Transformation of a Random Variable

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Monotonic increasing transformation

a continuous RV X

Definition

A transform T is called monotonically increasing
if $T(x_1) < T(x_2)$ for any $x_1 < x_2$

$$y_0 = T(x_0) \qquad \{Y \le y_0\}$$

$$x_0 = T^{-1}(y_0) \qquad \{X \le x_0\}$$

Monotonic decreasing transformation

a continuous RV X

Definition

A transform T is called monotonically decreasing
if $T(x_1) > T(x_2)$ for any $x_1 < x_2$

$$y_0 = T(x_0) \qquad \{Y \leq y_0\}$$

$$x_0 = T^{-1}(y_0) \qquad \{X \leq x_0\}$$

Cumulative Density Functions

monotonic increasing transform T, a continous RV X

- $y_0 = T(x_0) \qquad \{Y \le y_0\}$
- $F_Y(y_0) = P\{Y \le y_0\}$

- $x_0 = T^{-1}(y_0) \qquad \{X \le x_0\}$
- $F_X(x_0) = P\{X \le x_0\}$

- $P\{Y \le y_0\} = P\{X \le x_0\}$
- $F_Y(y_0) = F_X(x_0)$

Probability Density Functions

monotonic increasing transform T, a continous RV X

$$F_Y(y_0) = P\{Y \le y_0\} = P\{X \le x_0\} = F_X(x_0)$$

$$F_Y(y_0) = \int_{-\infty}^{y_0} f_Y(y)dy = \int_{-\infty}^{x_0 = T^{-1}(y_0)} f_X(x)dx = F_X(x_0)$$

$$f_Y(y_0)dy_0 = f_X[T^{-1}(y_0)]dx_0$$

$$f_Y(y_0) = f_X[T^{-1}(y_0)]\frac{dT^{-1}(y_0)}{dy_0}$$

monotonic increasing transform T

$$\boxed{f_Y(y) = f_X[T^{-1}(y)]\frac{dT^{-1}(y)}{dy}}$$

Cumulative Density Functions

monotonic decreasing transform T, a continous RV X

- $y_0 = T(x_0) \qquad \{Y \leq y_0\}$
- $x_0 = T^{-1}(y_0) \qquad \{X \geq x_0\}$

- $F_Y(y_0) = P\{Y \leq y_0\}$
- $F_X(x_0) = P\{X \geq x_0\} = 1 - P\{X \leq x_0\}$

- $P\{Y \leq y_0\} = 1 - P\{X \leq x_0\}$
- $F_Y(y_0) = 1 - F_X(x_0)$

Probability Density Functions

monotonic decreasing transform T, a continous RV X

$$F_Y(y_0) = P\{Y \le y_0\} = 1 - P\{X \le x_0\} = 1 - F_X(x_0)$$

$$F_Y(y_0) = \int_{-\infty}^{y_0} f_Y(y)dy = 1 - \int_{-\infty}^{x_0 = T^{-1}(y_0)} f_X(x)dx = 1 - F_X(x_0)$$

$$f_Y(y_0)dy_0 = f_X[T^{-1}(y_0)]dx_0$$

$$f_Y(y_0) = -f_X[T^{-1}(y_0)]\frac{dT^{-1}(y_0)}{dy_0}$$

monotonic decreasing transform T

$$\boxed{f_Y(y)dy = -f_X[T^{-1}(y)]\frac{dT^{-1}(y)}{dy}}$$

Probability Density Functions
monotonic transform T, a continous RV X

monotonic increasing T $$f_Y(y) = f_X[T^{-1}(y)] \frac{dT^{-1}(y)}{dy}$$

monotonic decreasing T $$f_Y(y)dy = -f_X[T^{-1}(y)] \frac{dT^{-1}(y)}{dy}$$

$$\boxed{f_Y(y_0)dy = \left|f_X[T^{-1}(y_0)]\right| \frac{dT^{-1}(y_0)}{dy_0}}$$

$$\boxed{f_Y(y) = f_X[x] \left|\frac{dx}{dy}\right|}$$

Non-monotonic Transformation
a continous RV X

Definition

$\{x|Y \le y_0\}$=$\{x$ values which yields $Y \le y_0\}$
may have more than one interval of X
Assume $\{Y \le y_0\}$ corresponds to$\{X \le x_1\} \cup \{x_2 \le X \le x_3\}$

$$f_Y(y_0) = \frac{d}{dy_0} \int_{\{x|Y \le y_0\}} f_X(x)dx$$

$$f_Y(y_0) = \sum_n \frac{f_X(x_n)}{\left|\frac{dT(x)}{dx}\right|_{x=x_n}}$$

Non-monotonic Transformation

a continous RV X

- $f_Y(y_0) = \dfrac{d}{dy_0} \int_{\{x|Y \le y_0\}} f_X(x)dx$
- $f_Y(y_0) = \sum_n \dfrac{f_X(x_n)}{\left|\dfrac{dT(x)}{dx}\right|_{x=x_n}}$

- $f_Y(y_0) = f_X[T^{-1}(y_0)] \dfrac{dT^{-1}(y_0)}{dy_0}$
- $f_Y(y_0) = f_X[x_0] \dfrac{dx_0}{dy_0} = \dfrac{f_X[x_0]}{\dfrac{dy_0}{dx_0}}$

Transformation of a Discrete Random Variables

monotonic continuous transform T, a discrete RV X

Definition

A discrete random variable X
a continuous transformation $Y = T(X)$

$$f_X(x) = \sum_n P(x_n)\delta(x - x_n) \quad \Longrightarrow \quad f_Y(y) = \sum_n P(y_n)\delta(y - y_n)$$

$$F_X(x) = \sum_n P(x_n)u(x - x_n) \quad \Longrightarrow \quad F_Y(y) = \sum_n P(y_n)u(y - y_n)$$

$$n = 1, 2, \cdots$$

one-to-one correspondence between $\{x_n\}$ and $\{y_n\}$

- $y_n = T(x_n)$
- $P(y_n) = P(x_n)$