ODE Background: Integral (2A)

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12/23/15

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Derivative and Integral of Trigonometric Functions

Differentiation & Integration of sinusoidal functions

$$\frac{d}{dx} f(x) = \cos(x)$$ ***leads*** $$f(x) = \sin(x)$$

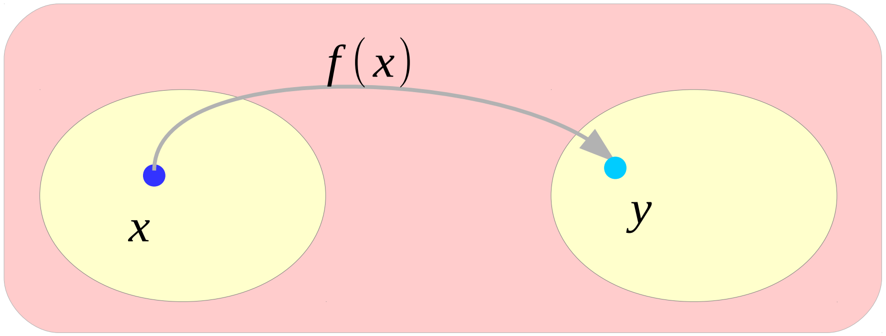
$$\frac{d}{dx} g(x) = -\sin(x)$$ ***leads*** $$g(x) = \cos(x)$$

$$\int f(x)\,dx = -\cos(x) + C$$ ***lags*** $$f(x) = \sin(x)$$

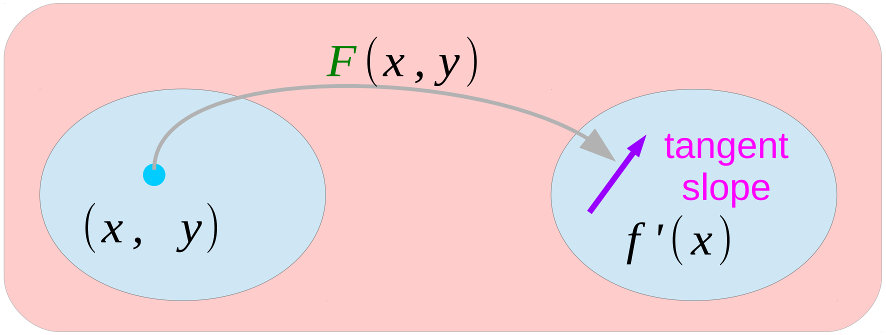
$$\int g(x)\,dx = \sin(x) + C$$ ***lags*** $$g(x) = \cos(x)$$

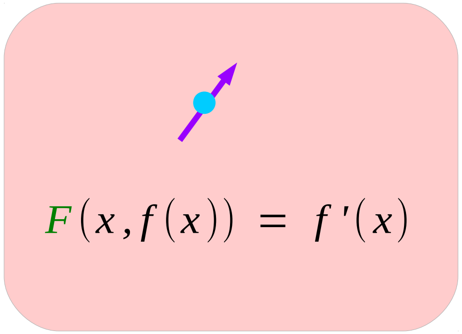
Plotting Lineal Elements

a single variable function



a two variable function

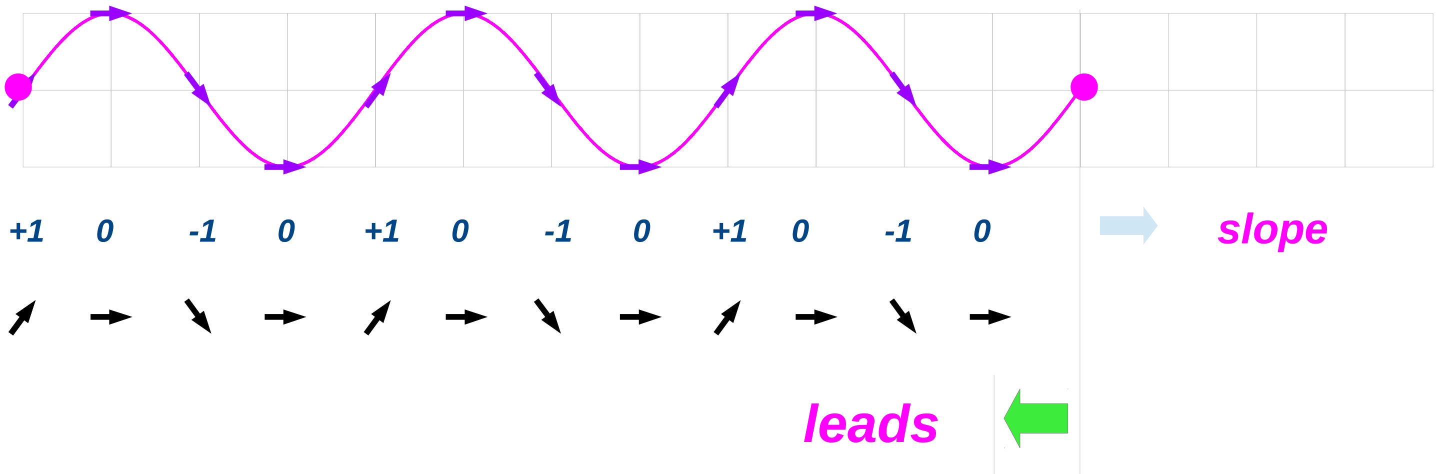




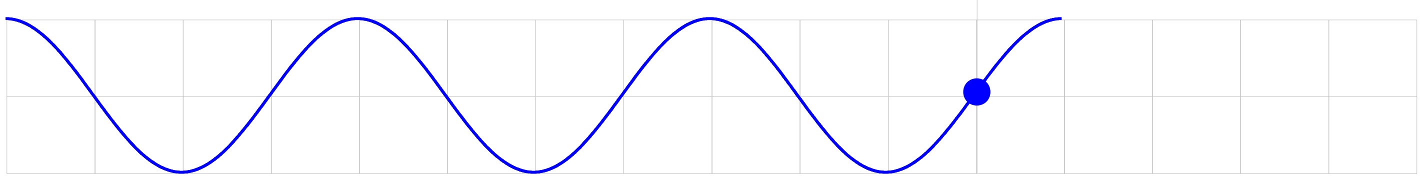
Derivative of sin(x)

A1

$$f(x) = \sin(x)$$



$$\frac{d}{dx} f(x) = \cos(x)$$



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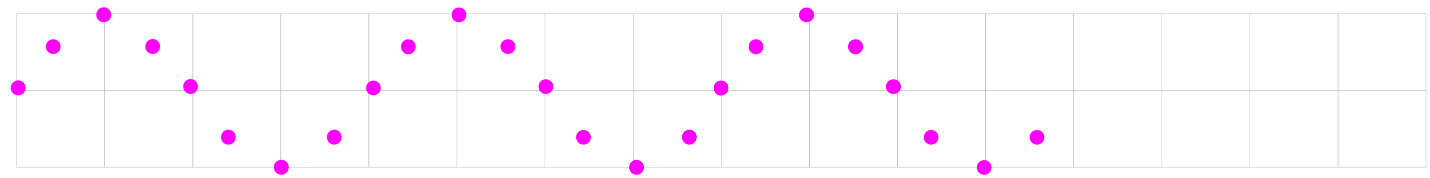
Plot of F(x,y) = f'(x) (= cos(x))

A2

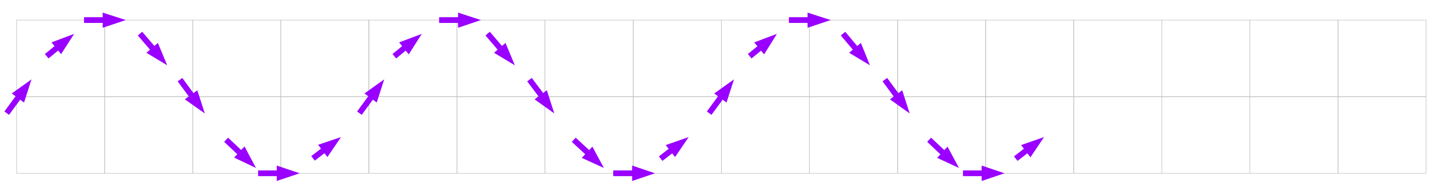
$(x,\ y) = (x, f(x)) = (x, \sin(x))$ $x \Rightarrow y$



$f(x) = \sin(x)$

+1 0 -1 0 +1 0 -1 0 +1 0 -1 0 ***slope***

$(x,\ y) \Rightarrow y'$



$F(x,\ y) = f'(x)$

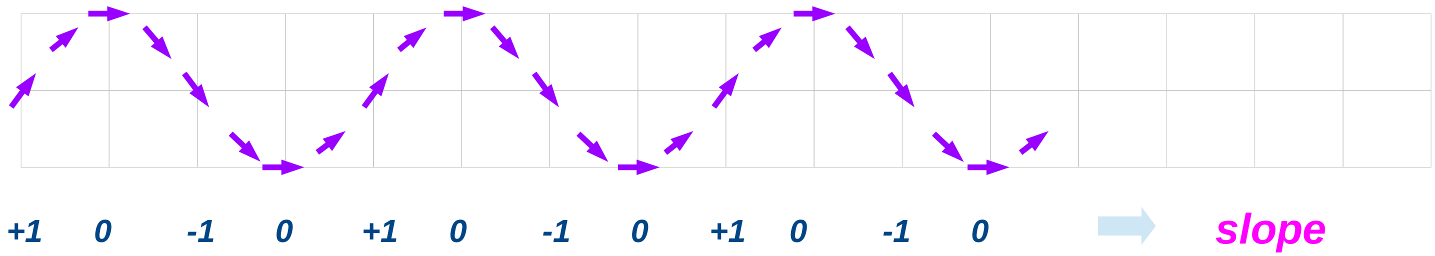
+1 0 -1 0 +1 0 -1 0 +1 0 -1 0 ***slope m***

$(x,\ y) \Rightarrow$ ***m*** *= slope of a tangent* $f'(x)$

$F(x,\ \sin(x)) = \cos(x)$

Plot of f'(x)=cos(x) from a lineal element plot

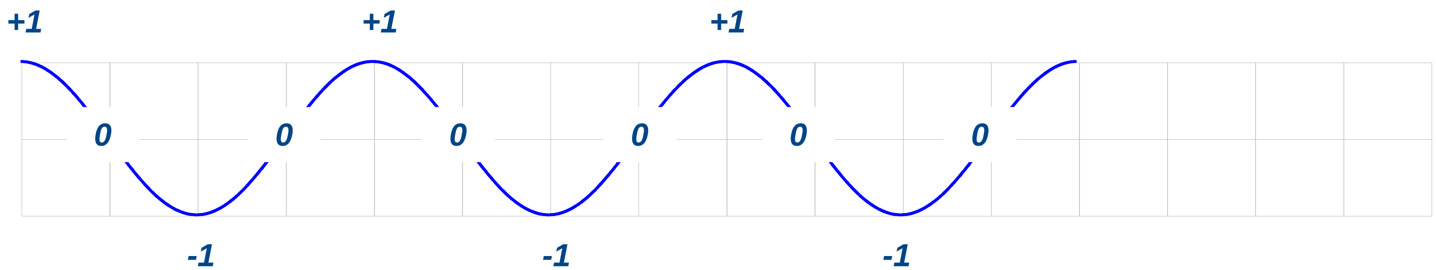
A3



$$F(x,\ y) = f'(x)$$

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

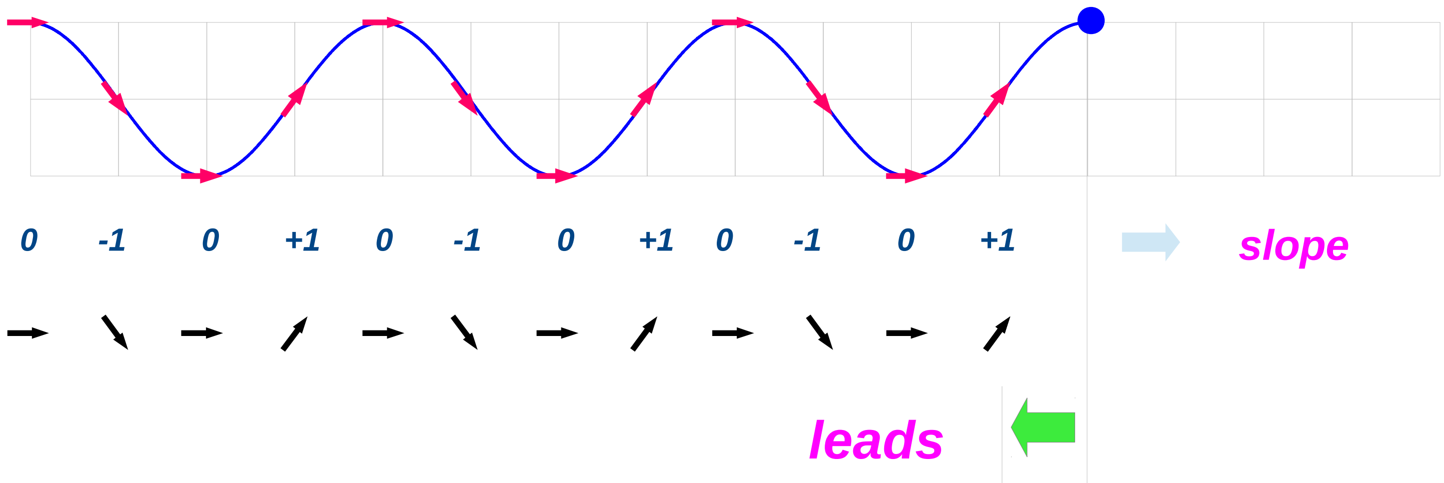


$$f(x) = \cos(x)$$

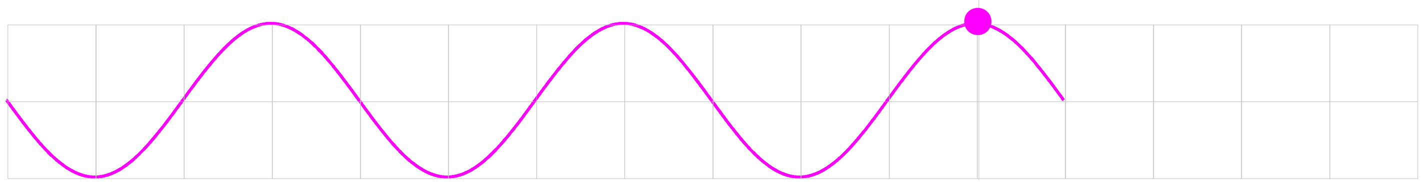
Derivative of cos(x)

B1

$$f(x) = \cos(x)$$



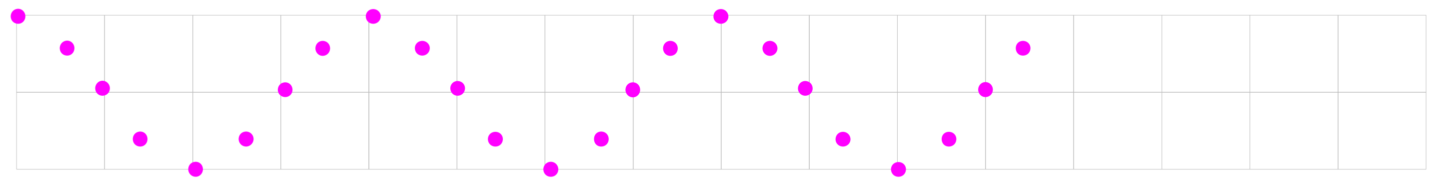
$$\frac{d}{dx} f(x) = -\sin(x)$$



Plot of F(x,y) = f'(x) (= -sin(x))

B2

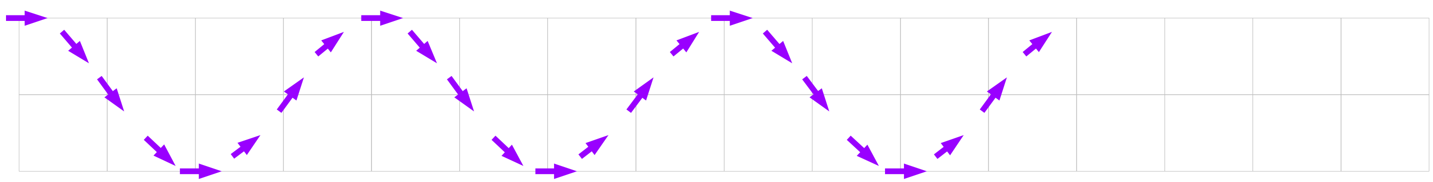
$(x, y) = (x, f(x)) = (x, \cos(x))$ $x \Rightarrow y$



$f(x) = \cos(x)$

+1 0 -1 0 +1 0 -1 0 +1 0 -1 0 **slope**

$(x, y) \Rightarrow y'$



$F(x, y) = f'(x)$

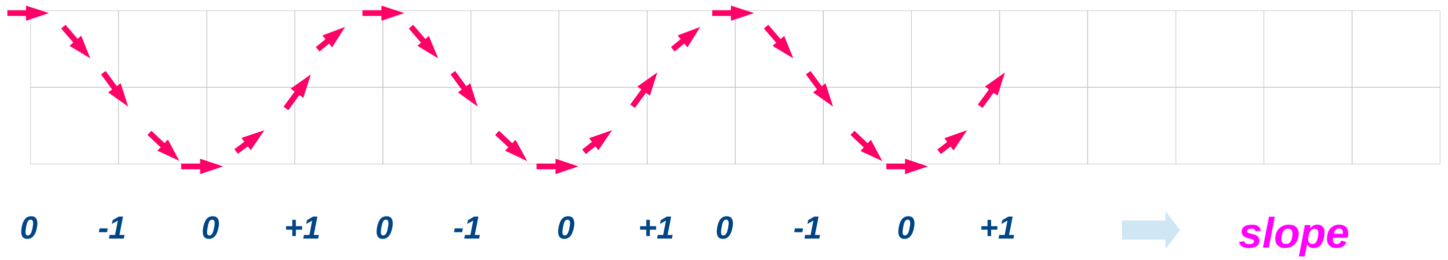
+1 0 -1 0 +1 0 -1 0 +1 0 -1 0 **slope m**

$(x, y) \Rightarrow$ ***m** = slope of a tangent* $f'(x)$

$F(x, \cos(x)) = -\sin(x)$

Plot of f'(x)=-sin(x) from a lineal element plot

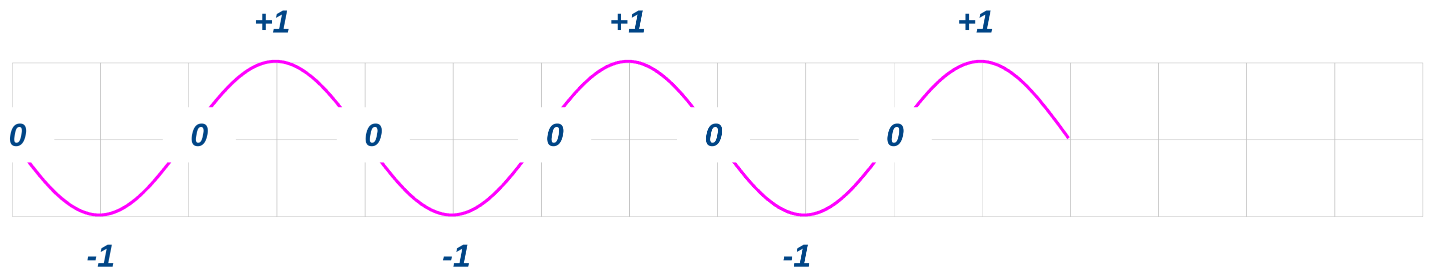
B3



$\cos(x)$

$$f(x) = \cos(x)$$

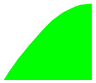
$$f'(x) = -\sin(x)$$



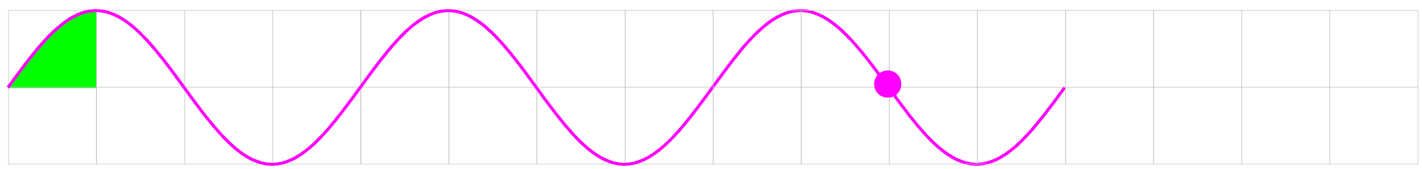
$f(x) = -\sin(x)$

C1

$$f(x) = \sin(x)$$



$$\int_0^{\pi/2} \sin(t)\,dt = 1$$

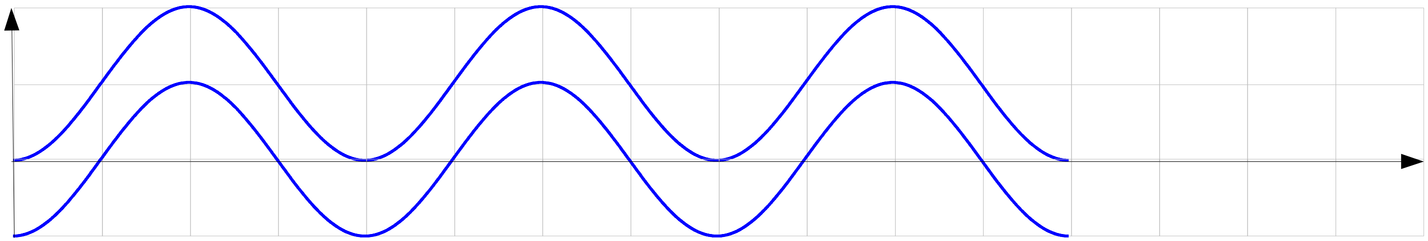


$\int_0^x \sin(t)\,dt$ 0   1   2   1   0   1   2   1   0   1   2   1   ***area + 0***

$= [-\cos(t)]_0^x$ $= -\cos(x) + 1$

$\int_{-\pi/2}^x \sin(t)\,dt$ -1   0   +1   0   -1   0   +1   0   -1   0   +1   0   ***area - 1***

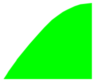
$= [-\cos(t)]_{-\pi/2}^x$ $= -\cos(x) + 0$



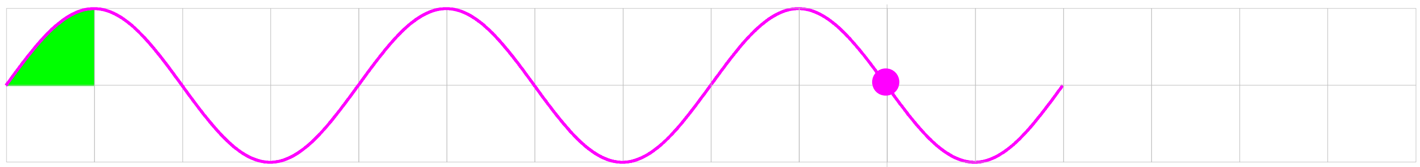
Indefinite Integrals of sin(x)

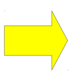
C2

$$f(x) = \sin(x)$$

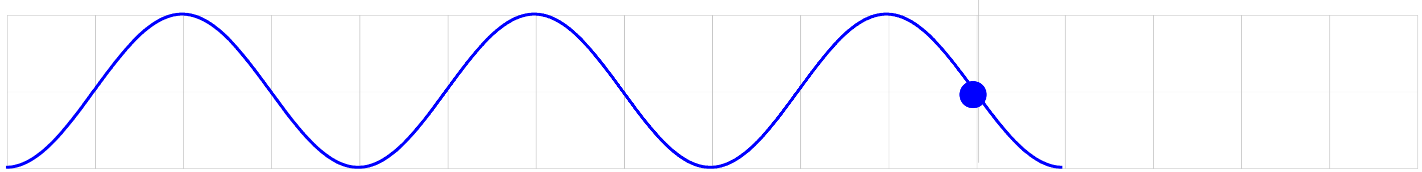


$$\int_0^{\pi/2} \sin(t)\, dt = 1$$





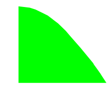
$$\int f(x)\, dx = -\cos(x) + C$$



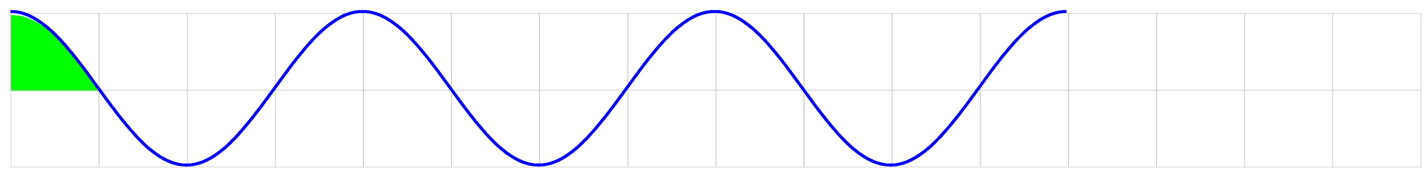
Definite Integrals of cos(x)

D1

$$f(x) = \cos(x)$$



$$\int_0^{\pi/2} \cos(x)\, dx = 1$$



$$\int_0^x \cos(t)\, dt$$

0 1 0 -1 0 1 0 -1 0 1 0 -1

area - 0

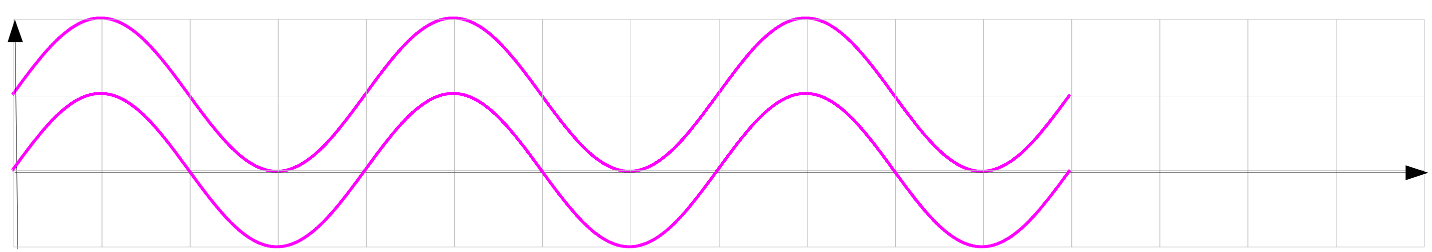
$$= [\sin(t)]_0^x \qquad = \sin(x) - 0$$

$$\int_{-\pi/2}^x \cos(t)\, dt$$

1 2 1 0 1 2 1 0 1 2 1 0

area + 1

$$= [\sin(t)]_{-\pi/2}^x \qquad = \sin(x) + 1$$



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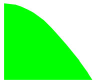
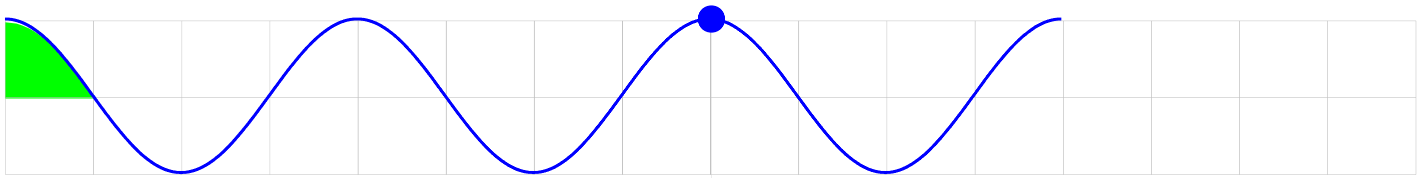
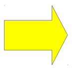
D2

$$f(x) = \cos(x)$$



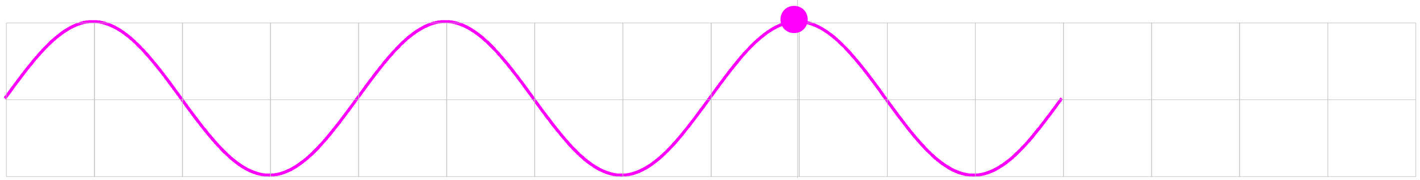
$$\int_0^{\pi/2} \cos(x)\, dx = 1$$



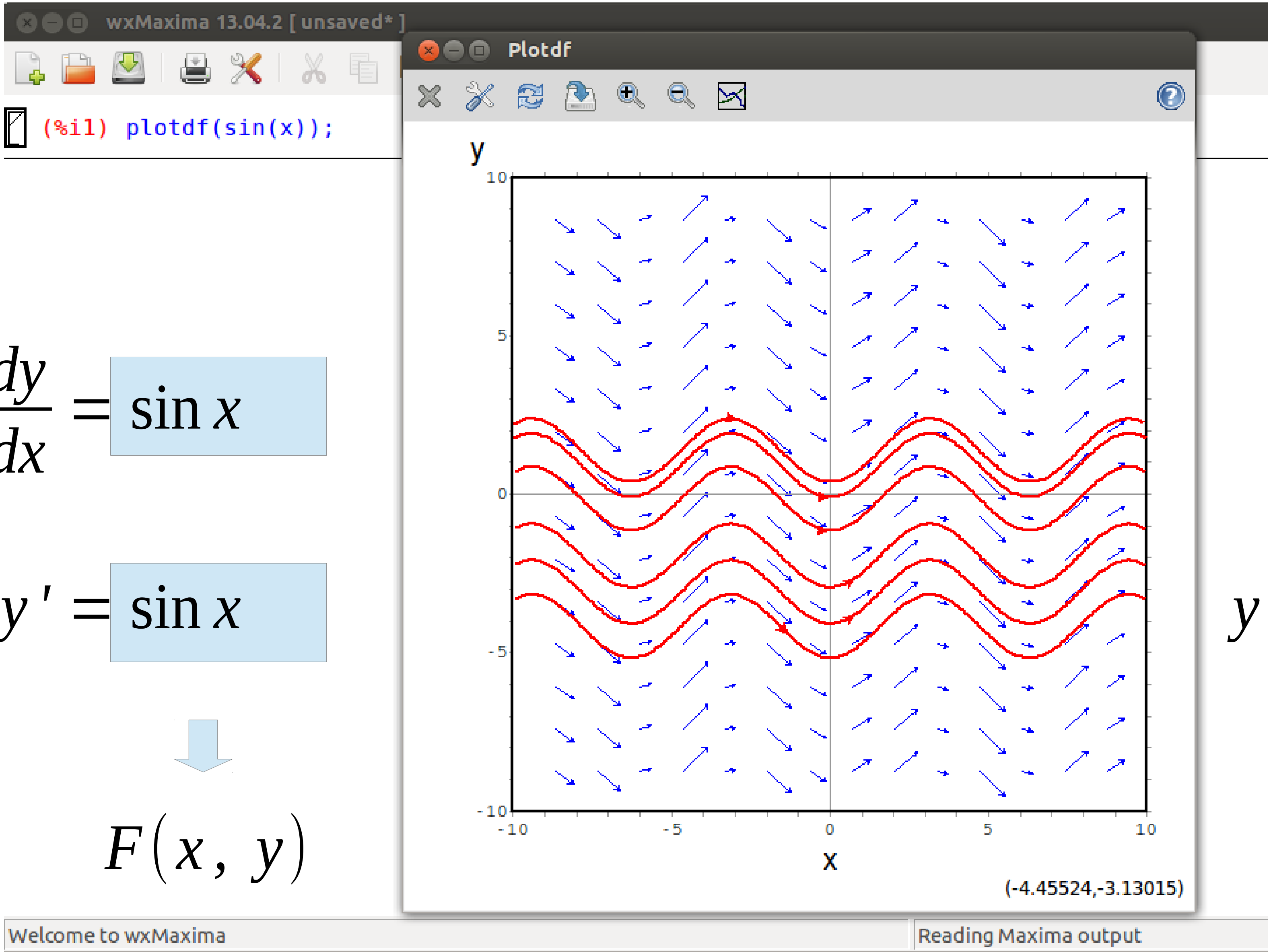


lags

$$\int f(x)\, dx = \sin(x) + C$$



Direction Field (1)



$$\frac{dy}{dx} = \sin x$$

$$y' = \sin x$$

$$F(x, y)$$

$$y = -\cos x$$

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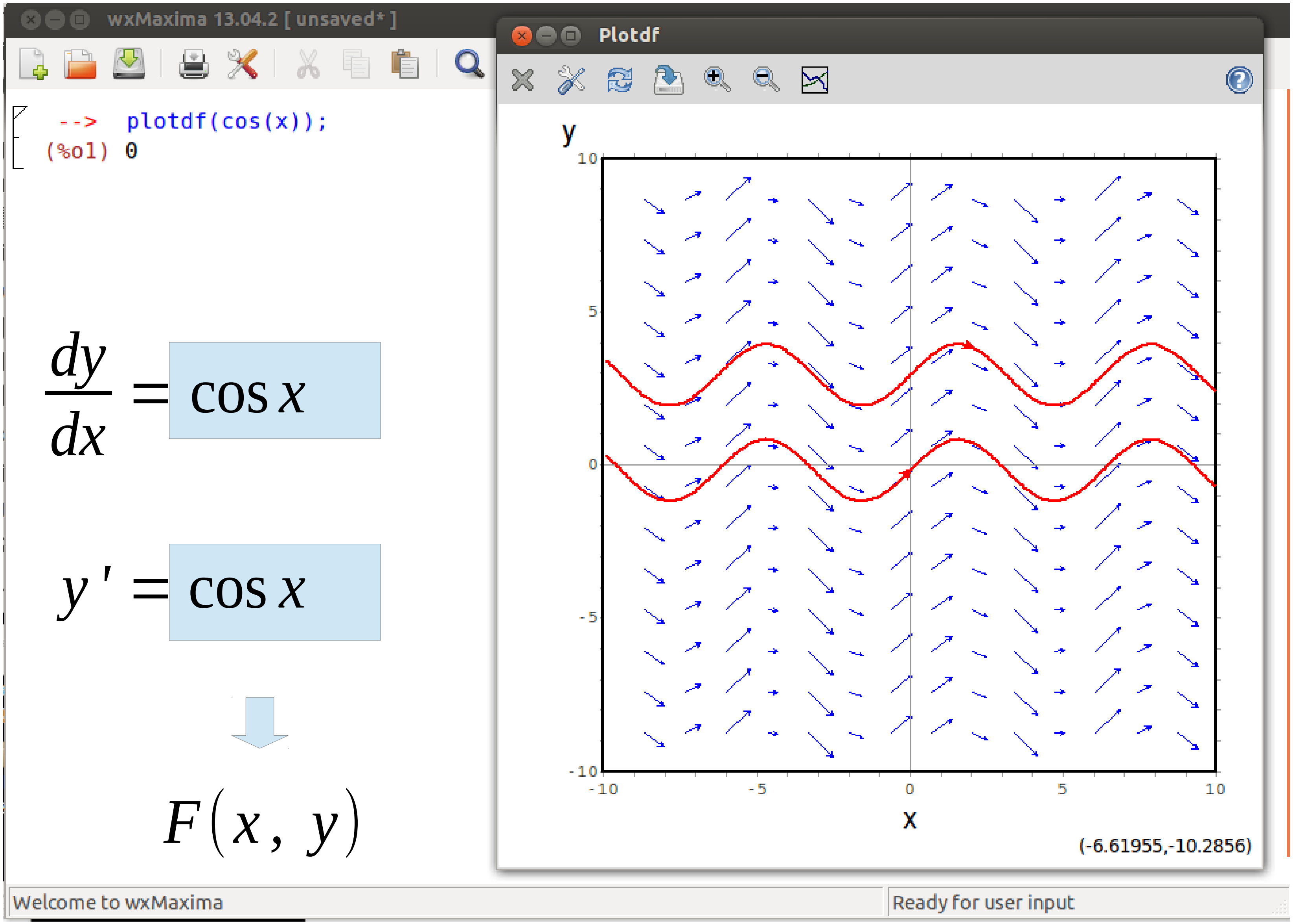
Direction Field (2)



$$\frac{dy}{dx} = \cos x$$

$$y' = \cos x$$

$$F(x, y)$$

$$y = \sin x$$

Derivative and Integral of Exponential Functions

The Euler constant e

$$\frac{d}{dx} a^x = \lim_{h\to 0} \frac{a^{(x+h)} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} \Rightarrow a^x$$

such a,
we call e

$$\lim_{h\to 0} \frac{a^h - 1}{h} = 1 \iff \lim_{h\to 0} \frac{a^h - a^0}{h - 0} = 1 \iff f'(0) = 1$$

$$\frac{d}{dx} e^x = e^x$$

$$e = 2.71828\cdots$$

$$f(x) = e^x \rightarrow f'(x) = e^x \rightarrow f''(x) = e^x$$

The Euler constant e

$$\frac{d}{dx} e^x = e^x$$

$$e = 2.71828\cdots$$

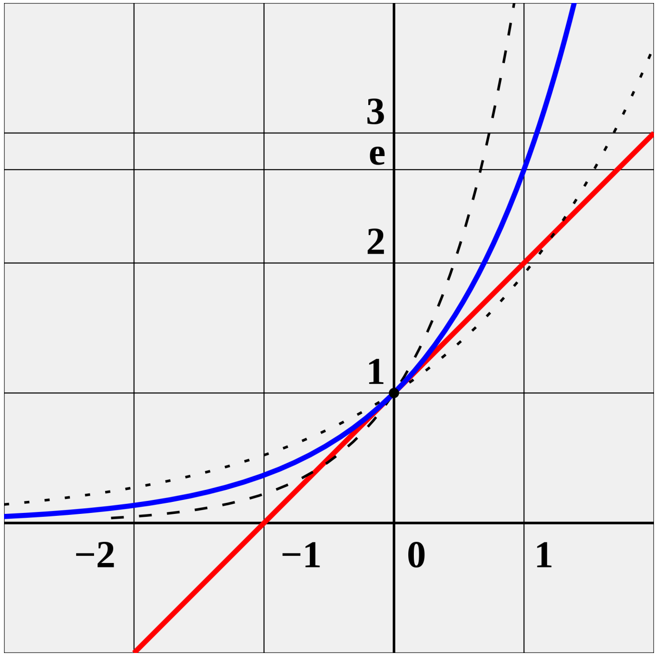
$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1 \qquad iif \quad a = e$$

$$f'(0) = 1 \qquad \lim_{h \to 0} \frac{a^h - a^0}{h - 0} = 1$$



Functions $f(x) = a^x$ are shown for several values of a. e is the unique value of a, such that the derivative of $f(x) = a^x$ at the point x = 0 is equal to 1. The blue curve illustrates this case, ex. For comparison, functions 2^x (dotted curve) and 4^x (dashed curve) are shown; they are not tangent to the line of slope 1 and y-intercept 1 (red).

http://en.wikipedia.org/wiki/Derivative

The Derivative of a^x

$$a^x = e^{\ln a^x} = e^{x \ln a}$$

$$\frac{d}{dx}\{a^x\} = \frac{d}{dx}\{e^{x \ln a}\}$$

$$= \{e^{x \ln a}\} \frac{d}{dx}\{x \ln a\}$$

$$\frac{d}{dx}\{a^x\} = \{a^x\} \ln a$$

$$\frac{d}{dx}\{e^x\} = \{e^x\} \ln e = \{e^x\}$$

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Differentiation and Integration (1)

$f(x)$ $\xrightarrow{\frac{d}{dx}}$ $f'(x)$

$\int f'(x)\,dx$ $\xleftarrow{\int \cdot\, dx}$ $f'(x)$

$= f'(x) + c_1$

e^x $\xrightarrow{\frac{d}{dx}}$ e^x

$e^x + c$ $\xleftarrow{\int \cdot\, dx}$ e^x

Differentiation and Integration (2)

$f(x)$	$\xrightarrow{\frac{d}{dx}}$	$f'(x)$	$\xrightarrow{\frac{d}{dx}}$	$f''(x)$
$\int f'(x)\,dx = f(x) + c_1$	$\xleftarrow{\int \cdot\, dx}$	$\int f''(x)\,dx = f'(x) + c_2$	$\xleftarrow{\int \cdot\, dx}$	$\int f^{(3)}(x)\,dx = f''(x) + c_3$

e^x	$\xrightarrow{\frac{d}{dx}}$	e^x	$\xrightarrow{\frac{d}{dx}}$	e^x
$e^x + c_1 x + c_2$	$\xleftarrow{\int \cdot\, dx}$	$e^x + c_1$	$\xleftarrow{\int \cdot\, dx}$	e^x

Differentiation and Integration (3)

$$f(x) \xrightarrow{\frac{d}{dx}} f'(x) \xrightarrow{\int \cdot \, dx} f(x) + c$$

$$f(x) \xrightarrow{\int \cdot \, dx} F(x) + c \xrightarrow{\frac{d}{dx}} f(x)$$

$$e^{x} \xrightarrow{\frac{d}{dx}} e^{x} \xrightarrow{\int \cdot \, dx} e^{x} + c$$

$$e^{x} \xrightarrow{\int \cdot \, dx} e^{x} + c \xrightarrow{\frac{d}{dx}} e^{x}$$

Chain Rule

Chain Rule

$$f(g(x)) \rightarrow \frac{d}{dx} \rightarrow f'(g(x)) \cdot g'(x)$$

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

$$\frac{df}{dg} = f'(g(x)) \qquad \frac{dg}{dx} = g'(x)$$

$$f(\Box) \rightarrow \frac{d}{dx} \rightarrow f'(\Box) \cdot \Box'$$

with respect to $\Box$ with respect to x

$$e^{\int P(x)dx} \rightarrow \frac{d}{dx} \rightarrow e^{\int P(x)dx} \frac{d}{dx}\left(\int P(x)dx\right) = e^{\int P(x)dx} P(x)$$

$$e^{g} \rightarrow \frac{d}{dx} \rightarrow e^{g} \cdot \frac{dg}{dx} \qquad \frac{df}{dg} \cdot \frac{dg}{dx}$$

Substitution Rule

Substitution Rule

$$f(g(x)) + C \Leftarrow \int \cdot \, dx \Leftarrow f'(g(x)) \cdot g'(x)$$

$$f(g(x)) + C = \int f'(g(x)) \cdot g'(x) dx$$

$$f(u) + C = \int f'(u) \cdot du$$

$$\int f'(g(x)) \cdot g'(x) \, dx = \int f'(g(x)) \cdot \frac{dg}{dx} dx$$

$$= \int f'(u) \, du$$

$$= f(u) + C$$

$u = g(x)$
$du = \frac{dg}{dx} dx$

Substitution Rule – the traditional form

$$f(g(x)) + C \Leftarrow \int \cdot\, dx \Leftarrow f'(g(x)) \cdot g'(x)$$

$$f(g(x)) + C = \int f'(g(x)) \cdot g'(x)\, dx$$

$f \Leftarrow f'$ *view (I)*

$$\int f(u)\, du = \int f(g(x)) \cdot g'(x)\, dx$$

$\int f \Leftarrow f$ *view (II)*

The Traditional Substitution Rule Formula

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du$$

$$u = g(x)$$

$$du = g'(x)\, dx$$

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Chain Rule and Substitution Rule Examples

Chain Rule and Substitution Rule

$$f(g(x)) \Rightarrow \frac{d}{dx} \Rightarrow f'(g(x)) \cdot g'(x)$$

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

$$f(g(x)) + C \Leftarrow \int \cdot\, dx \Leftarrow f'(g(x)) \cdot g'(x)$$

$$f(g(x)) + C = \int f'(g(x)) \cdot g'(x)\, dx$$

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Chain Rule and Substitution Rule Examples

$$e^{(x^2+2)} \quad \Rightarrow \quad \frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx} = e^g(2x) = e^{x^2+2}(2x)$$

$$\begin{cases} f(x) = e^x \Rightarrow f(g) = e^g \\ g(x) = x^2+2 \end{cases} \qquad \begin{cases} \dfrac{df}{dg} = e^g \\ \dfrac{dg}{dx} = 2x \end{cases}$$

$$\int e^{x^2+2}(2x)\,dx \quad \Rightarrow \quad \int \frac{df}{dg}dg = e^g = e^{x^2+2} + c \quad \text{or} \quad \int \frac{df}{dg}dg = e^g = e^{x^2+2} + c$$

view (I)

$$\begin{cases} f'(x) = e^x \Rightarrow f'(g) = e^g \\ g(x) = x^2+2 \end{cases} \qquad \begin{cases} \int \dfrac{df}{dg}dg \Rightarrow f'(g) = e^g + c \\ \dfrac{dg}{dx}dx = 2x\,dx \end{cases}$$

view (II)

$$\begin{cases} f(x) = e^x \Rightarrow f(g) = e^g \\ g(x) = x^2+2 \end{cases} \qquad \begin{cases} \int f(g)\,dg \Rightarrow F(g) = e^g + c \\ \dfrac{dg}{dx}dx = 2x\,dx \end{cases}$$

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Substitution Rule Examples (1)

$$f(g(x)) + C \Leftarrow \int \cdot\, dx \Leftarrow f'(g(x)) \cdot g'(x)$$

$$f(g(x)) + C = \int f'(g(x)) \cdot g'(x)\, dx$$

Ex 1:

$$\int e^{3x}\, dx$$

$$\int e^{g(x)} g'(x)\, dx = e^{g(x)}$$

$$g(x) = 3x \Rightarrow g'(x) = 3$$

$$\int e^{3x}\, dx = \frac{1}{3}\int e^{3x} \cdot 3\, dx$$

$$\Rightarrow \int e^{3x}\, dx = \frac{1}{3} e^{3x}$$

Ex 2:

$$\int e^{2y}\, dy$$

$$\int e^{h(y)} h'(y)\, dy = e^{h(y)}$$

$$h(y) = 2y \Rightarrow h'(y) = 2$$

$$\int e^{2y}\, dy = \frac{1}{2}\int e^{2y} \cdot 2\, dy$$

$$\Rightarrow \int e^{2y}\, dy = \frac{1}{2} e^{2y}$$

Substitution Rule Examples (2)

$$f(g(x)) + C \Leftarrow \int \cdot\, dx \Leftarrow f'(g(x)) \cdot g'(x)$$

$$f(g(x)) + C \quad = \quad \int f'(g(x)) \cdot g'(x)\, dx$$

Ex 3:

$$\int \frac{x}{(x^2-9)}\, dx = \int x(x^2-9)^{-1}\, dx$$

$$= \frac{1}{2}\int (x^2-9)^{-1} \cdot 2x\, dx$$

$$= \frac{1}{2}\int (x^2-9)^{-1} \cdot \left\{\frac{d}{dx}(x^2-9)\right\} dx$$

$$= \frac{1}{2}\int u^{-1}\, du = \frac{1}{2}\ln|u| = \ln|u|^{1/2}$$

$$= \ln(x^2-9)^{1/2} = \ln\sqrt{x^2-9}$$

for $(x^2 > 9)$

Ex 4:

$$p(y)\frac{dy}{dx} = g(x) \qquad y = \Phi(x)$$

$$p(\Phi(x))\Phi'(x) = g(x)$$

$$\int p(\Phi(x))\Phi'(x)\,dx = \int g(x)\,dx$$

$$dy = \Phi'(x)\,dx$$

$$\int p(y)\,dy = \int g(x)\,dx$$

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Substitution Rule Examples (3)

$$f(g(x)) + C \Leftarrow \int \cdot\, dx \Leftarrow f'(g(x)) \cdot g'(x)$$

$$f(g(x)) + C = \int f'(g(x)) \cdot g'(x)\, dx$$

Ex 5: $\int \frac{x}{(x-1)^2}\, dx$

$$= \int \frac{(x-1)+1}{(x-1)^2} \left\{ \frac{d}{dx}(x-1) \right\} dx$$

$$= \int \frac{u+1}{u^2}\, du$$

$$= \int \frac{1}{u} + \frac{1}{u^2}\, du$$

$$= \ln|u| - \frac{1}{u} + C$$

Derivative Product and Quotient Rule

Derivative Product and Quotient Rule

$$fg \;\rightarrow\; \boxed{\frac{d}{dx}} \;\rightarrow\; f'g + fg'$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

$f(x), \quad g(x)$

$$\frac{f}{g} \;\rightarrow\; \boxed{\frac{d}{dx}} \;\rightarrow\; \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \left(\frac{df}{dx}g - f\frac{dg}{dx}\right) / \, g^2$$

$f(x), \quad g(x)$

Integration By Parts

Integration by parts (1)

$$f(x)g(x) \xrightarrow{\frac{d}{dx}} f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

differentiation

$$f(x)g(x) \xleftarrow{\int \cdot\, dx} f'(x)g(x) + f(x)g'(x)$$

$$fg = \int f'g\,dx + \int fg'\,dx$$

integration

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

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Integration by parts (2)

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$\int x e^x\,dx = x e^x - \int e^x dx = x e^x - e^x + c_1 = (x-1)e^x + c_1$$

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x dx = x^2 e^x - 2x e^x + 2e^x + c_2 = (x^2 - 2x + 2)e^x + c_2$$

$$\int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c_3 = (x^3 - 3x^2 + 6x - 6)e^x + c_3$$

$$\int x e^x\,dx = (x-1)e^x + c_1$$

$$\int x^2 e^x\,dx = (x^2 - 2x + 2)e^x + c_2$$

$$\int x^3 e^x\,dx = (x^3 - 3x^2 + 6x - 6)e^x + c_3$$

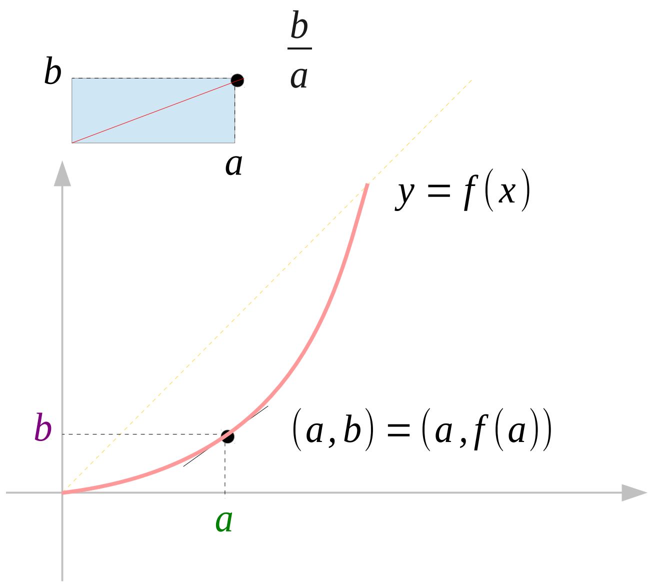
$$\int e^x dx = \int \left\{\frac{d}{dx} e^x\right\} dx = e^x + c$$

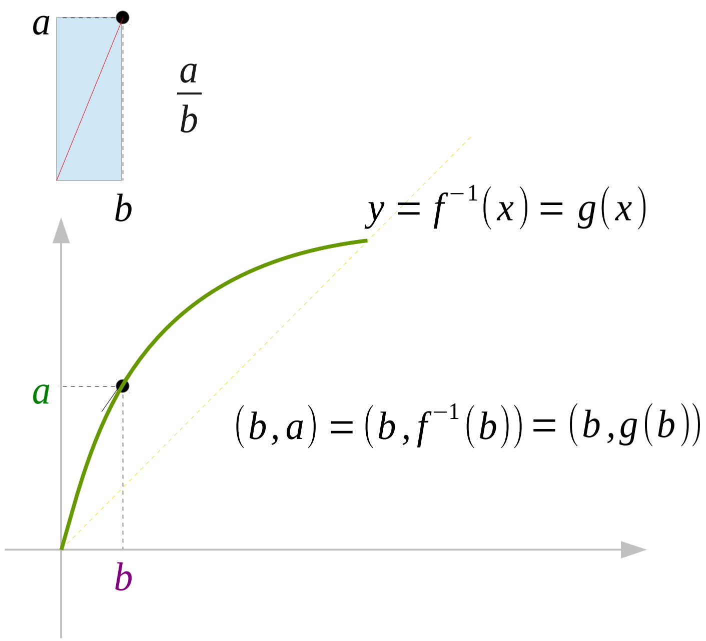
$$\int x e^{x^2/2} dx = \int \left\{\frac{d}{dx} e^{x^2/2}\right\} dx = e^{x^2/2} + c$$

$$\int x^2 e^{x^3/3} dx = \int \left\{\frac{d}{dx} e^{x^3/3}\right\} dx = e^{x^3/3} + c$$

Derivative of Inverse Functions

Derivatives of Inverse Functions





$$m_1 = f'(a)$$

$$m_2 = g'(b)$$

$$m_1 m_2 = f'(a) g'(b) = 1 \quad \Rightarrow \quad g'(b) = \frac{1}{f'(a)} \quad \Rightarrow \quad g'(b) = \frac{1}{f'(g(b))}$$

$$g(b) = a$$

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Derivatives of Inverse Functions

$$m_1 m_2 = f'(a) g'(b) = 1 \Rightarrow g'(b) = \frac{1}{f'(a)} \Rightarrow g'(b) = \frac{1}{f'(g(b))}$$

$$g(b) = a$$

$$\Downarrow$$

$$g'(x) = \frac{1}{f'(g(x))}$$

To find g'(x)
(1) find f'(x)
(2) find 1 / f'(x)
(3) substitute x with g(x)

	$\frac{d}{dx}\ln x \Rightarrow \frac{1}{x}$	$\frac{d}{dx}e^x \Rightarrow e^x$
$f'(x)$	$(e^x)' \Rightarrow e^x$	$(\ln x)' = \frac{1}{x}$
$\frac{1}{f'(x)}$	➡ $\frac{1}{e^x}$	➡ $\frac{1}{1/x} = x$
$g'(x)$	➡ $\frac{1}{e^{\ln x}} = \frac{1}{x}$	➡ e^x

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Derivative of ln x

$$f(x) = e^x$$

$$g(x) = \ln x$$

$$g'(x) = \frac{1}{f'(g(x))}$$

$$\frac{d}{dx}\ln x \Rightarrow \frac{1}{x}$$

$f'(x)$: $(e^x)' \Rightarrow e^x$

$\frac{1}{f'(x)}$: ➡ $\frac{1}{e^x}$

$g'(x)$: ➡ $\frac{1}{e^{\ln x}} = \frac{1}{x}$

$$y = \ln x$$

$$e^y = x$$

$$\frac{d}{dx}e^y = \frac{d}{dx}x$$

chain rule ➡ $e^y \frac{dy}{dx} = 1$

➡ $\frac{dy}{dx} = \frac{1}{e^y}$

$= \frac{1}{x}$ $\qquad e^y = x$

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Derivative of ln |x|

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad x > 0$$

$$\frac{d}{dx}\ln(-x) = \frac{1}{(-x)}\cdot(-1) = \frac{1}{x} \qquad x < 0$$

$\ln x \qquad x > 0 \quad \xrightarrow{\frac{d}{dx}} \quad \frac{1}{x}$

$\ln(-x) \qquad x < 0 \quad \xrightarrow{\frac{d}{dx}} \quad \frac{1}{x}$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad x \neq 0$$

Derivative of ln |x|

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad x>0$$

$$\frac{d}{dx}\ln(-x) = \frac{1}{(-x)}\cdot(-1) = \frac{1}{x} \qquad x<0$$

$\ln x \qquad x>0 \qquad \xrightarrow{\frac{d}{dx}} \qquad \frac{1}{x}$

$\ln(-x) \qquad x<0 \qquad \xrightarrow{\frac{d}{dx}} \qquad \frac{1}{x}$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad x\neq 0$$

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Indefinite Integral of ln |x|

$$\frac{d}{dx} \ln|x| = \frac{1}{x} \qquad x \neq 0$$

$$\begin{cases} \ln x & (x > 0) \\ \ln(-x) & (x < 0) \end{cases} \quad \xleftarrow{\int \cdot \, dx} \quad \frac{1}{x}$$

$$\ln|x| = \int \frac{1}{x}\, dx \qquad (x \neq 0)$$

References

[1] http://en.wikipedia.org/
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[3] E. Kreyszig, "Advanced Engineering Mathematics"
[4] D. G. Zill, W. S. Wright, "Advanced Engineering Mathematics"