

Transformation of a Random Variable

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May 14, 2020

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

- 1 Monotonic Transformation : Continuous Cases
- 2 Monotonic Transformation : Discrete Cases

Monotonic **increasing** transformation

a continuous RV X

Definition

A transform T is called monotonically **increasing** if $T(x_1) < T(x_2)$ for any $x_1 < x_2$

$$y_0 = T(x_0) \quad \{Y \leq y_0\}$$

$$x_0 = T^{-1}(y_0) \quad \{X \leq x_0\}$$

Monotonic **decreasing** transformation

a continuous RV X

Definition

A transform T is called monotonically **decreasing** if $T(x_1) > T(x_2)$ for any $x_1 < x_2$

$$y_0 = T(x_0) \quad \{Y \leq y_0\}$$

$$x_0 = T^{-1}(y_0) \quad \{X \leq x_0\}$$

Cumulative Density Functions

monotonic **increasing** transform T , a continuous RV X

- $y_0 = T(x_0) \quad \{Y \leq y_0\}$

- $F_Y(y_0) = P\{Y \leq y_0\}$

- $x_0 = T^{-1}(y_0) \quad \{X \leq x_0\}$

- $F_X(x_0) = P\{X \leq x_0\}$

- $P\{Y \leq y_0\} = P\{X \leq x_0\}$

- $F_Y(y_0) = F_X(x_0)$

Probability Density Functions

monotonic **increasing** transform T , a continuous RV X

$$F_Y(y_0) = P\{Y \leq y_0\} = P\{X \leq x_0\} = F_X(x_0)$$

$$F_Y(y_0) = \int_{-\infty}^{y_0} f_Y(y) dy = \int_{-\infty}^{x_0=T^{-1}(y_0)} f_X(x) dx = F_X(x_0)$$

$$f_Y(y_0) dy_0 = f_X[T^{-1}(y_0)] dx_0$$

$$f_Y(y_0) = f_X[T^{-1}(y_0)] \frac{dT^{-1}(y_0)}{dy_0}$$

monotonic **increasing** transform T

$$f_Y(y) = f_X[T^{-1}(y)] \frac{dT^{-1}(y)}{dy}$$

Cumulative Density Functions

monotonic **decreasing** transform T , a continuous RV X

- $y_0 = T(x_0) \quad \{Y \leq y_0\}$
- $x_0 = T^{-1}(y_0) \quad \{X \geq x_0\}$

- $F_Y(y_0) = P\{Y \leq y_0\}$
- $F_X(x_0) = P\{X \geq x_0\} = 1 - P\{X \leq x_0\}$

- $P\{Y \leq y_0\} = 1 - P\{X \leq x_0\}$
- $F_Y(y_0) = 1 - F_X(x_0)$

Probability Density Functions

monotonic **decreasing** transform T , a continous RV X

$$F_Y(y_0) = P\{Y \leq y_0\} = 1 - P\{X \leq x_0\} = 1 - F_X(x_0)$$

$$F_Y(y_0) = \int_{-\infty}^{y_0} f_Y(y) dy = 1 - \int_{-\infty}^{x_0 = T^{-1}(y_0)} f_X(x) dx = 1 - F_X(x_0)$$

$$f_Y(y_0) dy_0 = f_X[T^{-1}(y_0)] dx_0$$

$$f_Y(y_0) = -f_X[T^{-1}(y_0)] \frac{dT^{-1}(y_0)}{dy_0}$$

monotonic **decreasing** transform T

$$f_Y(y) dy = -f_X[T^{-1}(y)] \frac{dT^{-1}(y)}{dy}$$

Probability Density Functions

monotonic transform T , a continuous RV X

monotonic **increasing** T $f_Y(y) = f_X[T^{-1}(y)] \frac{dT^{-1}(y)}{dy}$

monotonic **decreasing** T $f_Y(y)dy = -f_X[T^{-1}(y)] \frac{dT^{-1}(y)}{dy}$

$$f_Y(y_0)dy = |f_X[T^{-1}(y_0)]| \frac{dT^{-1}(y_0)}{dy_0}$$

$$f_Y(y) = f_X[x] \left| \frac{dx}{dy} \right|$$

Non-monotonic Transformation

a continuous RV X

Definition

$\{x | Y \leq y_0\} = \{x \text{ values which yields } Y \leq y_0\}$

may have more than one interval of X

Assume $\{Y \leq y_0\}$ corresponds to $\{X \leq x_1\} \cup \{x_2 \leq X \leq x_3\}$

$$f_Y(y_0) = \frac{d}{dy_0} \int_{\{x | Y \leq y_0\}} f_X(x) dx$$

$$f_Y(y_0) = \sum_n \frac{f_X(x_n)}{\left| \frac{dT(x)}{dx} \right|_{x=x_n}}$$

Non-monotonic Transformation

a continuous RV X

- $f_Y(y_0) = \frac{d}{dy_0} \int_{\{x|Y \leq y_0\}} f_X(x) dx$
- $f_Y(y_0) = \sum_n \frac{f_X(x_n)}{\left| \frac{dT(x)}{dx} \right|_{x=x_n}}$

- $f_Y(y_0) = f_X[T^{-1}(y_0)] \frac{dT^{-1}(y_0)}{dy_0}$
- $f_Y(y_0) = f_X[x_0] \frac{dx_0}{dy_0} = \frac{f_X[x_0]}{\frac{dy_0}{dx_0}}$

Transformation of a Discrete Random Variables

monotonic continuous transform T , a discrete RV X

Definition

A discrete random variable X

a continuous transformation $Y = T(X)$

$$f_X(x) = \sum_n P(x_n) \delta(x - x_n) \implies f_Y(y) = \sum_n P(y_n) \delta(y - y_n)$$

$$F_X(x) = \sum_n P(x_n) u(x - x_n) \implies F_Y(y) = \sum_n P(y_n) u(y - y_n)$$

$$n = 1, 2, \dots$$

one-to-one correspondence between $\{x_n\}$ and $\{y_n\}$

- $y_n = T(x_n)$
- $P(y_n) = P(x_n)$

