

Characteristics of Multiple Random Variables

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles, Jr. and B. Shi

Outline

1 Linear Transformation of Gaussian Random Variables

Linear Transform

N Gaussian random variables

Definition

linear transform of Gaussian random variables X_1, X_2, \dots, X_N

$$Y_1 = a_{11}X_1 + a_{12}X_2 + \dots + a_{1N}X_N$$

$$Y_2 = a_{21}X_1 + a_{22}X_2 + \dots + a_{2N}X_N$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$Y_N = a_{N1}X_1 + a_{N2}X_2 + \dots + a_{NN}X_N$$

$$[Y] = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix}, [T] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix}, [X] = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

Linear Transform

N Gaussian random variables

Corollary

linear transform of Gaussian random variables X_1, X_2, \dots, X_N

$$\begin{aligned} [Y] &= [T][X], & [Y - \bar{Y}] &= [T][X - \bar{X}] \\ [X] &= [T]^{-1}[Y], & [X - \bar{X}] &= [T]^{-1}[Y - \bar{Y}] \end{aligned}$$

$$[Y] = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix}, \quad [\bar{Y}] = \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \\ \vdots \\ \bar{Y}_N \end{bmatrix}, \quad [X] = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}, \quad [\bar{X}] = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \vdots \\ \bar{X}_N \end{bmatrix}$$

$[T]$ is non-singular

Linear Transform

N Gaussian random variables

Corollary

linear transform of Gaussian random variables X_1, X_2, \dots, X_N

$$X_i = T_i^{-1}(Y_1, \dots, Y_N) = a^{i1}Y_1 + \dots + a^{iN}Y_N$$

$$\frac{\partial X_i}{\partial Y_j} = \frac{\partial T_i^{-1}}{\partial Y_j} = a^{ij}$$

$$X_i - \bar{X}_i = a^{i1}(Y_1 - \bar{Y}_1) + \dots + a^{iN}(Y_N - \bar{Y}_N)$$

a^{ij} denotes the ij -th element of $[T]^{-1}$

Using Jacobian (1)

 N Gaussian random variables

Corollary

linear transform of Gaussian random variables X_1, X_2, \dots, X_N

$$|J| = \left| \left| [T]^{-1} \right| \right| = \frac{1}{\left| \left| [T] \right| \right|}$$

$$C_{X_i X_j} = E[(X_i - \bar{X}_i)(X_j - \bar{X}_j)]$$

$$= \sum_{k=1}^N a^{ik} \sum_{m=1}^N a^{jm} E[(Y_k - \bar{Y}_k)(Y_m - \bar{Y}_m)]$$

$$= \sum_{k=1}^N a^{ik} \sum_{m=1}^N a^{jm} C_{Y_k Y_m}$$

Using Jacobian (2)

 N Gaussian random variables

Corollary

$C_{X_i X_j}$ is the ij -th element of the covariance matrix $[C_X]$
and $C_{Y_k Y_m}$ is the km -th element of the covariance matrix $[C_Y]$

$$[C_X] = [T]^{-1}[C_Y]([T]^t)^{-1}$$

$$[C_X]^{-1} = [T]^t[C_Y]^{-1}[T]$$

$$\|[C_X]^{-1}\| = \|[C_Y]^{-1}\| [T]^2$$

Using Jacobian (3)

 N Gaussian random variables

Corollary

$$\begin{aligned}
 & f_{X_1 \dots X_N}(x_1, \dots, x_N) \\
 &= f_{X_1 \dots X_N}(T_1^{-1}(Y_1, \dots, Y_N), \dots, T_N^{-1}(Y_1, \dots, Y_N)) \\
 &= \frac{\| [T] \| \| [C_Y]^{-1} \|^{1/2}}{(2\pi)^{N/2}} \exp \left\{ -\frac{[x - \bar{X}] [T]^t [C_Y]^{-1} [T] [x - \bar{X}]}{2} \right\}
 \end{aligned}$$

Using Jacobian (4)

 N Gaussian random variables

Corollary

$$\begin{aligned}
 f_{X_1 \dots X_N}(x_1, \dots, x_N) &= \frac{\| [T] \| \| [C_Y]^{-1} \|^{1/2}}{(2\pi)^{N/2}} \exp \left\{ -\frac{[x - \bar{X}] [T]^t [C_Y]^{-1} [T] [x - \bar{X}]}{2} \right\} \\
 &= \frac{\| [C_Y]^{-1} \|^{1/2}}{(2\pi)^{N/2}} \exp \left\{ -\frac{[y - \bar{Y}]^t [C_Y]^{-1} [y - \bar{Y}]}{2} \right\}
 \end{aligned}$$

Using Jacobian (5)

N Gaussian random variables

Corollary

*a linear transformation of gaussian random variables produces gaussian random variables
the new variables have the mean values*

$$\bar{Y}_j = \sum_{k=1}^N a_{jk} \bar{X}_k$$

and covariance given by the elements of the covariance matrix

$$[C_Y] = [T][C_X][T]^t$$

