CLTI Differential Equations (3B)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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- Three Differential Equations

Three Differential Equations

y_p **particular** solutions

EQ 1

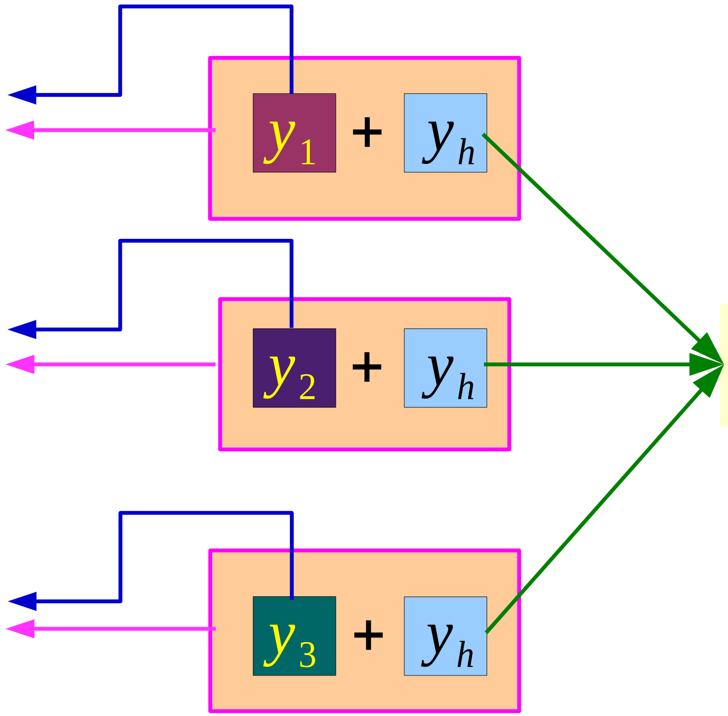
$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_2 y(t) = x_1(x)$$

EQ 2

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_2 y(t) = x_2(x)$$

EQ 3

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_2 y(t) = x_3(x)$$



$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_2 y(t) = 0$$

general solutions

y_h **homogeneous** solution

First Order ODE – general solution examples (1)

$$\frac{dy}{dt} + y = 1$$

$$e^t \frac{dy}{dt} + e^t y = e^t$$

$$\frac{d}{dt}\left[e^t y\right] = e^t$$

$$e^t y = \int e^t dt + c$$

$$y = 1 + c e^{-t}$$

$$\frac{dy}{dt} + y = t$$

$$e^t \frac{dy}{dt} + e^t y = x e^t$$

$$\frac{d}{dt}\left[e^t y\right] = t e^t$$

$$e^t y = \int t e^t dt + c$$

$$e^x y = x e^x - e^x + c$$

$$y = (t-1) + c e^{-t}$$

$$\frac{d}{dt}\left[t e^t\right] = e^t + t e^t$$

$$t e^t = \int e^t dt + \int t e^t dt$$

$$\frac{dy}{dt} + y = t^2$$

$$e^t \frac{dy}{dt} + e^t y = t^2 e^t$$

$$\frac{d}{dt}\left[e^t y\right] = t^2 e^t$$

$$e^t y = \int t^2 e^t dt + c$$

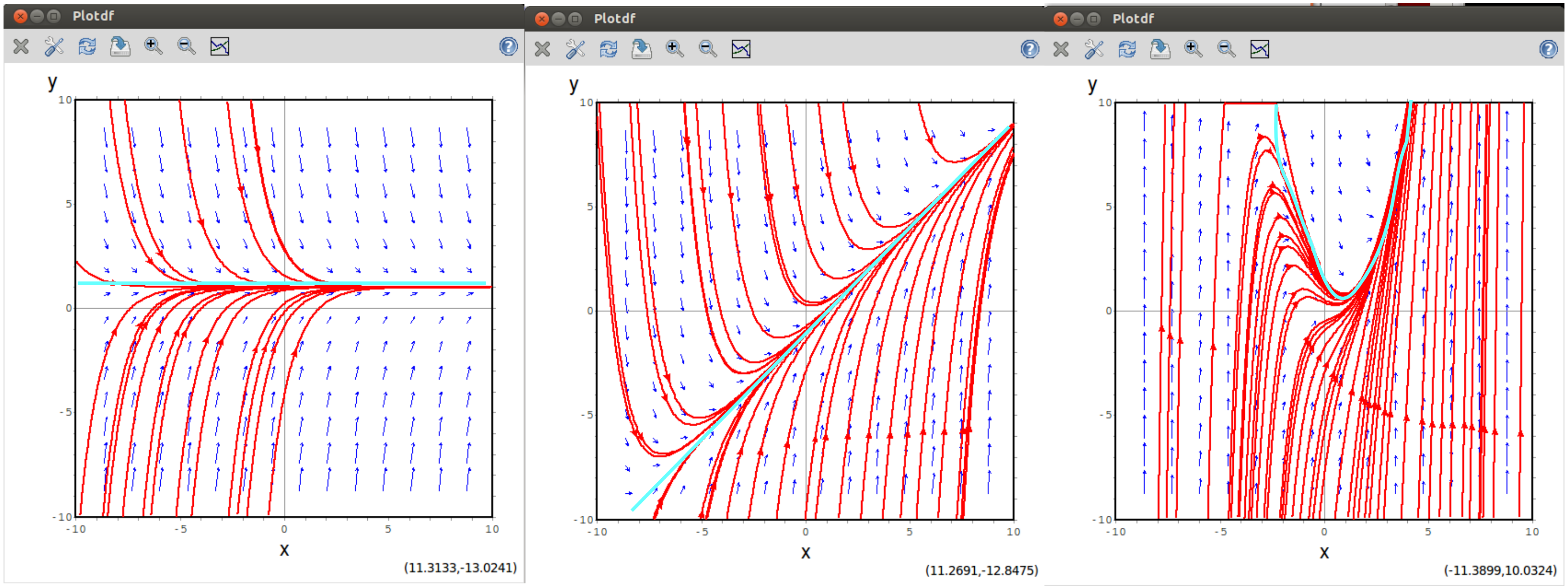
$$e^t y = t^2 e^t - 2(t e^t - e^t) + c$$

$$y = t^2 - 2t + 2 + c e^{-t}$$

$$\frac{d}{dt}\left[t^2 e^t\right] = 2t e^t + t^2 e^t$$

$$t^2 e^t = 2\int t e^t dx + \int t^2 e^t dt$$

plotdf in wxMaxima (2)



$$\frac{dy}{dt} + y = 1$$

$$\frac{dy}{dt} + y = t$$

$$\frac{dy}{dt} + y = t^2$$

$$y = 1 + ce^{-t}$$

$$y = (t-1) + ce^{-t}$$

$$y = t^2 - 2t + 2 + ce^{-t}$$

Causal LTI System Equations

$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy}{dt} + a_N y(t) = b_0 \frac{d^N x}{dt^N} + b_1 \frac{d^{N-1} x}{dt^{N-1}} + \cdots + b_{N-1} \frac{dx}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$$Q(D) = (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)$$

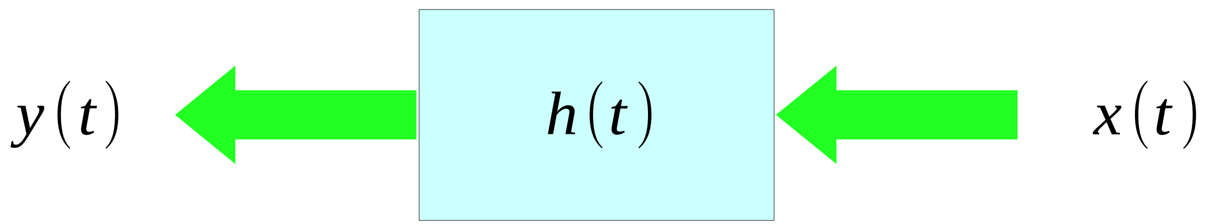
$$P(D) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)$$

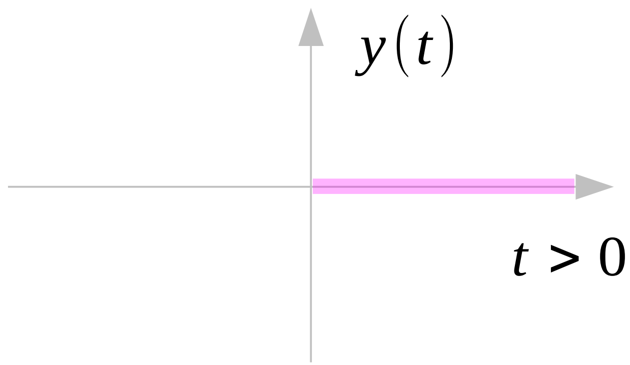
- **Zero Input Response**
- **Zero State Response (Convolution with h(t))**

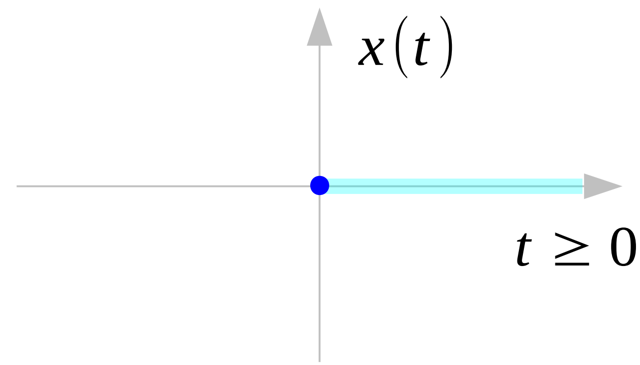
- **Natural Response (Homogeneous Solution)**
- **Forced Response (Particular Solution)**

Interval of Validity

$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy}{dt} + a_N y(t) = b_0 \frac{d^N x}{dt^N} + b_1 \frac{d^{N-1} x}{dt^{N-1}} + \cdots + b_{N-1} \frac{dx}{dt} + b_N x(t)$$







- **Zero Input Response**
- **Zero State Response (Convolution with h(t))**

Natural Response

- **Natural Response**

Homogeneous Solution

$$\frac{d^2 y_n(t)}{dt^2} + a_1 \frac{d\,y_n(t)}{dt} + a_2 y_n(t) = 0$$

Homogeneous Solution

$$Q(D)\,y_n(t) = 0$$

characteristic modes response

$$Q(\lambda) = (\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) = 0$$

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ \cdots\ (\lambda - \lambda_N) = 0$$

$$y_n(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} \cdots + K_N e^{\lambda_N t} = \sum_i K_i e^{\lambda_i t}$$ **Natural**

$$y_n(t) + y_p(t) \quad \{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\} \Rightarrow K_i$$

$$y_{zi}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \cdots + c_N e^{\lambda_N t} = \sum_i c_i e^{\lambda_i t}$$ **ZIR**

$$y_{zi}(t) \quad \{y^{(N-1)}(0^-), \cdots, y^{(1)}(0^-), y(0^-)\} \Rightarrow c_i$$

linear combination of the **characteristic modes**.

the same form as that of the **zero input response**

only its constants are different
← different initial conditions

Forced Response – undetermined coefficients

- **Forced Response**

Particular Solution

$$\frac{d^2 y_p(t)}{dt^2} + a_1 \frac{d\,y_p(t)}{dt} + a_2 y_p(t) = b_0 \frac{d^2 x(t)}{dt^2} + b_1 \frac{d\,x(t)}{dt} + b_2 x(t)$$

Particular Solution

$$Q(D)\,y_p(t) = P(D)\,x(t)$$

non-characteristic mode response

$y_p(t) = \beta$	⟸	$x(t) = k$			
$y_p(t) = \beta e^{\zeta t}$	⟸	$x(t) = e^{\zeta t}$	$\zeta \neq \lambda_i$		
$y_p(t) = \beta t e^{\zeta t}$	⟸	$x(t) = e^{\zeta t}$	$\zeta = \lambda_i$	$e^{\zeta t}$	ch. mode
$y_p(t) = \beta t^2 e^{\zeta t}$	⟸	$x(t) = e^{\zeta t}$	$\zeta = \lambda_i$	$e^{\zeta t}, t e^{\zeta t}$	ch. mode
$y_p(t) = (t^r + \beta_{r-1} t^{r-1} + \cdots + \beta_1 t + \beta_0) e^{\zeta t}$	⟸	$x(t) = (t^r + \alpha_{r-1} t^{r-1} + \cdots + \alpha_1 t + \alpha_0) e^{\zeta t}$			
$y_p(t) = \beta \cos(\omega t + \Phi)$	⟸	$x(t) = \cos(\omega t + \theta)$			

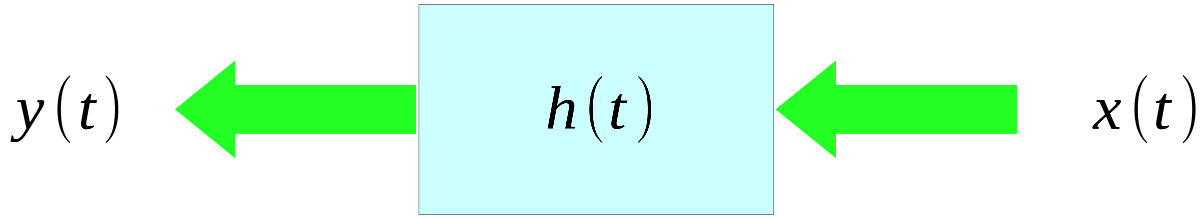
coefficients β_i are determined
by substituting the possible $y_p(t)$
into the given differential equation, then
equating the similar terms

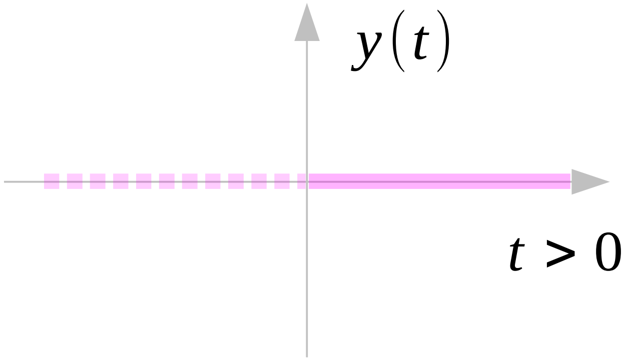
only for inputs with the finite derivatives

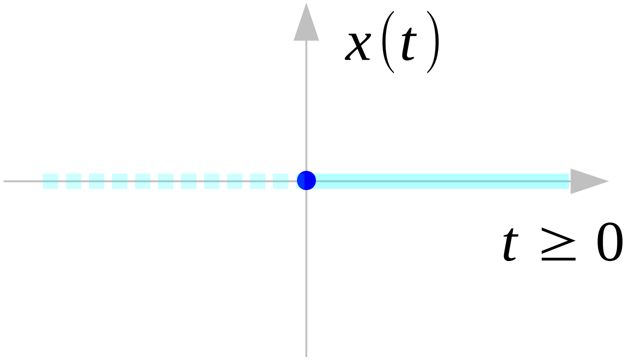
$$Q(D)\,y_p(t) = P(D)\,x(t)$$

Interval of Validity

$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy}{dt} + a_N y(t) = b_0 \frac{d^N x}{dt^N} + b_1 \frac{d^{N-1} x}{dt^{N-1}} + \cdots + b_{N-1} \frac{dx}{dt} + b_N x(t)$$







- **Natural Response (Homogeneous Solution)**
- **Forced Response (Particular Solution)**

- Three Initial Value Problems

Three Initial Value Problems

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0^+) = y_0$$
$$y'(x_0^+) = y_1$$

$$=$$

$$y'' + P(x)y' + Q(x)y = 0$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

Homogeneous DEQ — *Zero Input Response*

$$+$$

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$

Nonhomogeneous DEQ — *Zero State Response*

Zero Initial Conditions

Initially at rest

See "Differential Equations with Boundary-Value Problems", by Dennis Zill, Warren Wright

* Green's functions and impulse responses

Green's Function and IVP's

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y'(x_0) = 0$$
$$y(x_0) = 0$$

$$u_1'(x) = -\frac{y_2(x)f(x)}{W(x)}$$

$$u_1(x) = \int_{x_0}^{x} -\frac{y_2(t)f(t)}{W(t)}\,dt$$

$$u_2'(x) = \frac{y_1(x)f(x)}{W(x)}$$

$$u_2(x) = \int_{x_0}^{x} \frac{y_1(t)f(t)}{W(t)}\,dt$$

$$y_p = u_1(x)y_1 + u_2(x)y_2 = \left[\int_{x_0}^{x} -\frac{y_2(t)f(t)}{W(t)}\,dt\right] y_1(x) + \left[\int_{x_0}^{x} \frac{y_1(t)f(t)}{W(t)}\,dt\right] y_2(x)$$

$$= \left[\int_{x_0}^{x} -\frac{y_1(x)y_2(t)}{W(t)} f(t)\,dt\right] + \left[\int_{x_0}^{x} \frac{y_1(t)y_2(x)}{W(t)} f(t)\,dt\right]$$

$$= \int_{x_0}^{x} \left[\frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}\right] f(t)\,dt$$

$$= \int_{x_0}^{x} G(x, t) f(t)\,dt$$

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Green's Function

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y'' + P(x)y' + Q(x)y = g(x)$$

$$y'' + P(x)y' + Q(x)y = h(x)$$

$y_1, \; y_2$ $\qquad W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$

the same Green's function

$$G(x,\; t) = \left[\frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}\right]$$

$$y_p = u_1(x)y_1 + u_2(x)y_2 = \int_{x_0}^{x} G(x,\; t)f(t)\,dt = \int_{x_0}^{x} \left[\frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}\right] f(t)\,dt$$

General Solutions of the Initial Value Problem

$$y'' + 5y' + 6y = f(x)$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$

ZSR

$$m^2 + 5m + 6 = (m+2)(m+3) = 0$$

$$m = -2, \; -3$$

$$y_1 = e^{-2t} \qquad y_2 = e^{-3t}$$

$$W(t) = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$$

$$G(x, t) = \left[\frac{y_1(t)\,y_2(x) - y_1(x)\,y_2(t)}{W(t)}\right]$$

$$= \left[\frac{e^{-2t}e^{-3x} - e^{-2x}e^{-3t}}{-e^{-5t}}\right]$$

$$= \left[-e^{3t}e^{-3x} + e^{-2x}e^{+2t}\right]$$

$$= \left[e^{-2(x-t)} - e^{-3(x-t)}\right]$$

LTI $\quad = h(x-t) \quad$ Impulse Response

$$y_p = \int_{x_0}^{x} h(x-t)\,f(t)\,dt$$

$$= (h * f)(t)$$ convolution

Impulse Response by the Green's function

$$y'' + 3y' + 2y = x'$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$
ZSR

$$y'' + 3y' + 2y = x$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$
ZSR

$$m^2 + 3m + 2 = (m+1)(m+2) = 0$$

$$m = -1, \ -2$$

$$y_1 = e^{-t} \qquad y_2 = e^{-2t}$$

$$W(t) = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = -e^{-3t}$$

$$G(x,\ t) = \left[\frac{y_1(t)\,y_2(x) - y_1(x)\,y_2(t)}{W(t)}\right]$$

$$= \left[\frac{e^{-t}e^{-2x} - e^{-1x}e^{-2t}}{-e^{-3t}}\right]$$

$$= \left[-e^{2t}e^{-2x} + e^{-x}e^{+t}\right]$$

$$= \left[e^{-(x-t)} - e^{-2(x-t)}\right]$$

$$= g(x-t)$$

$$g(t) = \left[e^{-t} - e^{-2t}\right]$$

$$h(t) = [D\,g(t)]\,u(t)$$

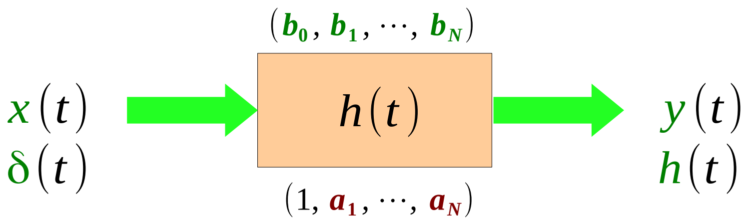
$$= -e^{-t} + 2e^{-2t}$$

- Superposition of the derivatives of $x(t)$

General and Base Systems

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

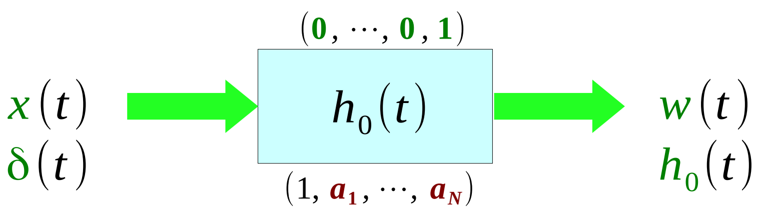
General System S



$h(t)$: the impulse response of **S**

$$w^{(N)} + a_1 w^{(N-1)} + \cdots + a_{N-1} w^{(1)} + a_N w = x$$

Base System S0



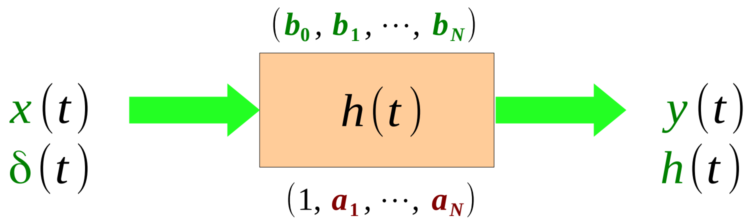
$h_0(t)$: the impulse response of **S0**

Differential Equations of S & S_0

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

$y(t) \Leftarrow x(t)$

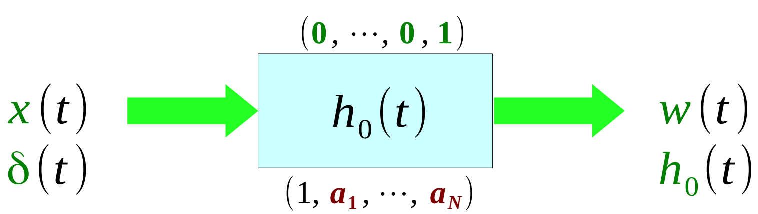


General System S

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot w(t) = x(t)$$

$$w^{(N)} + a_1 w^{(N-1)} + \cdots + a_{N-1} w^{(1)} + a_N w = x$$

$w(t) \Leftarrow x(t)$



Base System S0

Impulse Responses of S & S_0

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

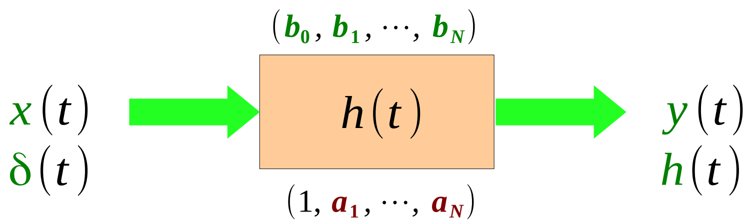
$y(t) \Leftarrow x(t)$

$$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h = b_0 \delta^{(N)} + b_1 \delta^{(N-1)} + \cdots + b_{N-1} \delta^{(1)} + b_N \delta$$

$h(t) \Leftarrow \delta(t)$

$$h = \frac{1}{a_N}\left(b_0 \delta^{(N)} + b_1 \delta^{(N-1)} + \cdots + b_{N-1} \delta^{(1)} + b_N \delta - h^{(N)} - a_1 h^{(N-1)} - \cdots - a_{N-1} h^{(1)}\right)$$

the impulse response
in terms of its derivatives



General System S

$$w^{(N)} + a_1 w^{(N-1)} + \cdots + a_{N-1} w^{(1)} + a_N w = x$$

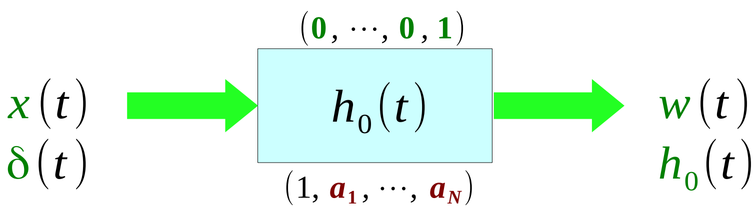
$w(t) \Leftarrow x(t)$

$$h_0^{(N)} + a_1 h_0^{(N-1)} + \cdots + a_{N-1} h_0^{(1)} + a_N h_0 = \delta$$

$h_0(t) \Leftarrow \delta(t)$

$$h_0 = \frac{1}{a_N}\left(\delta - h_0^{(N)} - a_1 h_0^{(N-1)} - \cdots - a_{N-1} h_0^{(1)}\right)$$

the impulse response
in terms of its derivatives

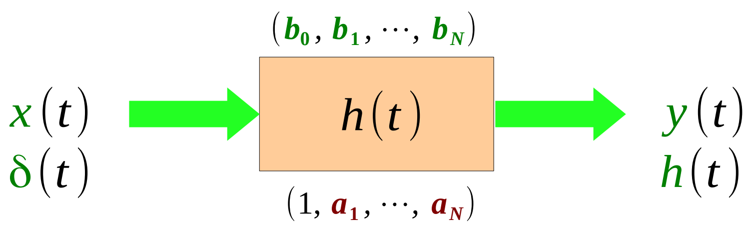


Base System S0

Polynomial Forms of S & S_0

$$Q(D) = (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)$$

$$P(D) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)$$

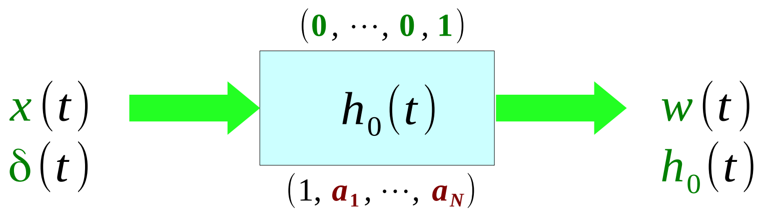


$$Q(D)\, y(t) = \boxed{P(D)\, x(t)}$$

$$Q(D)\, h(t) = \boxed{P(D)\, \delta(t)}$$

$$Q(D) = (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)$$

$$P(D) = (1)$$



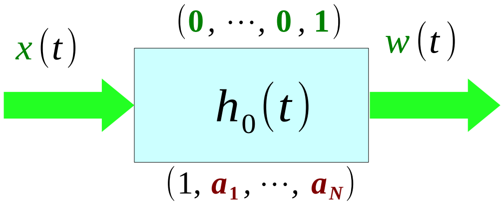
$$Q(D)\, w(t) = \boxed{x(t)}$$

$$Q(D)\, h_0(t) = \boxed{\delta(t)}$$

Base Systems with derivatives of x(t)

The base system **S0** with the input of x(t)

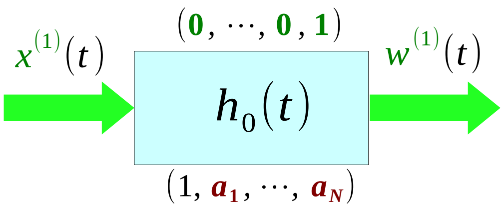
$$w^{(N)} + a_1 w^{(N-1)} + \cdots + a_{N-1} w^{(1)} + a_N w = x$$



The base system **S0** with the input of $x^{(1)}(t)$

$$w^{(N+1)} + a_1 w^{(N)} + \cdots + a_{N-1} w^{(2)} + a_N w^{(1)} = x^{(1)}$$

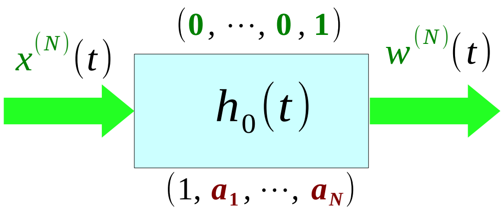
$$\{w^{(1)}\}^{(N)} + a_1 \{w^{(1)}\}^{(N-1)} + \cdots + a_{N-1} \{w^{(1)}\}^{(1)} + a_N \{w^{(1)}\} = x^{(1)}$$



The base system **S0** with the input of $x^{(N)}(t)$

$$w^{(2N)} + a_1 w^{(2N-1)} + \cdots + a_{N-1} w^{(N+1)} + a_N w^{(N)} = x^{(N)}$$

$$\{w^{(N)}\}^{(N)} + a_1 \{w^{(N)}\}^{(N-1)} + \cdots + a_{N-1} \{w^{(N)}\}^{(1)} + a_N \{w^{(N)}\} = x^{(N)}$$

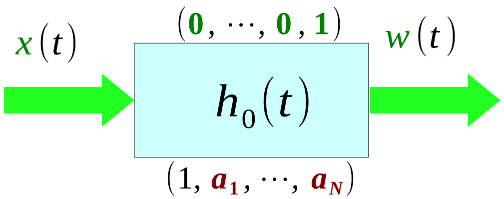


$$w^{(2N-1)} = \frac{d^{2N-1}}{dt^{2N-1}}\{w(t)\} \iff \{w^{(N)}\}^{(N-1)} = \frac{d^{N-1}}{dt^{N-1}}\left\{\frac{d^N}{dt^N} w(t)\right\}$$

Base System S0 with derivatives of x(t)

The base system **S0** with the input of x(t)

$$w^{(N)} + a_1 w^{(N-1)} + \cdots + a_{N-1} w^{(1)} + a_N w = x$$



$$
\begin{aligned}
b_N \quad & w^{(N)} + a_1 w^{(N-1)} + \cdots + a_{N-1} w^{(1)} + a_N w = x && 0 \\
b_{N-1} \quad & w^{(N+1)} + a_1 w^{(N)} + \cdots + a_{N-1} w^{(2)} + a_N w^{(1)} = x^{(1)} && 1 \\
& \vdots \\
b_0 \quad & w^{(2N)} + a_1 w^{(2N-1)} + \cdots + a_{N-1} w^{(N+1)} + a_N w^{(N)} = x^{(N)} && N
\end{aligned}
$$

$$
\begin{aligned}
& b_N w^{(N)} + a_1 b_N w^{(N-1)} + \cdots + a_{N-1} b_N w^{(1)} + a_N b_N w = b_N x && 0 \\
& b_{N-1} w^{(N+1)} + a_1 b_{N-1} w^{(N)} + \cdots + a_{N-1} b_{N-1} w^{(2)} + a_N b_{N-1} w^{(1)} = b_{N-1} x^{(1)} && 1 \\
& \vdots \\
& b_0 w^{(2N)} + a_1 b_0 w^{(2N-1)} + \cdots + a_{N-1} b_0 w^{(N+1)} + a_N b_0 w^{(N)} = b_0 x^{(N)} && N
\end{aligned}
$$

General System S with derivatives of x(t)

Let $y(t) = b_N w(t) + b_{N-1} w^{(1)}(t) + \cdots + b_0 w^{(N)}(t)$

$$b_N w^{(N)} + a_1 b_N w^{(N-1)} + \cdots + a_{N-1} b_N w^{(1)} + a_N b_N w = b_N x \quad 0$$

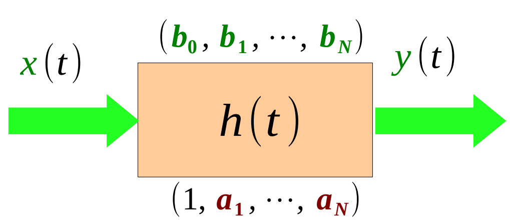
$$b_{N-1} w^{(N+1)} + a_1 b_{N-1} w^{(N)} + \cdots + a_{N-1} b_{N-1} w^{(2)} + a_N b_{N-1} w^{(1)} = b_{N-1} x^{(1)} \quad 1$$

$$\vdots$$

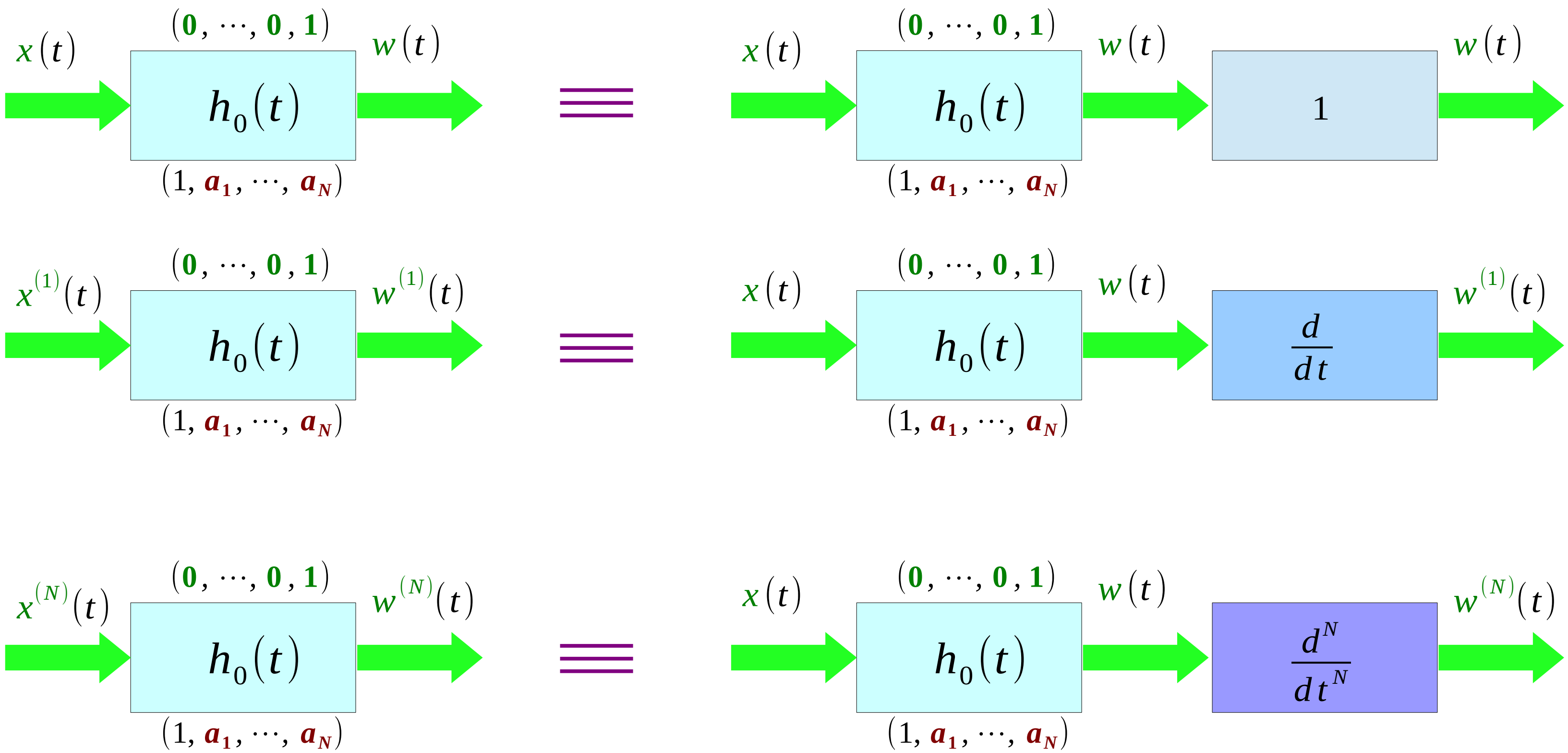
$$b_0 w^{(2N)} + a_1 b_0 w^{(2N-1)} + \cdots + b_0 a_{N-1} w^{(N+1)} + a_N b_0 w^{(N)} = b_0 x^{(N)} \quad N$$

$$y^{(N)} \qquad y^{(N-1)} \qquad y^{(1)} \qquad y$$

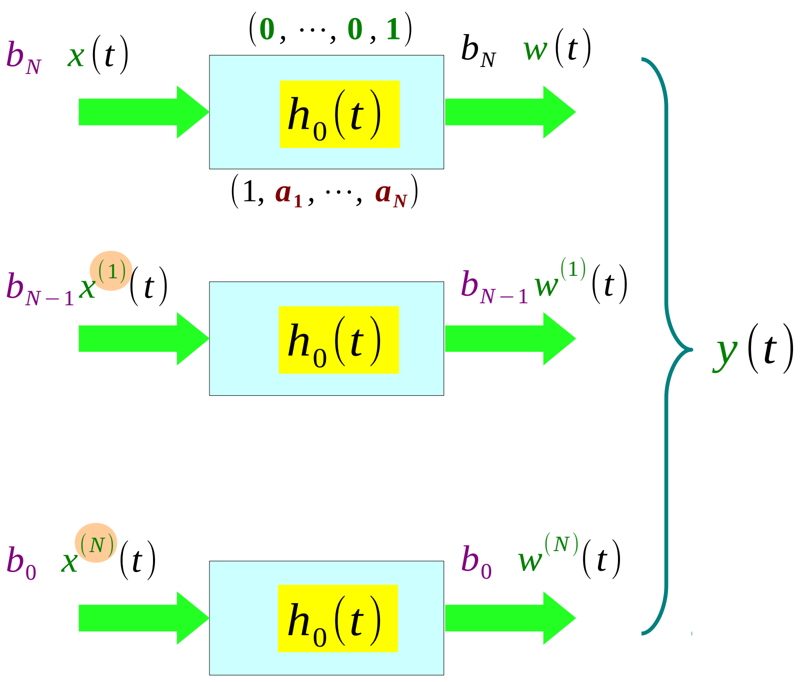
$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

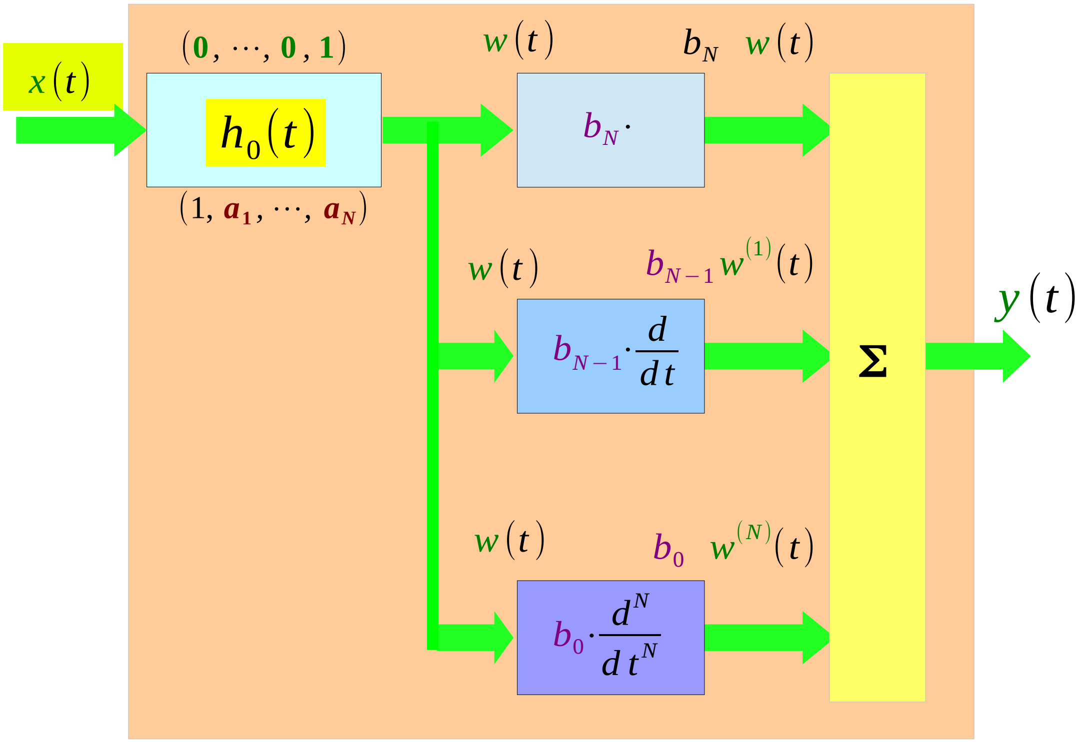


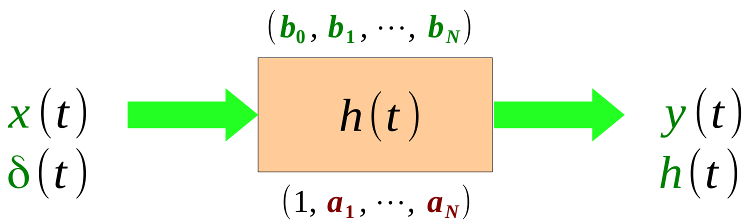
Equivalent Impulse Responses



Superposition of Base System Outputs





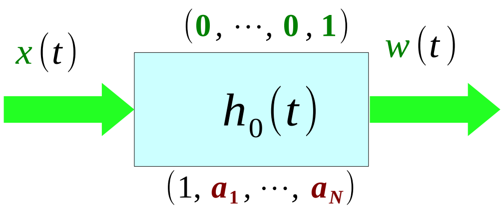


$$y(t) = b_N w(t) + b_{N-1} w^{(1)}(t) + \cdots + b_0 w^{(N)}(t)$$

Derivatives of Impulse Response

The base system **S0** with the input of x(t)

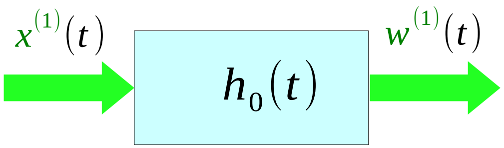
$$w^{(N)} + a_1 w^{(N-1)} + \cdots + a_{N-1} w^{(1)} + a_N w = x$$



The base system **S0** with the input of $x^{(1)}(t)$

$$w^{(N+1)} + a_1 w^{(N)} + \cdots + a_{N-1} w^{(2)} + a_N w^{(1)} = x^{(1)}$$

$$\{w^{(1)}\}^{(N)} + a_1 \{w^{(1)}\}^{(N-1)} + \cdots + a_{N-1} \{w^{(1)}\}^{(1)} + a_N \{w^{(1)}\} = x^{(1)}$$

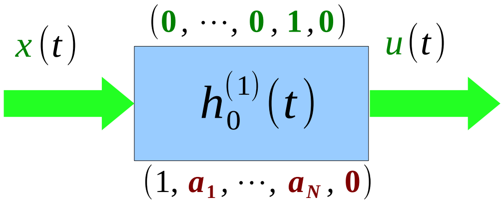


$$w(t) = h_0(t) * x(t)$$

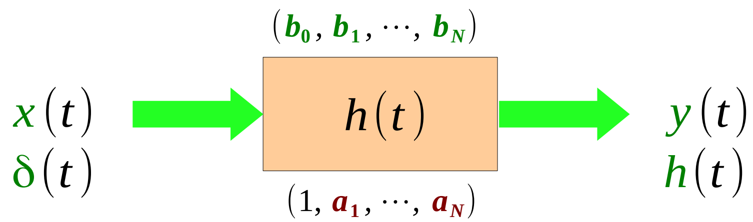
$$w^{(1)}(t) = h_0(t) * x^{(1)}(t)$$
$$= h_0^{(1)}(t) * x(t)$$

$$u(t) = w^{(1)}$$

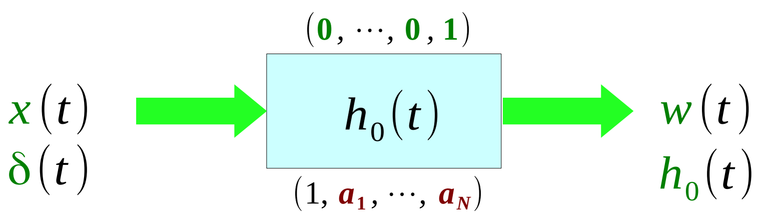
$$u(t) = y_n^{(1)}(t) * x(t)$$



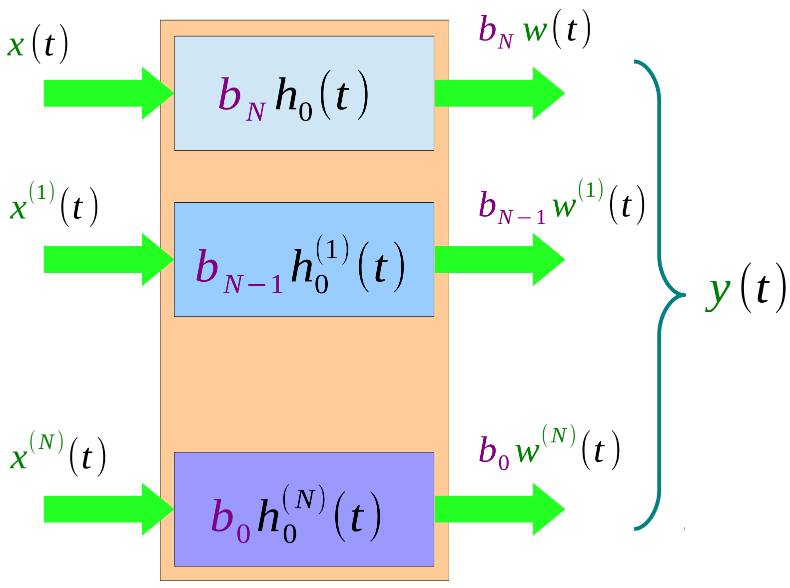
General System via Base System Responses



$$y = b_0 w^{(N)} + b_1 w^{(N-1)} + \cdots + b_N w$$
$$h = b_0 y_n^{(N)} + b_1 y_n^{(N-1)} + \cdots + b_N y_n$$



$$w = x^{(N)} + a_1 x^{(N-1)} + \cdots + a_N x$$
$$h_0 = \delta^{(N)} + a_1 \delta^{(N-1)} + \cdots + a_N \delta$$



$$Q(D) w(t) = x(t)$$
$$Q(D) \boxed{P(D) w(t)} = \boxed{P(D) x(t)}$$
$$y(t) = P(D) w(t)$$

$$Q(D) h_0(t) = \delta(t)$$
$$Q(D) \boxed{P(D) h_0(t)} = \boxed{P(D) \delta(t)}$$
$$h(t) = P(D) h_0(t)$$

- Initial Value Problems and System Responses

Decomposing an Initial Value Problem

$$y''(t) + a_1 y'(t) + a_2 y(t) = b_0 x''(t) + b_1 x'(t) + b_2 x(t)$$

$$y(0^+) = k_0 \quad y'(0^+) = k_1$$

Target Initial Value Problem

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

=

$$y''(t) + a_1 y'(t) + a_2 y(t) = 0$$

$$y_{zi}(0^+) = k_0 \quad y_{zi}'(0^+) = k_1$$

Zero Input

$$y_{zi}(t)$$

Zero Input Response

+

$$y''(t) + a_1 y'(t) + a_2 y(t) = b_0 y''(t) + b_1 y'(t) + b_2 x(t)$$

$$y_{zs}(0^+) = 0 \quad y_{zs}'(0^+) = 0$$

Zero State

$$y_{zs}(t) = x(t) * h(t)$$

Zero State Response

Decomposing a Differential Equation

$$y''(t) + a_1 y'(t) + a_2 y(t) = 0$$

$y_n(t)$ Natural Response

+

$$y''(t) + a_1 y'(t) + a_2 y(t) = b_0 y''(t) + b_1 y'(t) + b_2 x(t)$$

$y_p(t)$ Forced Response

=

$$y''(t) + a_1 y'(t) + a_2 y(t) = b_0 x''(t) + b_1 x'(t) + b_2 x(t)$$

$y(0^+) = y_0$ $\quad y'(0^+) = y_1$

Target Initial Value Problem

$$y(t) = y_n(t) + y_p(t)$$

References

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[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] B.P. Lathi, Linear Systems and Signals (2nd Ed)
[4] X. Xu, http://ecse.bd.psu.edu/eebd410/ltieqsol.pdf