Double Integrals (5A)

- Double Integral
- Double Integrals in Polar Coordinates
- Green's Theorem

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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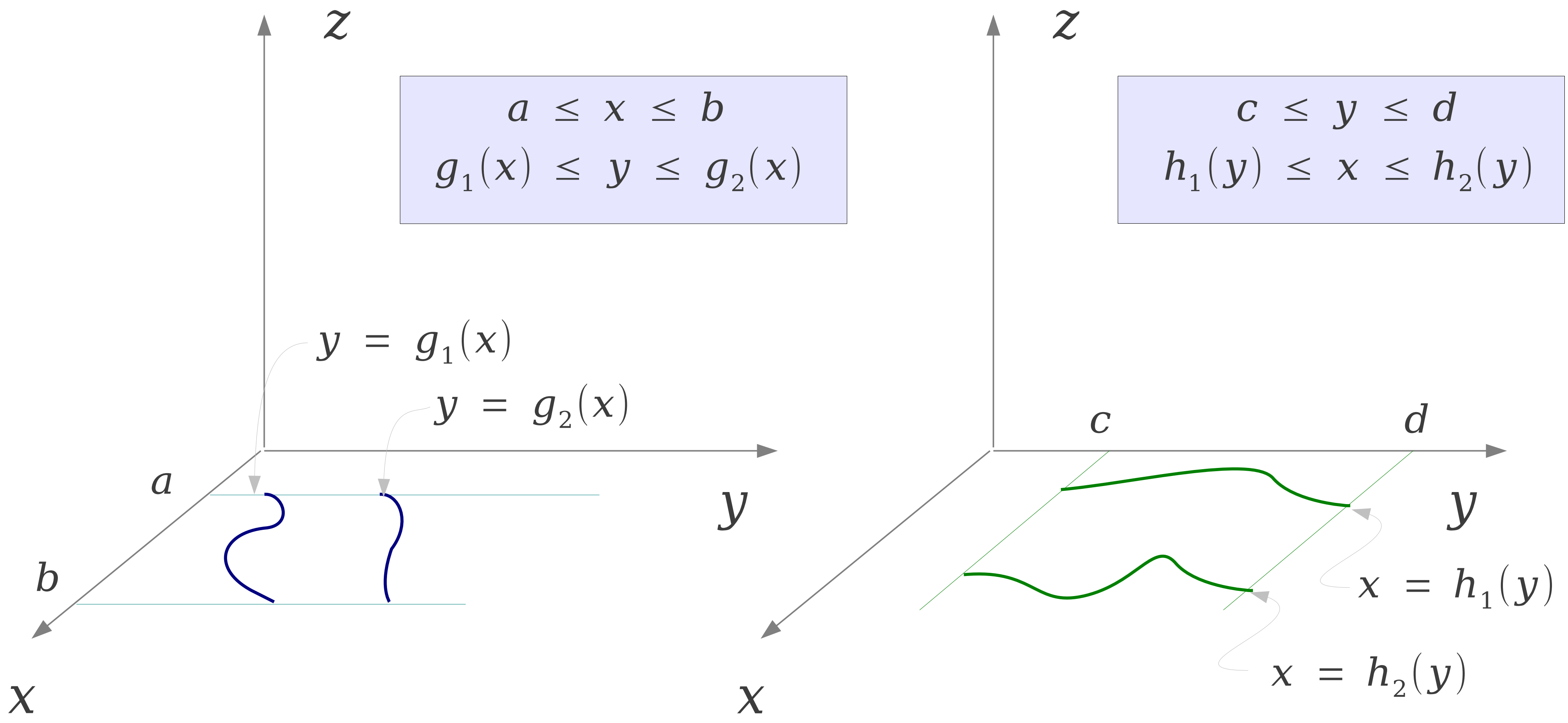
Area and Volume

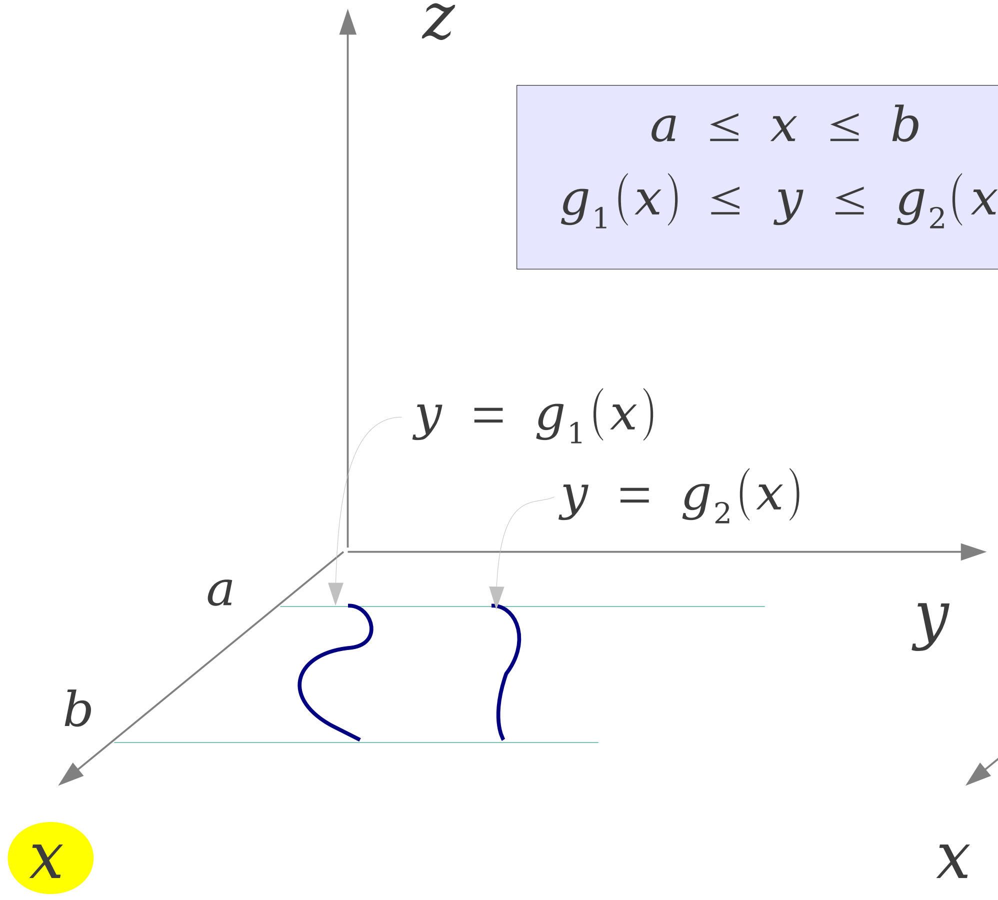
$$A = \iint_R dA$$

$$V = \iint_R f(x, y) dA$$

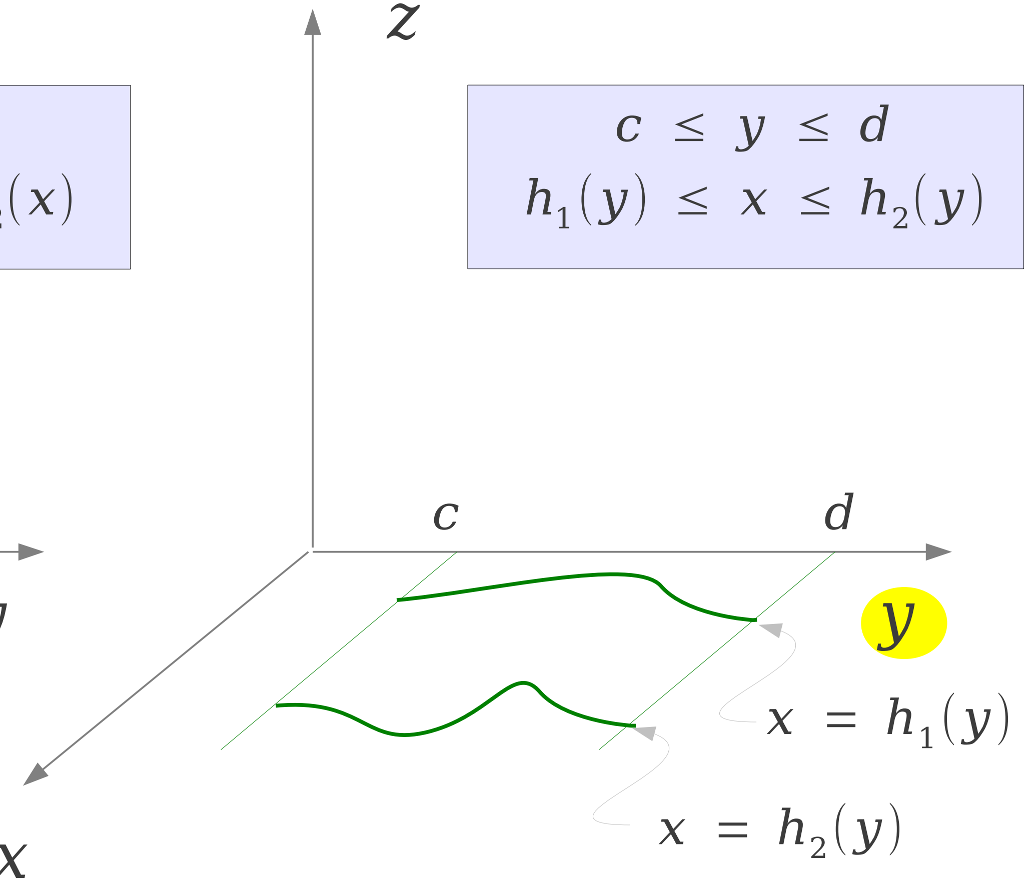
Type I and Type II



Fubini's Theorem



$$a \le x \le b$$
$$g_1(x) \le y \le g_2(x)$$

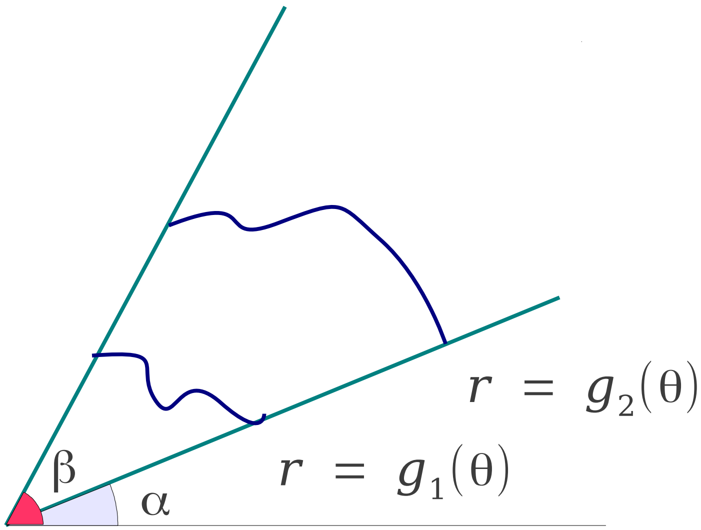


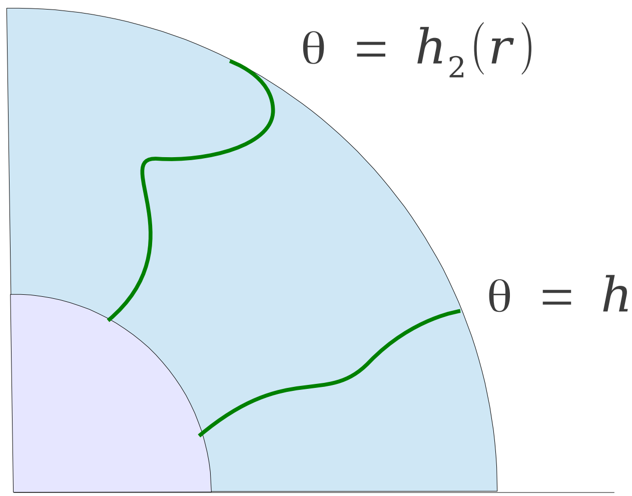
$$c \le y \le d$$
$$h_1(y) \le x \le h_2(y)$$

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Type A and Type B





$$\iint_R f(r,\theta)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

$$\iint_R f(r,\theta)\, dA = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r,\theta)\, d\theta\, r\, dr$$

Work using an Arc Length Parameter s

$$W = \boldsymbol{F} \cdot \boldsymbol{d}$$

A force field $\boldsymbol{F}(x, y) = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j}$

A smooth curve $C : x = f(t),\ \ y = g(t),\ \ a \le t \le b$

Work done by **F** along C $W = \int_c \boldsymbol{F}(x, y) \cdot d\boldsymbol{r}$

$$= \int_C P(x, y)\, dx + Q(x, y)\, dy$$

Unit Tangent Vector

$$\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{ds}\frac{ds}{dt} \qquad d\boldsymbol{r} = \frac{d\boldsymbol{r}}{ds} ds \qquad d\boldsymbol{r} = \boldsymbol{T}\, ds$$

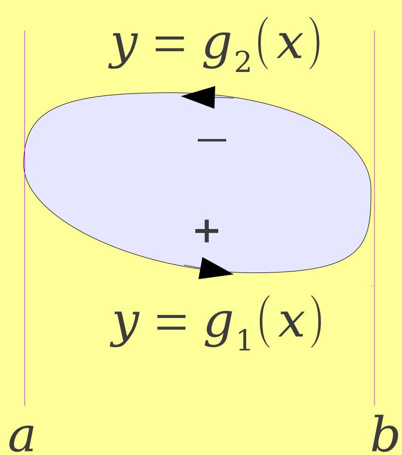
$$W = \int_c \boldsymbol{F} \cdot d\boldsymbol{r} = \int_c \boldsymbol{F} \cdot \boldsymbol{T}\, ds$$

Green's Theorem in the Plane (1)

C: a piecewise simple closed curve bounding by a simply connected region **R**

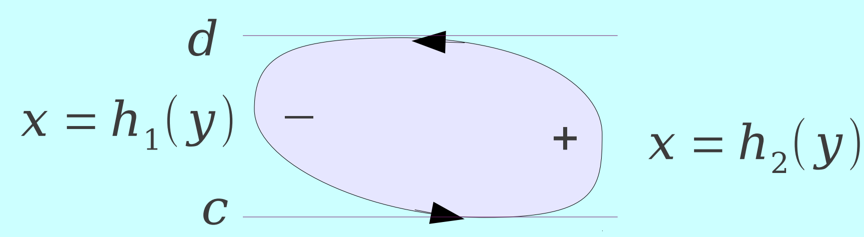
$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Line Integral — Double Integral



$$\iint_R -\frac{\partial P}{\partial y}\,dy\,dx$$

$$= \int_a^b P\,dx$$



$$\iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d Q\,dy$$

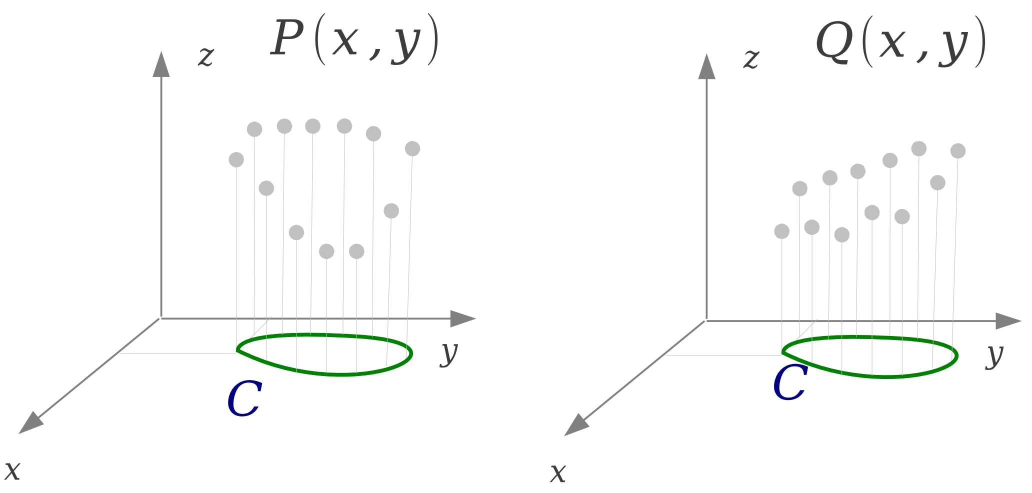
$$\int_{y_1}^{y_2} f'(y)\,dy = f(y_2) - f(y_1)$$

$$\int_{x_1}^{x_2} f'(x)\,dx = f(x_2) - f(x_1)$$

Line Integral in the Plane (2)

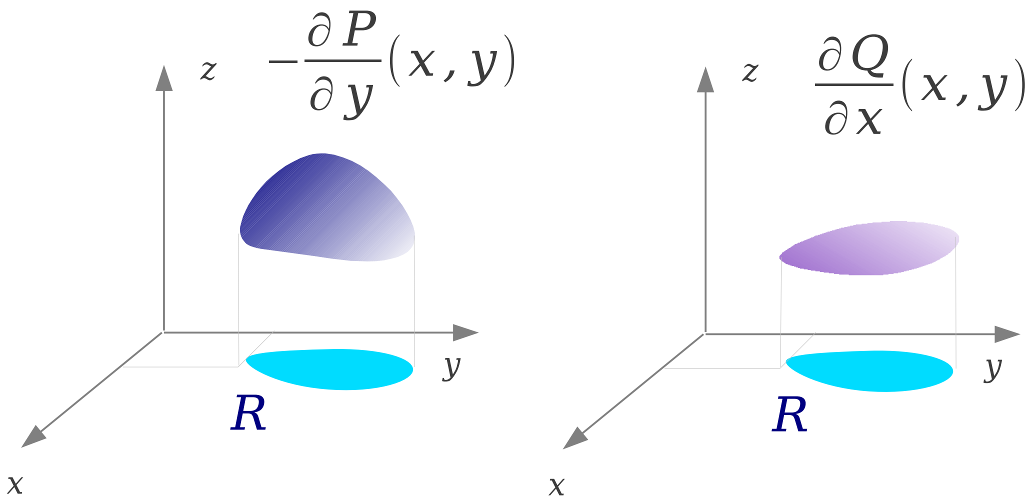
Line Integral

$$\oint_C P\,dx + Q\,dy$$



Double Integral

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

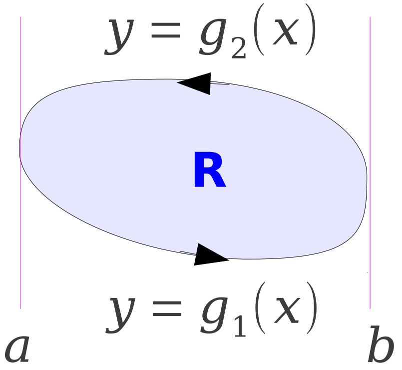


Green's Theorem in the Plane (3)

C: a piecewise c simple closed curve

R: a simply connected bounding region

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



$$\oint_C P\,dx$$

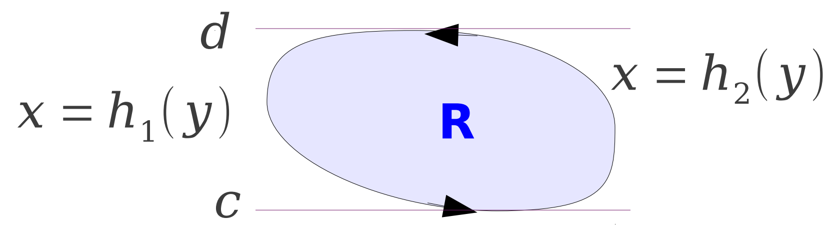
$$\iint_R -\frac{\partial P}{\partial y}\,dA$$

$$= -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx$$

$$= -\int_a^b \left[P(x, g_2(x)) - P(x, g_1(x))\right] dx$$

$$= \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx$$

$$= \oint_C P\,dx$$



$$\oint_C Q\,dy$$

$$\iint_R \frac{\partial Q}{\partial x}\,dA$$

$$= -\int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial x}\,dx\,dy$$

$$= -\int_c^d \left[Q(h_2(y), y) - Q(h_1(y), y)\right] dy$$

$$= \int_c^d Q(h_1(y), y)\,dx - \int_a^b P(h_2(y), y)\,dx$$

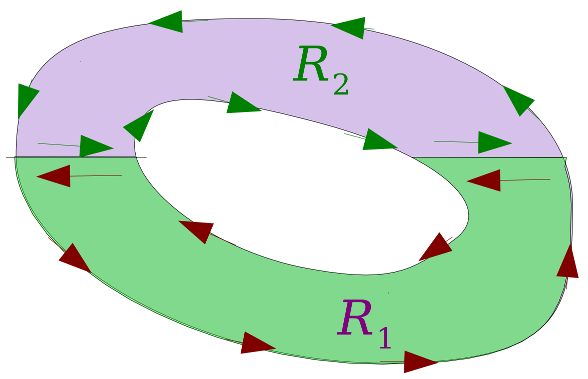
$$= \oint_C Q\,dy$$

Region with Holes

C: a piecewise simple closed curve bounding by a simply connected region **R**

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Line Integral | Double Integral



$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$R_1 \cup R_2$ $$= \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$C_1 \cup C_2$ $$= \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy$$

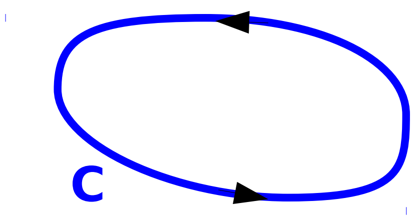
C $$= \oint_C P\,dx + Q\,dy$$

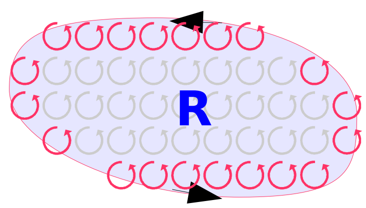
Vector Form of Green's Theorem – Curl

C: a piecewise simple closed curve bounding by a simply connected region **R**

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Line Integral | Double Integral





curl **F**

$$\text{curl } \mathbf{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}$$

$$\oint_C P\,dx + Q\,dy = \iint_R (\nabla \times \boldsymbol{F}) \cdot k\, dA$$

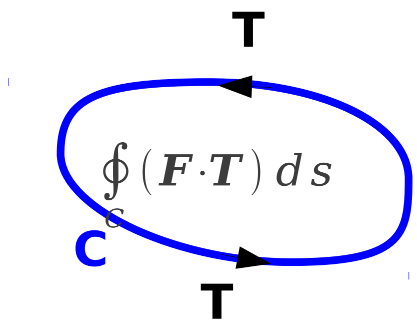
$$(\nabla \times \boldsymbol{F}) \cdot \boldsymbol{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

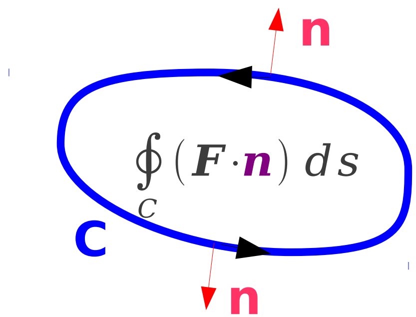
Vector Form of Green's Theorem – Div (1)

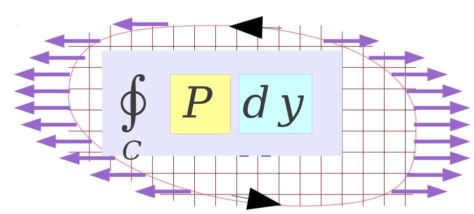
C: a piecewise simple closed curve bounding by a simply connected region **R**

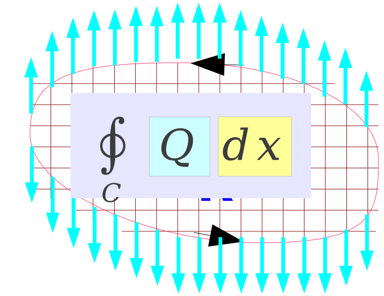
$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

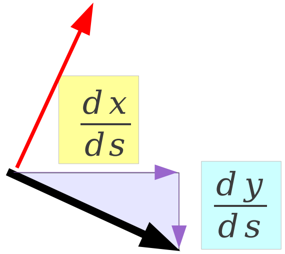
Line Integral   Double Integral

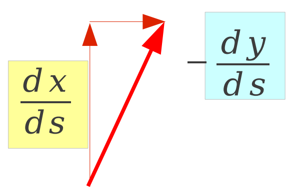












$$\boldsymbol{T} = \frac{dx}{ds}\boldsymbol{i} + \frac{dy}{ds}\boldsymbol{j}$$

$$\boldsymbol{n} = \frac{dy}{ds}\boldsymbol{i} - \frac{dx}{ds}\boldsymbol{j}$$

$$\oint_C (\boldsymbol{F}\cdot\boldsymbol{T})\,ds = \oint_C P\,dx + Q\,dy$$

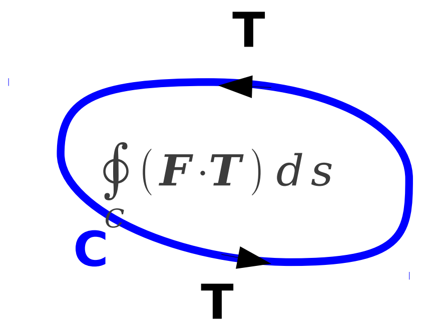
$$\oint_C (\boldsymbol{F}\cdot\boldsymbol{n})\,ds = \oint_C P\,dy - Q\,dx$$

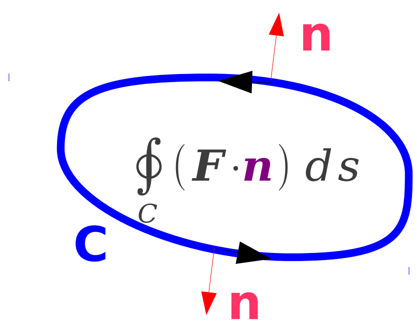
Vector Form of Green's Theorem – Div (2)

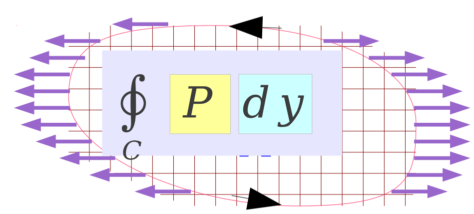
C: a piecewise simple closed curve bounding by a simply connected region **R**

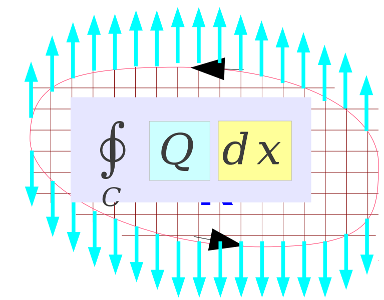
$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Line Integral　　　　Double Integral









$$\oint_C (\boldsymbol{F}\cdot\boldsymbol{T})\,ds = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

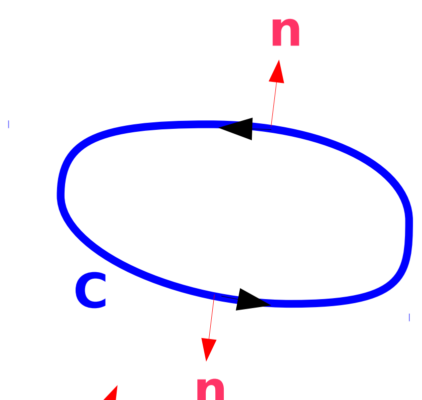
$$\oint_C (\boldsymbol{F}\cdot\boldsymbol{n})\,ds = \oint_C P\,dy - Q\,dx = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

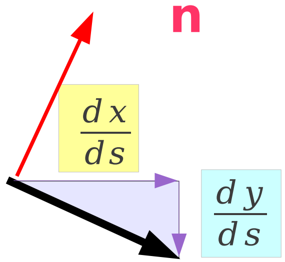
Vector Form of Green's Theorem – Div

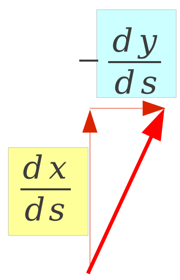
C: a piecewise simple closed curve bounding by a simply connected region **R**

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Line Integral | Double Integral

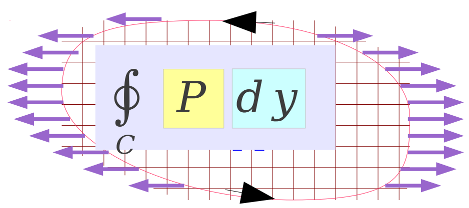


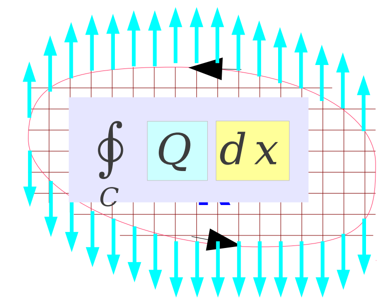




$$\boldsymbol{T} = \frac{dx}{ds}\boldsymbol{i} + \frac{dy}{ds}\boldsymbol{j}$$

$$\boldsymbol{n} = \frac{dy}{ds}\boldsymbol{i} - \frac{dx}{ds}\boldsymbol{j}$$





div **F** $= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}$

$$(\nabla\times \boldsymbol{F})\cdot\boldsymbol{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

2-Divergence

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y + \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x \quad = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$

Flux density

$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

Divergence of **F** Flux Density

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”