Propositional Logic – Semantics (3A)

Young W. Lim
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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Based on

Contemporary Artificial Intelligence,
R.E. Neapolitan & X. Jiang

Logic and Its Applications,
Burkey & Foxley

Semantics

gives meaning to the propositions

consists of rules for assigning either the value **T** or **F** to every proposition

The truth value of a proposition

If a proposition has truth value **T**, we say it is true
Otherwise, we say it is false

Semantic Rules

1. the logical value **True** ← the value T always
   the logical value **False** ← the value F always

2. Every **atomic proposition** ← a value T or F

   The set of all these assignments constitues a **model** or **possible world**

   All **possible worlds** (assignments) are **permissable**

3. The truth values of arbitrary propositions connected with **connectives**
   are given by the connective **truth tables**.

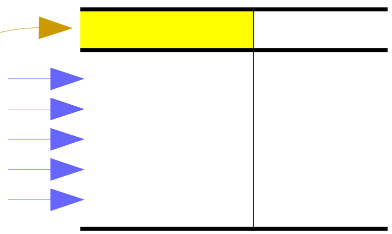
4. The truth value for **compound propositions** are determined recursively
   using the truth tables according to the following rules

   (a) the grouping () has highest precedence
   (b) the precedence order : ¬, ∧, ∨, → , ↔
   (c) binary connectives : from left to right

A Model

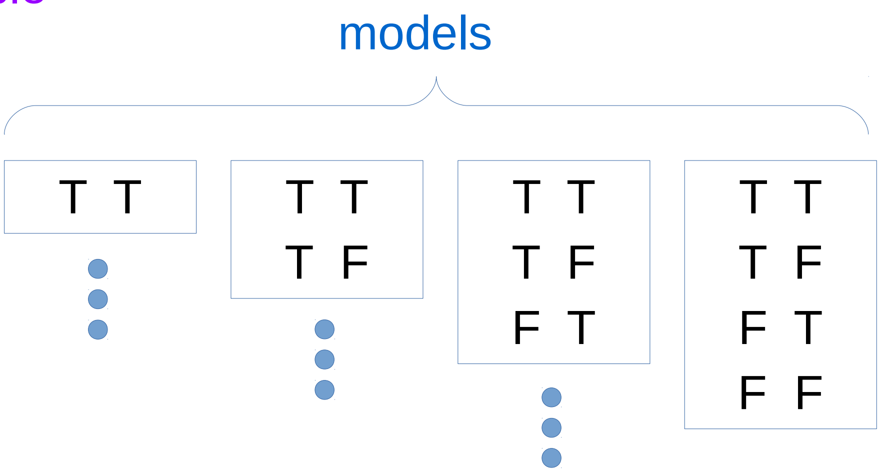
A **model** or a **possible world**:

Every **atomic proposition** is assigned a value **T** or **F**



The set of **all** these assignments constitutes
A **model** or a **possible world**

All possible worlds (assignments) are permissible



A	B	A∧B	A∧B ⇒ A
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Every atomic proposition : A, B

$$2^4 = 16$$

Interpretation

Semantics : the meaning of formulas

Truth values are assigned to the atoms of a formula
in order to evaluate the truth value of the formula

	A	B
Interpretation $\mathbf{I}_1$ ➡	T	T
Interpretation $\mathbf{I}_2$ ➡	T	F
Interpretation $\mathbf{I}_3$ ➡	F	T
Interpretation $\mathbf{I}_4$ ➡	F	F

An **interpretation** for A is a **total function** I_A: $P_A \rightarrow \{\mathbf{T}, \mathbf{F}\}$
that assigns the truth values **T** or F to every atom in P_A

$A \in F$ a formula
P_A the set of atoms in A

https://en.wikipedia.org/wiki/Syntax_(logic)#Syntactic_consequence_within_a_formal_system

The Definition of a Model

A set of formulas U = {A_1, A_2, … }
is *simultaneously* **satisfiable**

⟺

there exists an interpretation ***I***
such that $v_I(A_i)$ = **T** for all ***I***

$v_I(A_i)$: the value of
a formula A_i under
an interpretation ***I***

this satisfying interpretation ***I*** is a **model** of U

U is **unsatisfiable**

⟺

for every interpretation ***I***,
there exists an ***I*** such that $v_I(A_i)$ = **F**

Mathematical Logic for Computer Science
M. Ben-Arie

All possible worlds

A	B	
T	T	T
T	F	
F	T	
F	F	

A	B	
T	T	
T	F	T
F	T	
F	F	

A	B	
T	T	
T	F	
F	T	T
F	F	

A	B	
T	T	
T	F	
F	T	
F	F	T

A	B	
T	T	T
T	F	T
F	T	
F	F	

A	B	
T	T	T
T	F	
F	T	T
F	F	

A	B	
T	T	T
T	F	
F	T	
F	F	T

A	B	
T	T	
T	F	T
F	T	T
F	F	

A	B	
T	T	
T	F	T
F	T	
F	F	T

A	B	
T	T	
T	F	
F	T	T
F	F	T

A	B	
T	T	T
T	F	T
F	T	T
F	F	

A	B	
T	T	T
T	F	T
F	T	
F	F	T

A	B	
T	T	T
T	F	
F	T	T
F	F	T

A	B	
T	T	
T	F	T
F	T	T
F	F	T

A	B	
T	T	T
T	F	T
F	T	T
F	F	T

A	B	
T	T	
T	F	T
F	T	T
F	F	

Interpretations and all possible worlds

	A	B	
Interpretation I_1 ➡	T	T	
Interpretation I_2 ➡	T	F	
Interpretation I_3 ➡	F	T	
Interpretation I_4 ➡	F	F	

	A	B	
	T	T	
a model ➡	T	F	T
a model ➡	F	T	T
	F	F	

All possible worlds

$$\{I_1\},\ \{I_2\},\ \{I_3\},\ \{I_4\},$$

$$\{I_1,I_2\},\ \{I_1,I_3\},\ \{I_1,I_4\},\ \{I_2,I_3\},\ \{I_2,I_4\},\ \{I_3,I_4\},$$

$$\{I_1,I_2,I_3\},\ \{I_1,I_2,I_4\},\ \{I_2,I_3,I_4\},$$

$$\{I_1,I_2,I_3,I_4\},\ \emptyset$$

All possible worlds

The set of all these assignments constitues a **model** or **possible world**

All **possible worlds** (assignments) are **permissable**

A proposition is called a **tautology**
If and only if it is **true** in all possible worlds

A proposition is called a **contradiction**
If and only if it is **false** in all possible worlds

Satisfiability and Validity

$A \in F \quad (a\ set\ of\ formulas)$

A is ***satisfiable***

$v_I(A) = T$ *for some interpretation* ***I***

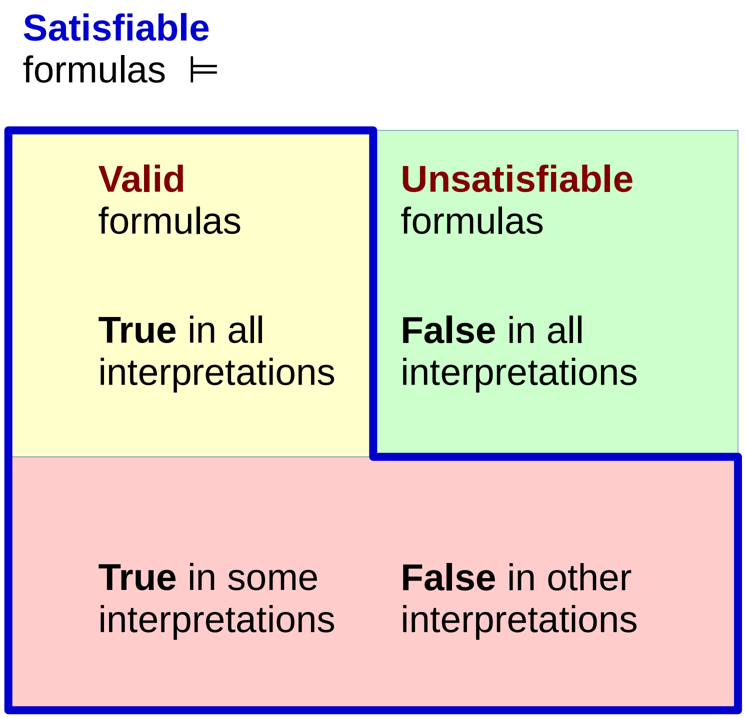
A satisfying interpretation is
a model for A

A is ***valid***

$\models A$ ⇔

$v_I(A) = T$ *for all interpretation* ***I***

A valid propositional formula
is called a tautology



Mathematical Logic for Computer Science
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Connective Truth Tables

from en.wikipedia.org

P	Q	$P \wedge Q$	$P \vee Q$	$P \underline{\vee} Q$	$P \underline{\wedge} Q$	$P \Rightarrow Q$	$P \Leftarrow Q$	$P \Leftrightarrow Q$
T	T	T	T	F	T	T	T	T
T	F	F	T	T	F	F	T	F
F	T	F	T	T	F	T	F	F
F	F	F	F	F	T	T	T	T

T = true, F = false

$\wedge$ = AND (logical conjunction)

$\vee$ = OR (logical disjunction)

$\underline{\vee}$ = XOR (exclusive or)

$\underline{\wedge}$ = XNOR (exclusive nor)

$\rightarrow$ = conditional "if-then"

$\leftarrow$ = conditional "(then)-if"

$\Longleftrightarrow$ biconditional or "if-and-only-if" is logically equivalent to $\underline{\wedge}$: XNOR (exclusive nor).

Semantic Rule Purposes

The semantics for propositional logic

- assign truth values to all propositions

- could use different truth tables from the conventional ones
- but it must provide a way to reflect the real world
  - ➔ to allow reasoning

- The purpose is to make statements about the real world
- and to reason with these statements
- the semantics must reflect the way humans reason
  with the statements in the world

- some difficulty with $A \rightarrow B$

Clause

a **clause** is an expression formed from a finite collection of **literals**

(**variables** or their **negations**) – **atoms**

A **clause** is **true**

either whenever **at least one** of the literals that form it is true

(a disjunctive clause, the most common use of the term), or

when **all** of the literals that form it are **true**

(a conjunctive clause, a less common use of the term).

it is a finite disjunction or conjunction of **literals**, depending on the context.

CNF

DNF

https://en.wikipedia.org/wiki/Clause_(logic)

Interpretation

Semantics allows you

to *relate* the **symbols** in the logic

to the **domain** you're trying to **model**.

	A	B	
Interpretation $\mathbf{I}_1$ ➡	T	T	
Interpretation $\mathbf{I}_2$ ➡	T	F	
Interpretation $\mathbf{I}_3$ ➡	F	T	
Interpretation $\mathbf{I}_4$ ➡	F	F	

An **interpretation I** assigns **a truth value** to **each atom**

A **body** b1∧b2 is true in **I** **iff** b1 is true in **I** and b2 is true in **I**.

A **rule** h←b is false in **I** **iff** b is true in **I** and h is false in **I**.

A **knowledge base** KB is true in **I** **iff** every clause in KB is true in **I**.

https://www.cs.ubc.ca/~kevinlb/teaching/cs322%20-%202009-10/Lectures/Logic2.pdf

Model

A **model** of a set of clauses is

an **interpretation** in which **all** the clauses are **true**. (a set of formulas)

A **satisfying interpretation**

If **KB** is **a set of clauses** and **g** is **a conjunction of atoms**,

If **g** is **true in every model of KB**,

then **g** is a **logical consequence of KB**, written **KB** $\models$ **g**.

https://www.cs.ubc.ca/~kevinlb/teaching/cs322%20-%202009-10/Lectures/Logic2.pdf

Model Example

KB: p ← q, q, r ← s → a set of clauses

	p	q	r	s	p←q	q	r←s	
I1	T	T	T	T	**T**	**T**	**T**	→ a **model** of **KB**
I2	F	F	F	F	T	F	T	**not** a model of **KB**
I3	T	T	F	F	**T**	**T**	**T**	→ a **model** of **KB**
I4	T	T	T	F	**T**	**T**	**T**	→ a **model** of **KB**
I5	T	T	F	T	T	T	F	**not** a model of **KB**

Interpretation I1, I2, I3, I4, I5

An **interpretation I** assigns **a truth value** to **each atom**

model I1**,** I3**,** I4

A **satisfying interpretation**

https://www.cs.ubc.ca/~kevinlb/teaching/cs322%20-%202009-10/Lectures/Logic2.pdf

Entailment Example

KB

p ← q
q
r ← s

KB ⊨ p

KB ⊨ q

KB ⊭ r

KB ⊭ s

KB ⊨ p ← q

KB ⊨ r ← s

	p	q	r	s	p←q	q	r←s	
I1	T	T	T	T	T	T	T	a **model** of **KB**
I2	F	F	F	F	T	F	T	
I3	T	T	**F**	**F**	T	T	T	a **model** of **KB**
I4	T	T	T	**F**	T	T	T	a **model** of **KB**
I5	T	T	F	T	T	T	F	

https://www.cs.ubc.ca/~kevinlb/teaching/cs322%20-%202009-10/Lectures/Logic2.pdf

Entailment Example

	p	q	r	s	p←q	q	r←s
I2	F	F	F	F	T	F	T
I5	T	T	F	T	T	T	F
I1	T	T	T	T	T	T	T
I3	T	T	**F**	**F**	T	T	T
I4	T	T	T	**F**	T	T	T

a **model** of **KB**

a **model** of **KB**

a **model** of **KB**

every model of **KB**

KB ⊨ q

KB ⊨ p

KB ⊭ r

KB ⊭ s

KB ⊨ p ← q

KB ⊨ q

KB ⊨ r ← s

https://www.cs.ubc.ca/~kevinlb/teaching/cs322%20-%202009-10/Lectures/Logic2.pdf

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