{-1} \quad \Big\downarrow \boxed{-1}$	$\Big\downarrow \boxed{-1} \quad \Big\downarrow \boxed{-1}$
$\boxed{a^{\boxed{-1}} \quad z^{\boxed{-1}}}$ a^{n}	$\boxed{a \quad z^{\boxed{-1}}}$ a^{-n}

the same formula,
different representations

Geometric Series

common ratio **common ratio**

$$\boxed{az} \rightarrow \frac{1}{1-az} = \frac{a^{-1}z^{-1}}{a^{-1}z^{-1}-1} \qquad \boxed{az}$$

$$\boxed{a^{-1}z^{-1}} \rightarrow \frac{1}{1-a^{-1}z^{-1}} = \frac{az}{az-1} \qquad \boxed{a^{-1}z^{-1}}$$

$$\boxed{a^{-1}z} \rightarrow \frac{1}{1-a^{-1}z} = \frac{az^{-1}}{az^{-1}-1} \qquad \boxed{a^{-1}z}$$

$$\boxed{az^{-1}} \rightarrow \frac{1}{1-az^{-1}} = \frac{a^{-1}z}{a^{-1}z-1} \qquad \boxed{a^{-1}z^{-1}}$$

two equivalent representations
of geometric series

the same formula
with different ROCs

different Geometric Series

complementary ranges

common ratio			inversed common ratio
az	causal u(n) $\frac{1}{1-az}$ $\lvert z\rvert<a^{-1}$ $a^0z^0+a^1z^1+a^2z^2+\cdots$	anti-causal u(-n-1) $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert>a^{-1}$ $-(a^{-1}z^{-1}+a^{-2}z^{-2}+a^{-3}z^{-3}+\cdots)$	$a^{-1}z^{-1}$
$a^{-1}z^{-1}$	anti-causal u(-n) $\frac{1}{1-a^{-1}z^{-1}}$ $\lvert z\rvert>a^{-1}$ $a^0z^0+a^{-1}z^{-1}+a^{-2}z^{-2}+\cdots$	causal u(n-1) $-\frac{az}{1-az}$ $\lvert z\rvert<a^{-1}$ $-(a^1z^1+a^2z^2+a^3z^3+\cdots)$	az
$a^{-1}z$	causal u(n) $\frac{1}{1-a^{-1}z}$ $\lvert z\rvert<a$ $a^0z^0+a^{-1}z^1+a^{-2}z^2+\cdots$	anti-causal u(-n-1) $-\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert>a$ $-(a^1z^{-1}+a^2z^{-2}+a^3z^{-3}+\cdots)$	az^{-1}
az^{-1}	anti-causal u(-n) $\frac{1}{1-az^{-1}}$ $\lvert z\rvert>a$ $a^0z^0+a^1z^{-1}+a^2z^{-2}+\cdots$	causal u(n-1) $-\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert<a$ $-(a^{-1}z^1+a^{-2}z^2+a^{-3}z^3+\cdots)$	$a^{-1}z$

Representation I	**Representation II**
geometric series starting with a unit term	geometric series starting with a non-unit term
non-shifted range u(n), u(-n)	shifted range u(n-1), u(-n-1)

the different formula with the same ROC

different Geometric Series

common ratio				inversed common ratio
	causal u(n)		left shifted u(-n-1)	
$a\,z$	$\frac{1}{1-az}$ $\quad |z|<a^{-1}$		$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z|>a^{-1}$	$a^{-1}z^{-1}$
	$a^0z^0 + a^1z^1 + a^2z^2 + \cdots$		$-(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$	
	anti-causal u(-n)		right shifted u(n-1)	
$a^{-1}z^{-1}$	$\frac{1}{1-a^{-1}z^{-1}}$ $\quad |z|>a^{-1}$		$-\frac{az}{1-az}$ $\quad |z|<a^{-1}$	$a\,z$
	$a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots$		$-(a^1z^1 + a^2z^2 + a^3z^3 + \cdots)$	
	causal u(n)		left shifted u(-n-1)	
$a^{-1}z$	$\frac{1}{1-a^{-1}z}$ $\quad |z|<a$		$-\frac{az^{-1}}{1-az^{-1}}$ $\quad |z|>a$	$a\,z^{-1}$
	$a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots$		$-(a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots)$	
	anti-causal u(-n)		right shifted u(n-1)	
$a\,z^{-1}$	$\frac{1}{1-az^{-1}}$ $\quad |z|>a$		$-\frac{a^{-1}z}{1-a^{-1}z}$ $\quad |z|<a$	$a^{-1}z$
	$a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots$		$-(a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \cdots)$	

Representation I	**Representation II**
geometric series starting with a unit term	geometric series starting with a non-unit term
non-shifted range u(n), u(-n)	shifted range u(n-1), u(-n-1)

Geometric Power Series Property (1)

Each representation has it own ROC
(Region of Convergence)

common ratio $a z \longrightarrow |z| < a^{-1}$ ROC

common ratio $a^{-1} z^{-1} \longrightarrow |z| > a^{-1}$ ROC

common ratio $a^{-1} z \longrightarrow |z| < a$ ROC

common ratio $a z^{-1} \longrightarrow |z| > a$ ROC

Geometric Power Series Property (2)

Starting terms

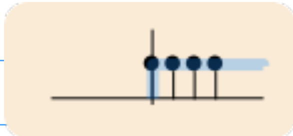
		geometric series starting with **a unit term**	geometric series starting with **a non-unit term (common ratio)**		
z	causal	$\frac{1}{1-az}$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	anti-causal	z^{-1}
z^{-1}	anti-causal	$\frac{1}{1-a^{-1}z^{-1}}$	$-\frac{az}{1-az}$	causal	z
z	causal	$\frac{1}{1-a^{-1}z}$	$-\frac{az^{-1}}{1-az^{-1}}$	anti-causal	z^{-1}
z^{-1}	anti-causal	$\frac{1}{1-az^{-1}}$	$-\frac{a^{-1}z}{1-a^{-1}z}$	causal	z
		Representation I	**Representation II**		

related to shifting

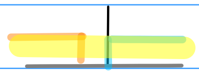
Geometric Power Series Property (3)

Complementary Ranges

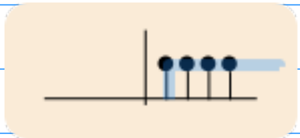
u[n]



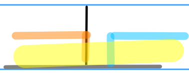
u[-n-1]



u[n-1]



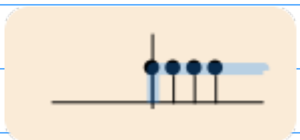
u[-n]



Shifted Ranges

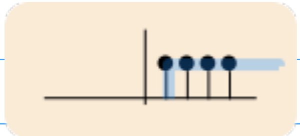
left shfited range

u[n]



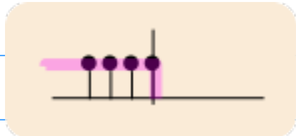
u[-n-1]

u[n-1]

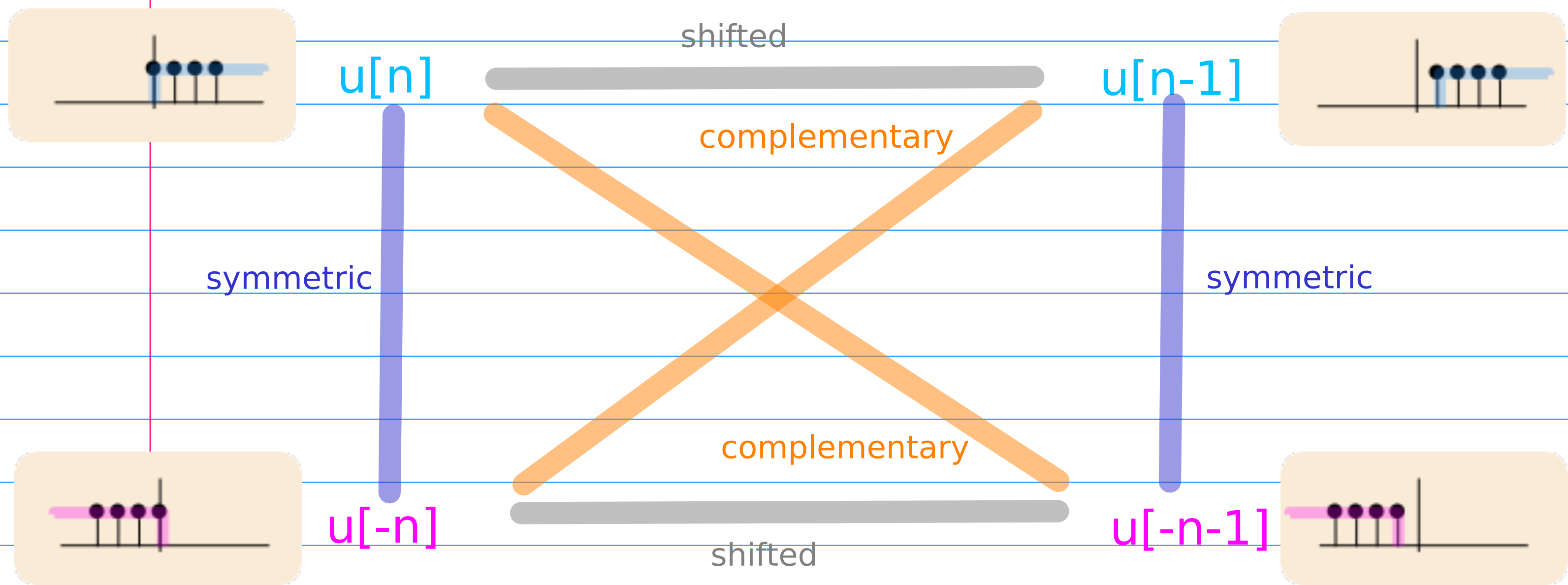


right shfited range

u[-n]

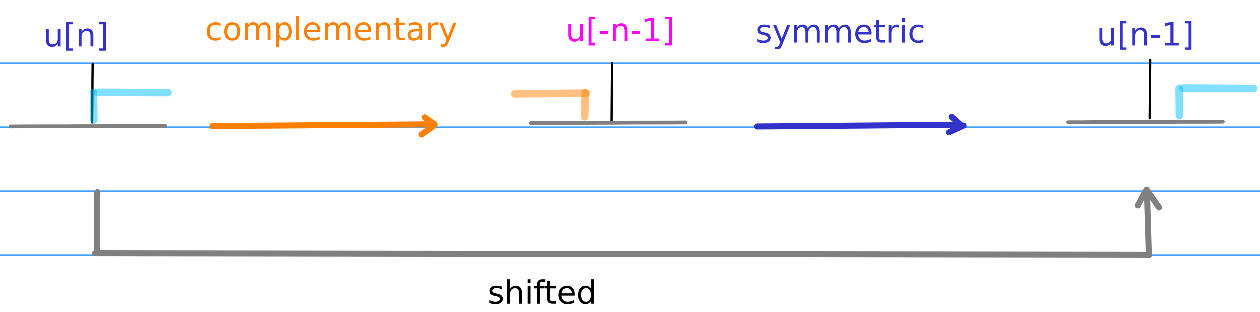


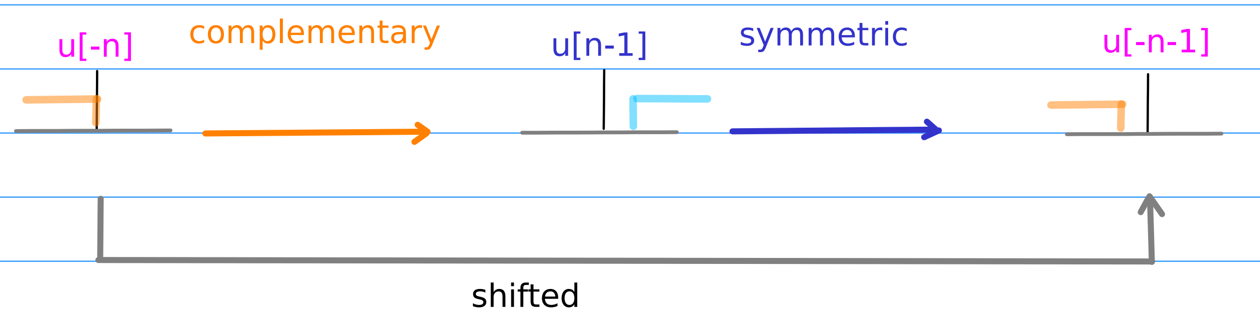
Geometric Power Series Property (4)



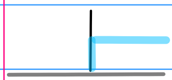
u[n] complementary u[-n-1] symmetric u[n-1]

u[-n] complementary u[n-1] symmetric u[-n-1]

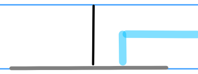




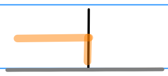
Geometric Power Series Property (5)



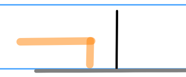
default range
u[n], u[-n]



shifted range
u[n-1], u[-n-1]



geometric series
starting with
a unit term



geometric series
starting with
a non-unit term
(common ratio)

u[n]	$\frac{1}{1-az}$	complementary	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	u[-n-1]
u[-n]	$\frac{1}{1-a^{-1}z^{-1}}$	complementary	$-\frac{az}{1-az}$	u[n-1]
u[n]	$\frac{1}{1-a^{-1}z}$	complementary	$-\frac{az^{-1}}{1-az^{-1}}$	u[-n-1]
u[-n]	$\frac{1}{1-az^{-1}}$	complementary	$-\frac{a^{-1}z}{1-a^{-1}z}$	u[n-1]

default range u(n), u(-n)

shifted range u(n-1), u(-n-1)

u[n]	$\frac{1}{1-az}$	shifted	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	u[-n-1]
u[-n]	$\frac{1}{1-a^{-1}z^{-1}}$	shifted	$-\frac{az}{1-az}$	u[n-1]
u[n]	$\frac{1}{1-a^{-1}z}$	shifted	$-\frac{az^{-1}}{1-az^{-1}}$	u[-n-1]
u[-n]	$\frac{1}{1-az^{-1}}$	shifted	$-\frac{a^{-1}z}{1-a^{-1}z}$	u[n-1]

A Common Ratio and a Exponent

$\boxed{a^{n}}$	$\boxed{a^{-n}}$
$\boxed{a^{1}z^{1}}$	$\boxed{a^{-1}z^{1}}$
$\updownarrow \boxed{-1} \quad \updownarrow \boxed{-1}$	$\updownarrow \boxed{-1} \quad \updownarrow \boxed{-1}$
$\boxed{a^{-1}z^{-1}}$	$\boxed{a^{1}z^{-1}}$
$\boxed{a^{n}}$	$\boxed{a^{-n}}$

Exponent

$1 / 1 \longrightarrow$	a^{n}
$-1 / -1 \longrightarrow$	a^{n}
$-1 / 1 \longrightarrow$	a^{-n}
$1 / -1 \longrightarrow$	a^{-n}

A Common Ratio and a Default Range

a^{n} u[n] causal

$a \, \boxed{z}$ $|z| < a^{-1}$

-1 -1

$a^{-1} \, \boxed{z^{-1}}$ $|z| > a^{-1}$

a^{n} u[-n] anti-causal

a^{-n} u[n] causal

$a^{-1} \, \boxed{z}$ $|z| < a$

-1 -1

$a \, \boxed{z^{-1}}$ $|z| > a$

a^{-n} u[-n] anti-causal

Default Ranges

$\boxed{z}$ u[n] causal

$\boxed{z^{-1}}$ u[-n] anti-causal

A Common Ratio and a Complementary Range

a^n u[n] u[-n-1]
left shifted range

$a z$ $|z| < a^{-1}$ $|z| > a^{-1}$

$\boxed{-1}$ $\boxed{-1}$

$a^{\boxed{-1}} z^{\boxed{-1}}$ $|z| > a^{-1}$ $|z| < a^{-1}$

a^n u[-n] u[n-1]
right shifted range

a^{-n} u[n] u[-n-1]
left shifted range

$a^{\boxed{-1}} z$ $|z| < a$ $|z| > a$

$\boxed{-1}$ $\boxed{-1}$

$a z^{\boxed{-1}}$ $|z| > a$ $|z| < a$

a^{-n} u[-n] u[n-1]
right shifted range

Complementary Ranges

default range
u[n] $\longleftrightarrow$ u[-n-1]

default range
u[-n] $\longleftrightarrow$ u[n-1]

A Common Ratio and a Symmetric Range

a^n — u[n], u[-n-1]

left shifted range

$a \; z$ — $|z| < a^{-1}$, $|z| > a^{-1}$

$\updownarrow \; -1 \qquad \updownarrow \; -1$

$a^{-1} \; z^{-1}$ — $|z| > a^{-1}$, $|z| < a^{-1}$

a^n — u[-n], u[n-1]

right shifted range

a^{-n} — u[n], u[-n-1]

left shifted range

$a^{-1} \; z$ — $|z| < a$, $|z| > a$

$\updownarrow \; -1 \qquad \updownarrow \; -1$

$a \; z^{-1}$ — $|z| > a$, $|z| < a$

a^{-n} — u[-n], u[n-1]

right shifted range

Symmetric Ranges

$a \; z$	a^n u[n]	$\leftrightarrow$	a^n u[-n]	$a^{-1} \; z^{-1}$
	a^n u[-n-1]	$\leftrightarrow$	a^n u[n-1]	
$a^{-1} \; z$	a^{-n} u[n]	$\leftrightarrow$	a^{-n} u[-n]	$a \; z^{-1}$
	a^{-n} u[-n-1]	$\leftrightarrow$	a^{-n} u[n-1]	

Common Ratio and ROC

default	★ $1 / (1 - a z)$	default	★ $1 / (1 - a^{-1} z)$
complementary	$a^{-1} z^{-1} / (1 - a^{-1} z^{-1})$	complementary	$a z^{-1} / (1 - a z^{-1})$

$\boxed{a z}$ $|z| < a^{-1}$ $|z| > a^{-1}$ $\boxed{a^{\boxed{-1}} z}$ $|z| < a$ $|z| > a$

$\updownarrow \boxed{-1}$ $\updownarrow \boxed{-1}$ $\updownarrow \boxed{-1}$ $\updownarrow \boxed{-1}$

$\boxed{a^{\boxed{-1}} z^{\boxed{-1}}}$ $|z| > a^{-1}$ $|z| < a^{-1}$ $\boxed{a z^{\boxed{-1}}}$ $|z| > a$ $|z| < a$

default	★ $1 / (1 - a^{-1} z^{-1})$	default	★ $1 / (1 - a z^{-1})$
complementary	$a z / (1 - a z)$	complementary	$a^{-1} z / (1 - a^{-1} z)$

Each common ratio has two represenations — Sequences

Each representation has it own ROC — Ranges

The two representations have complementary ROC's — complementary ROC's

Common Ratio and ROC

		1/1 = +1	
★ $1/(1-az)$	$\lvert z\rvert < a^{-1}$	causal (z)	a^{n} u[n] ★ default range
$a^{-1}z^{-1}/(1-a^{-1}z^{-1})$	$\lvert z\rvert > a^{-1}$	anti-causal	a^{n} u[-n-1] complementary
		-1/-1 = +1	
		-1/-1 = +1	
★ $1/(1-a^{-1}z^{-1})$	$\lvert z\rvert > a^{-1}$	anti-causal	a^{n} u[-n] ★ default range
$a\,z/(1-a\,z)$	$\lvert z\rvert < a^{-1}$	causal (z)	a^{n} u[n-1] complementary
		1/1 = +1	
		-1/1 = -1	
★ $1/(1-a^{-1}z)$	$\lvert z\rvert < a$	causal (z)	a^{-n} u[n] ★ default range
$a\,z^{-1}/(1-a\,z^{-1})$	$\lvert z\rvert > a$	anti-causal	a^{-n} u[-n-1] complementary
		1/-1 = -1	
		1/-1 = -1	
★ $1/(1-a\,z^{-1})$	$\lvert z\rvert > a$	anti-causal	a^{-n} u[-n] ★ default range
$a^{-1}z/(1-a^{-1}z)$	$\lvert z\rvert < a$	causal (z)	a^{-n} u[n-1] complementary
		-1/1 = -1	

Common Ratio and ROC - Summary

ordered by complementary relation

★ $1 / (1 - a z)$	$\lvert z \rvert < a^{-1}$	$a^{n}\, u[n]$	★ default range
$a^{-1} z^{-1} / (1 - a^{-1} z^{-1})$	$\lvert z \rvert > a^{-1}$	$a^{n}\, u[-n-1]$	complementary
★ $1 / (1 - a^{-1} z^{-1})$	$\lvert z \rvert > a^{-1}$	$a^{n}\, u[-n]$	★ default range
$a z / (1 - a z)$	$\lvert z \rvert < a^{-1}$	$a^{n}\, u[n-1]$	complementary
★ $1 / (1 - a^{-1} z)$	$\lvert z \rvert < a$	$a^{-n}\, u[n]$	★ default range
$a z^{-1} / (1 - a z^{-1})$	$\lvert z \rvert > a$	$a^{-n}\, u[-n-1]$	complementary
★ $1 / (1 - a z^{-1})$	$\lvert z \rvert > a$	$a^{-n}\, u[-n]$	★ default range
$a^{-1} z / (1 - a^{-1} z)$	$\lvert z \rvert < a$	$a^{-n}\, u[n-1]$	complementary

Common Ratio and ROC - Summary

ordered by symmetric relation

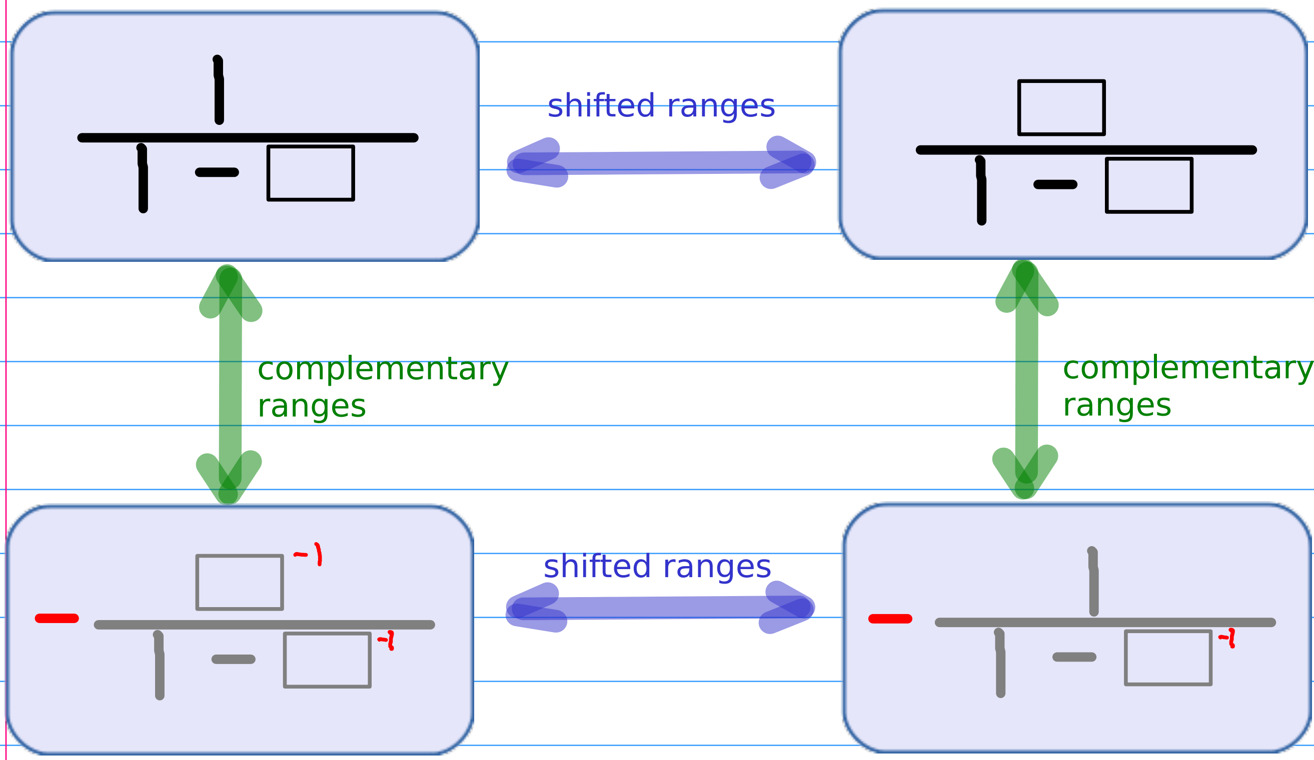
★ $1 / (1 - a z)$	$\lvert z \rvert < a^{-1}$	a^{n}	$u[n]$	★ default range
★ $1 / (1 - a^{-1} z^{-1})$	$\lvert z \rvert > a^{-1}$	a^{n}	$u[-n]$	★ default range
$a^{-1} z^{-1} / (1 - a^{-1} z^{-1})$	$\lvert z \rvert > a^{-1}$	a^{n}	$u[-n-1]$	complementary
$a z / (1 - a z)$	$\lvert z \rvert < a^{-1}$	a^{n}	$u[n-1]$	complementary
★ $1 / (1 - a^{-1} z)$	$\lvert z \rvert < a$	a^{-n}	$u[n]$	★ default range
★ $1 / (1 - a z^{-1})$	$\lvert z \rvert > a$	a^{-n}	$u[-n]$	★ default range
$a z^{-1} / (1 - a z^{-1})$	$\lvert z \rvert > a$	a^{-n}	$u[-n-1]$	complementary
$a^{-1} z / (1 - a^{-1} z)$	$\lvert z \rvert < a$	a^{-n}	$u[n-1]$	complementary

Common Ratio and ROC - Summary

ordered by shift relation

★ $1 / (1 - a z)$	$\lvert z \rvert < a^{-1}$	a^{n}	u[n]	★ default range
$a z / (1 - a z)$	$\lvert z \rvert < a^{-1}$	a^{n}	u[n-1]	complementary
★ $1 / (1 - a^{-1} z^{-1})$	$\lvert z \rvert > a^{-1}$	a^{n}	u[-n]	★ default range
$a^{-1} z^{-1} / (1 - a^{-1} z^{-1})$	$\lvert z \rvert > a^{-1}$	a^{n}	u[-n-1]	complementary
★ $1 / (1 - a^{-1} z)$	$\lvert z \rvert < a$	a^{-n}	u[n]	★ default range
$a^{-1} z / (1 - a^{-1} z)$	$\lvert z \rvert < a$	a^{-n}	u[n-1]	complementary
★ $1 / (1 - a z^{-1})$	$\lvert z \rvert > a$	a^{-n}	u[-n]	★ default range
$a z^{-1} / (1 - a z^{-1})$	$\lvert z \rvert > a$	a^{-n}	u[-n-1]	complementary

Common Ratios and Representations



az	$\lvert z\rvert < a^{-1}$
$a^{-1}z^{-1}$	$\lvert z\rvert > a^{-1}$
$a^{-1}z$	$\lvert z\rvert < a$
az^{-1}	$\lvert z\rvert > a$

$a z$ $\quad |z| < a^{-1}$

$\frac{1}{1-az}$	$\boxed{\frac{az}{1-az}}$
$-\boxed{\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}}$	$-\frac{1}{1-a^{-1}z^{-1}}$

Right Shifted

u(n)	u(n-1)
u(-n-1)	u(-n)

$a^{-1} z^{-1}$ $\quad |z| > a^{-1}$

$\frac{1}{1-a^{-1}z^{-1}}$	$\boxed{\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}}$
$-\boxed{\frac{az}{1-az}}$	$-\frac{1}{1-az}$

Left Shifted

u(-n)	u(-n-1)
u(n-1)	u(n)

$a^{-1} z$ $\quad |z| < a$

$\frac{1}{1-a^{-1}z}$	$\boxed{\frac{a^{-1}z}{1-a^{-1}z}}$
$-\boxed{\frac{az^{-1}}{1-az^{-1}}}$	$-\frac{1}{1-az^{-1}}$

Right Shifted

u(n)	u(n-1)
u(-n-1)	u(-n)

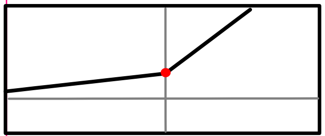
$a z^{-1}$ $\quad |z| > a$

$\frac{1}{1-az^{-1}}$	$\boxed{\frac{az^{-1}}{1-az^{-1}}}$
$-\boxed{\frac{a^{-1}z}{1-a^{-1}z}}$	$-\frac{1}{1-a^{-1}z}$

Left Shifted

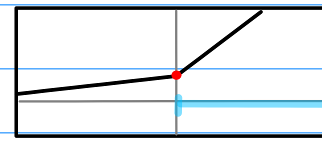
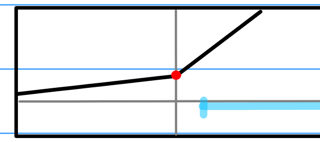
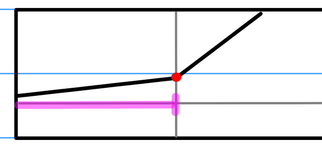
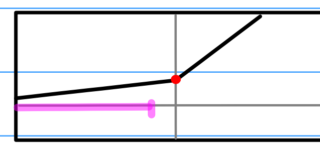
u(-n)	u(-n-1)
u(n-1)	u(n)

a^n



Geometric Series Combinations (1)

* inverted relation is ignored

Common Ratio	**2 Geometric Series**	**2 Sequences**
$\boxed{a\,z}$	$\frac{1}{1-az}$ $\quad \lvert z\rvert < a^{-1}$ a^n u[n]	
right shifted	$\frac{az}{1-az}$ $\quad \lvert z\rvert < a^{-1}$ a^n u[n-1]	
$\boxed{a^{\boxed{-1}}z^{\boxed{-1}}}$	$\frac{1}{1-a^{-1}z^{-1}}$ $\quad \lvert z\rvert > a^{-1}$ a^n u[-n]	
left shifted	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad \lvert z\rvert > a^{-1}$ a^n u[-n-1]	

$a^{n}\ u[n]$

$$\frac{1}{1-az} \qquad |z|<a^{-1}$$

$a^{-n}\ u[n]$

$$\frac{1}{1-a^{-1}z} \qquad |z|<a$$

$a^{n}\ u[n-1]$

$$\frac{az}{1-az} \qquad |z|<a^{-1}$$

$a^{-n}\ u[n-1]$

$$\frac{a^{-1}z}{1-a^{-1}z} \qquad |z|<a$$

$a^{n}\ u[-n]$

$$\frac{1}{1-a^{-1}z^{-1}} \qquad |z|>a^{-1}$$

$a^{-n}\ u[-n]$

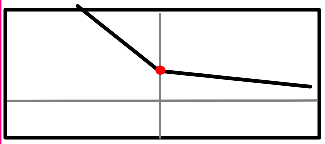
$$\frac{1}{1-az^{-1}} \qquad |z|>a$$

$a^{n}\ u[-n-1]$

$$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \qquad |z|>a^{-1}$$

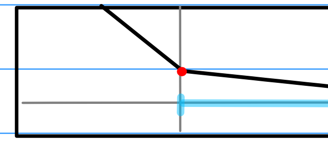
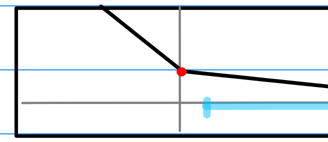
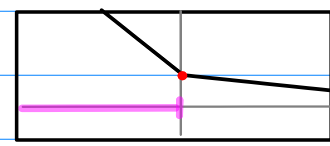
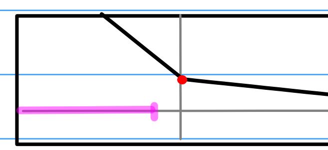
$a^{-n}\ u[-n-1]$

$$\frac{az^{-1}}{1-az^{-1}} \qquad |z|>a$$

a^{-n} 

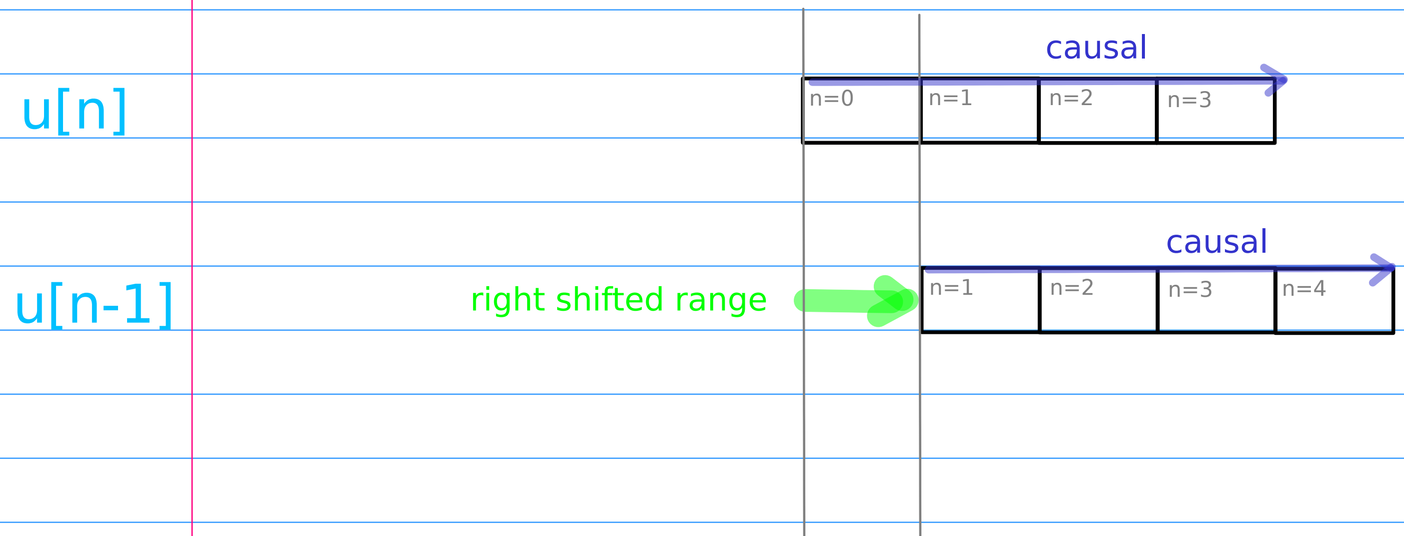
Geometric Series Combinations (2)

* inverted relation is ignored

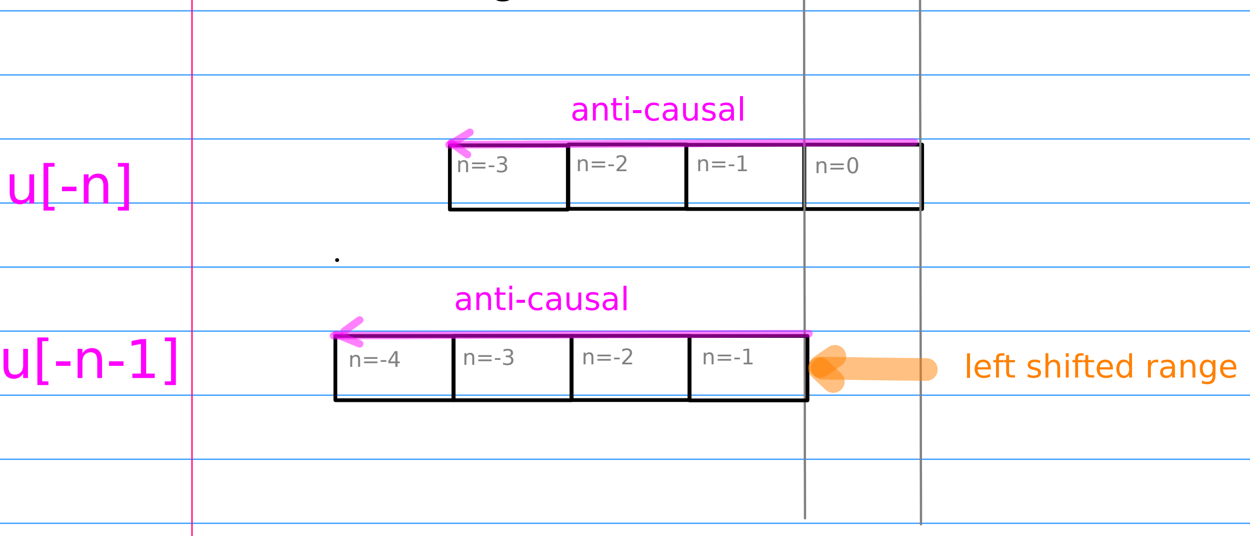
Common Ratio		**2 Geometric Series**	**2 Sequences**
$a^{\boxed{-1}} z$		$\frac{1}{1-a^{-1}z}$ $\quad \lvert z\rvert < a$ $a^{-n}\, u[n]$	 a^{-n}
	right shifted	$\frac{a^{-1}z}{1-a^{-1}z}$ $\quad \lvert z\rvert < a$ $a^{-n}\, u[n-1]$	 a^{-n}
$a\, z^{\boxed{-1}}$		$\frac{1}{1-a z^{-1}}$ $\quad \lvert z\rvert > a$ $a^{-n}\, u[-n]$	 a^{-n}
	left shifted	$\frac{a z^{-1}}{1-a z^{-1}}$ $\quad \lvert z\rvert > a$ $a^{-n}\, u[-n-1]$	 a^{-n}

Shift Relations of Ranges

Right Shifted Range Relation

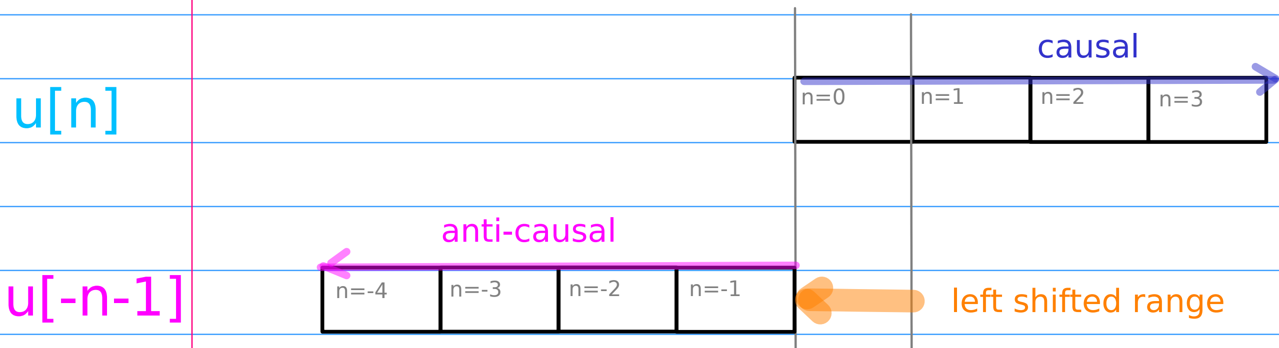


Left Shifted Range Relation

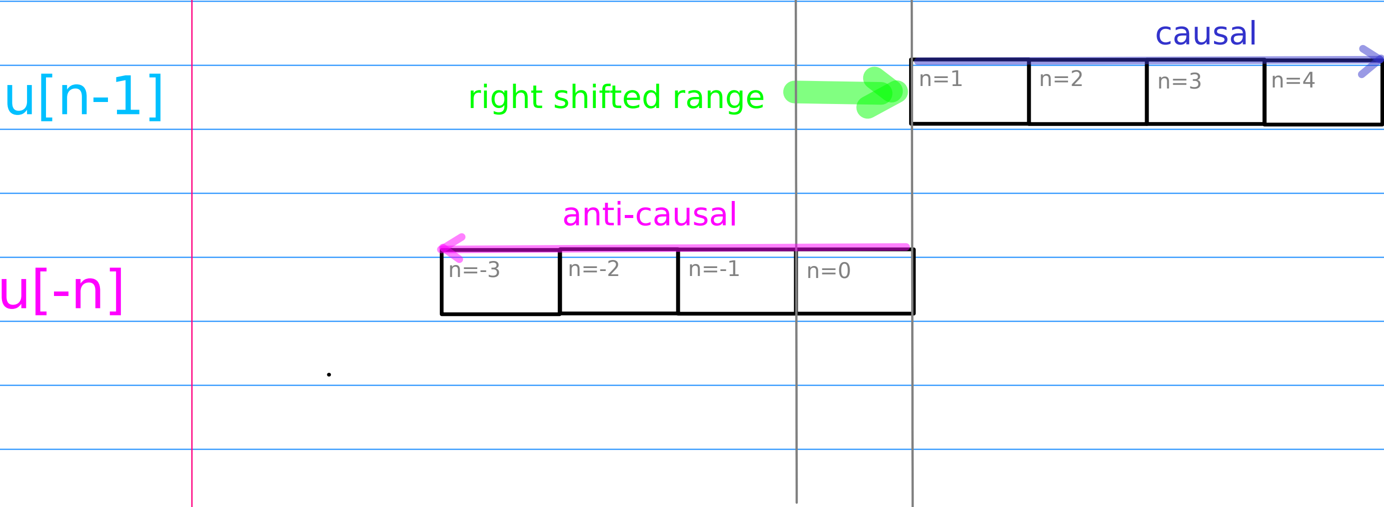


Complementary Relations of Ranges

Complementary Range Relation



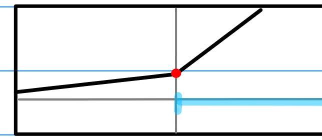
Complementary Range Relation



[Complementary Range & Inverted Relation]

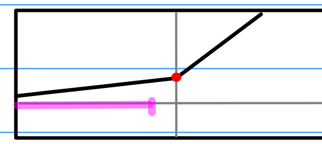
* inverted relation is ignored

$a\ z$ $\qquad \dfrac{1}{1-az}\quad |z|<a^{-1}$ $\qquad a^0z^0 + a^1z^1 + a^2z^2 + \cdots$ $\quad a^n$ u[n]

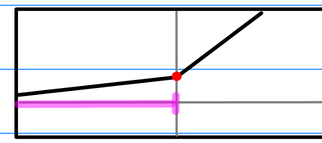


-1

$a^{\boxed{-1}}z^{\boxed{-1}}$ $\qquad \dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}\quad |z|>a^{-1}$ $\qquad a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots$ $\quad a^n$ u[-n-1]

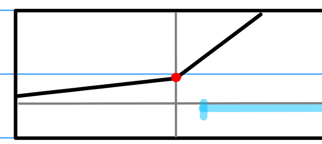


$a^{\boxed{-1}}z^{\boxed{-1}}$ $\qquad \dfrac{1}{1-a^{-1}z^{-1}}\quad |z|>a^{-1}$ $\qquad a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots$ $\quad a^n$ u[-n]

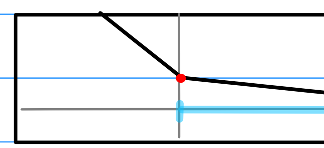


-1

$a\ z$ $\qquad \dfrac{az}{1-az}\quad |z|<a^{-1}$ $\qquad a^1z^1 + a^2z^2 + a^3z^3 + \cdots$ $\quad a^n$ u[n-1]

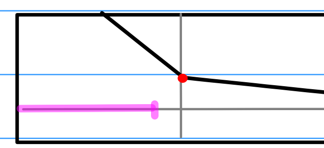


$a^{\boxed{-1}}z$ $\qquad \dfrac{1}{1-a^{-1}z}\quad |z|<a$ $\qquad a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots$ $\quad a^{-n}$ u[n]

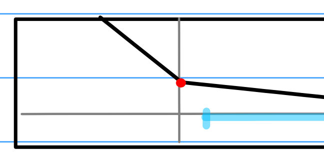


-1

$a\ z^{\boxed{-1}}$ $\qquad \dfrac{az^{-1}}{1-az^{-1}}\quad |z|>a$ $\qquad a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$ $\quad a^{-n}$ u[-n-1]

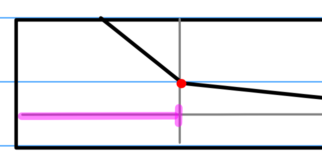


$a\ z^{\boxed{-1}}$ $\qquad \dfrac{1}{1-az^{-1}}\quad |z|>a$ $\qquad a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots$ $\quad a^{-n}$ u[n-1]



-1

$a^{\boxed{-1}}z$ $\qquad \dfrac{a^{-1}z}{1-a^{-1}z}\quad |z|<a$ $\qquad a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \cdots$ $\quad a^{-n}$ u[-n]

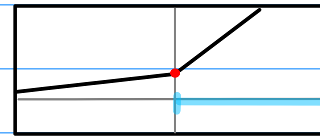


[Shifted Range Relation]

* inverted relation is ignored

$$a\ z$$
$$\frac{1}{1-az} \quad |z|<a^{-1}$$
$a^0z^0 + a^1z^1 + a^2z^2 + \cdots$

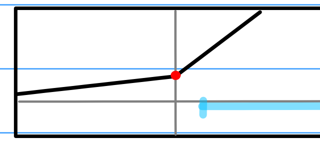
a^n u[n]



$$\|$$

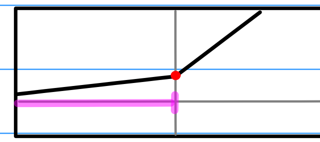
$$a\ z$$
$$\frac{az}{1-az} \quad |z|<a^{-1}$$
$a^1z^1 + a^2z^2 + a^3z^3 + \cdots$

a^n u[n-1]



$$a^{\boxed{-1}}\ z^{\boxed{-1}}$$
$$\frac{1}{1-a^{-1}z^{-1}} \quad |z|>a^{-1}$$
$a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots$

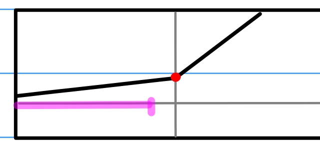
a^n u[-n]



$$\|$$

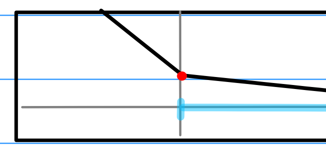
$$a^{\boxed{-1}}\ z^{\boxed{-1}}$$
$$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1}$$
$a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots$

a^n u[-n-1]



$$a^{\boxed{-1}}\ z$$
$$\frac{1}{1-a^{-1}z} \quad |z|<a$$
$a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots$

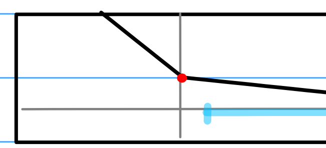
a^{-n} u[n]



$$\|$$

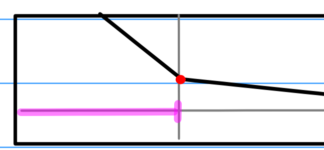
$$a^{\boxed{-1}}\ z$$
$$\frac{a^{-1}z}{1-a^{-1}z} \quad |z|<a$$
$a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \cdots$

a^{-n} u[n-1]



$$a\ z^{\boxed{-1}}$$
$$\frac{1}{1-az^{-1}} \quad |z|>a$$
$a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots$

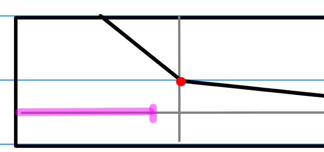
a^{-n} u[-n]



$$\|$$

$$a\ z^{\boxed{-1}}$$
$$\frac{az^{-1}}{1-az^{-1}} \quad |z|>a$$
$a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$

a^{-n} u[-n-1]



each formula has two geometric series
- two common ratios with inversed relation

$a \quad z$	$\frac{1}{1-az} \quad \lvert z\rvert < a^{-1}$	$a^{n}\,u[n]$
		Complementary Ranges
$a^{\boxed{-1}} z^{\boxed{-1}}$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad \lvert z\rvert > a^{-1}$	$-a^{n}\,u[-n-1]$
$a^{\boxed{-1}} z^{\boxed{-1}}$	$\frac{1}{1-a^{-1}z^{-1}} \quad \lvert z\rvert > a^{-1}$	$a^{n}\,u[-n]$
		Complementary Ranges
$a \quad z$	$-\frac{az}{1-az} \quad \lvert z\rvert < a^{-1}$	$-a^{n}\,u[n-1]$
$a^{\boxed{-1}} \quad z$	$\frac{1}{1-a^{-1}z} \quad \lvert z\rvert < a$	$a^{-n}\,u[n]$
		Complementary Ranges
$a \quad z^{\boxed{-1}}$	$-\frac{az^{-1}}{1-az^{-1}} \quad \lvert z\rvert > a$	$-a^{-n}\,u[-n-1]$
$a \quad z^{\boxed{-1}}$	$\frac{1}{1-az^{-1}} \quad \lvert z\rvert > a$	$a^{-n}\,u[-n]$
		Complementary Ranges
$a^{\boxed{-1}} \quad z$	$-\frac{a^{-1}z}{1-a^{-1}z} \quad \lvert z\rvert < a$	$-a^{-n}\,u[n-1]$

each common ratio is associated with
2 different sequences (represenations)

$a\,z$	$\dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^{n}\,u[n]$
			Shifted Ranges
$a\,z$	$-\dfrac{az}{1-az}$	$\lvert z\rvert < a^{-1}$	$-a^{n}\,u[n-1]$
$a^{-1}z^{-1}$	$\dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$a^{n}\,u[-n]$
			Shifted Ranges
$a^{-1}z^{-1}$	$-\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^{n}\,u[-n-1]$
$a^{-1}z$	$\dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$	$a^{-n}\,u[n]$
			Shifted Ranges
$a^{-1}z$	$-\dfrac{a^{-1}z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$-a^{-n}\,u[n-1]$
$a\,z^{-1}$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$	$a^{-n}\,u[-n]$
			Shifted Ranges
$a\,z^{-1}$	$-\dfrac{az^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$-a^{-n}\,u[-n-1]$