Multiple Random Variables

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

1. Central Limit Theorem
   - Unequal Distributions
   - Equal Distributions

Outline

Central Limit Theorem

Definition

the central limit theorem says that the probability distribution function of the sum of large number of random variables approaches a Gauassian distribution.
This theorem is known to apply some cases of statistically independent random variables.

Central Limit Theorem
Uneqaul Distribution Case

Definition

the sum Y of N independent random variables $X_1, X_2, ..., X_N$
Let $Y = X_1 + X_2 + \cdots + X_N$, then

$$\overline{Y}_N = \overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_N$$

$$\sigma_{Y_N}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \cdots + \sigma_{X_N}^2$$

the probability distribution of Y asymptotically approaches
to Gaussian distribution function as $N \to \infty$

Sufficient Conditions
Uneqaul Distribution Case

Definition

$$\sigma_{X_i}^2 > B_1 > 0 \qquad i = 1, 2, ..., N$$

$$E[|X_i - \overline{X}_i|^3] < B_2 \qquad i = 1, 2, ..., N$$

whre B_1 and B_2 are positive numbers
these conditions guarantee that no one random variable in the sum dominates

Distribution vs density functions
Uneqaul Distribution Case

the central limit theorem guarantees

- only that the distribution of the sum of random variables become Gaussian
- the density of the sum of random variables is not always Gaussian

- the sum of continuous random variables :
  - under certain conditions on individual random variables the density of the sum is always Gaussian
- the sum of discrete random variables :
  - the density function may contain impulses and thus is not Gaussian.

Discrete Random Variable Examples

distribution may contain impulses

the sum Y of N independent discrete random variables
$Y = X_1 + X_2 + ... + X_N$

- discrete random variable
- density function may contain impulses
- therefore the density function is not Gaussian
- although the distribution approaches Gaussian

- when the possible discrete values of each random variable are $kb, k = 0, \pm 1, \pm 2, ...,$ where b is a constant
  - the envelope of the impulses in the density of the sum will be Gaussian
  - with the mean Y_N and variance $\sigma^2_{Y_N}$

Central Limit Theorem (1)

Eqaul Distribution Case

Definition

the sum Y of N independent random variables $X_1, X_2, ..., X_N$
assume that $X_1, X_2, ..., X_N$ have the same distribution function.
Let $Y_N = X_1 + X_2 + \cdots + X_N$,
then $W_N = (Y_N - \overline{Y}_N)/\sigma_{Y_N}$ is
the zero-mean, unit-variance random variable

Central Limit Theorem (2)

Eqaul Distribution Case

Definition

Let $Y_N = X_1 + X_2 + \cdots + X_N$ and $W_N = (Y_N - \overline{Y}_N)/\sigma_{Y_N}$

$$
\begin{aligned}
W_N &= (Y_N - \overline{Y}_N)/\sigma_{Y_N} \\
&= \sum_{i=1}^{N} (X_i - \overline{X}_i) \Bigg/ \left[\sum_{i=1}^{N} \sigma_{X_i}^2 \right]^{1/2} \\
&= \frac{1}{\sqrt{N}\sigma_X} \sum_{i=1}^{N} (X_i - \overline{X}_i)
\end{aligned}
$$

where $\overline{X}_i = \overline{X}$ and $\sigma_{X_i}^2 = \sigma_{\overline{X}}$

Central Limit Theorem (3)

Eqaul Distribution Case

$$W_N = (Y_N - \overline{Y}_N) \Big/ \sigma_{Y_N}$$

$$= \left(X_i - \sum_{i=1}^{N} \overline{X}_i\right) \Big/ \left[\sum_{i=1}^{N} \sigma^2_{X_i}\right]^{1/2}$$

$$= \sum_{i=1}^{N} (X_i - \overline{X}_i) \Big/ \left[\sum_{i=1}^{N} \sigma^2_{X_i}\right]^{1/2}$$

$$= \sum_{i=1}^{N} (X_i - \overline{X}_i) \Big/ \left[N\sigma^2_X\right]^{1/2}$$

$$= \frac{1}{\sqrt{N}\sigma_X} \sum_{i=1}^{N} (X_i - \overline{X}_i)$$

where $\overline{X}_i = \overline{X}$ and $\sigma^2_{X_i} = \sigma_{\overline{X}}$
$\overline{Y}_N = \overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_N$ and $\sigma^2_{Y_N} = \sigma^2_{X_1} + \sigma^2_{X_2} + \cdots + \sigma^2_{X_N}$

Characteristic Function(1)

Eqaul Distribution Case

the characteristic function of W_N
a zero mean, unit variance Gaussian random variable

$$\Phi_{W_N}(\omega) = exp(-\omega^2/2)$$

$W_N$is the density of the Gaussian random variable
Fourier transforms are unique

$$\Phi_{W_N}(\omega) = E[e^{j\omega W_N}]$$

Characteristic Function(2)

Eqaul Distribution Case

$$W_N = \frac{1}{\sqrt{N}\sigma_X} \sum_{i=1}^{N} (X_i - \overline{X}_i)$$

$$\begin{aligned}
\Phi_{W_N}(\omega) &= E[e^{j\omega W_N}] = E\left[exp \left(\frac{j\omega}{\sqrt{N}\sigma_X} \sum_{i=1}^{N} (X_i - \overline{X}) \right) \right] \\
&= E\left[exp \left(\frac{j\omega}{\sqrt{N}\sigma_X} (X_1 - \overline{X}) \right) \cdots exp \left(\frac{j\omega}{\sqrt{N}\sigma_X} (X_N - \overline{X}) \right) \right] \\
&= \left\{ E\left[exp \left(\frac{j\omega}{\sqrt{N}\sigma_X} (X_1 - \overline{X}) \right) \right] \right\}^N
\end{aligned}$$

$$E[X_1] = E[X_2] = \cdots = E[X_N] = \overline{X}$$

$$E[(X_1 - \overline{X})^2] = E[(X_1 - \overline{X})^2] = \cdots = E[(X_N - \overline{X})^2] = \sigma_X^2$$

Characteristic Function (3)

Eqaul Distribution Case

$$\Phi_{W_N}(\omega) = \left\{ E\left[exp\left(\frac{j\omega}{\sqrt{N}\sigma_X}(X_1 - \overline{X}) \right) \right] \right\}^N$$

$$\ln[\Phi_{W_N}(\omega)] = N \ln \left\{ E\left[exp\left(\frac{j\omega}{\sqrt{N}\sigma_X}(X_1 - \overline{X}) \right) \right] \right\}$$

$$E\left[exp\left(\frac{j\omega}{\sqrt{N}\sigma_X}(X_1 - \overline{X}) \right) \right]$$

$$= E\left[1 + \frac{j\omega}{\sqrt{N}\sigma_X}(X_1 - \overline{X}) + \left(\frac{j\omega}{\sqrt{N}\sigma_X} \right)^2 (X_1 - \overline{X})^2 + \frac{R_N}{N} \right]$$

$$= 1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N}$$

where $E[R_N]$ approaches zero as $N \to \infty$

Characteristic Function (4)

Eqaul Distribution Case

$$\ln[\Phi_{W_N}(\omega)] = N \ln\left[1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N}\right]$$

$$\ln[1 - z] = -\left[z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right], |z| < 1$$

$$\ln[\Phi_{W_N}(\omega)] = -\frac{\omega^2}{2} + E[R_N] - \frac{N}{2}\left[\frac{\omega^2}{2N} + \frac{E[R_N]}{N}\right]^2 + \cdots$$

$$\lim_{N\to\infty} \ln[\Phi_{W_N}(\omega)] = \ln\left[\lim_{N\to\infty} \Phi_{W_N}(\omega)\right] = -\frac{\omega^2}{2}$$

$$\lim_{N\to\infty} \Phi_{W_N}(\omega) = e^{-\frac{\omega^2}{2}}$$