Spectrum Representation (2B)

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1/21/17

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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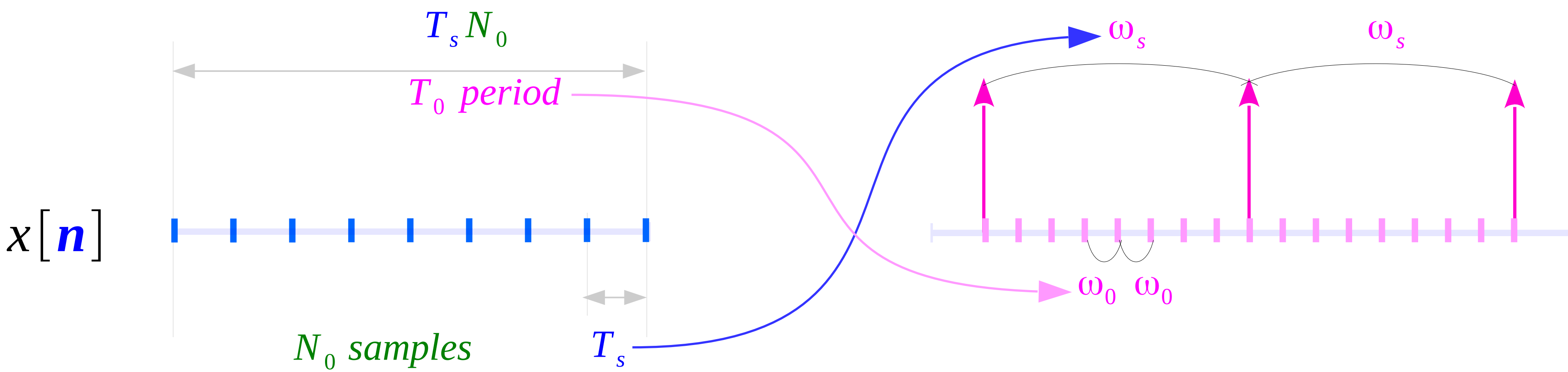
ω_s and ω_0

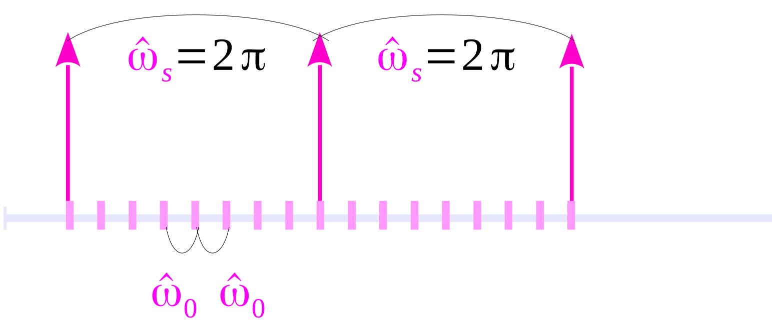


$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$\hat{\omega}_0 = \frac{2\pi}{N_0}$$

$$\hat{\omega}_s = \frac{2\pi}{1}$$



Fourier Transform Types

Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

Discrete Time Fourier **Series**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+jk\hat{\omega}_0 n}$$

Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Time Fourier **Transform**

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Leftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(j\hat{\omega})\, e^{+j\hat{\omega} n}\, d\hat{\omega}$$

Computation at $k\omega_0$

CTFS **Periodic x(t)**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \qquad k\omega_0$$

$$@ \quad k\omega_0 = k\left(\frac{2\pi}{T}\right) \quad rad/sec$$

CTFT **Aperiodic x(t)**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \qquad \omega \leftarrow k\omega_0$$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{N T_s}\right) \quad rad/sec$$

DTFS **Periodic x[n]**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \qquad k\hat{\omega}_0$$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{N T_s}\right) \quad rad/sec$$

DTFT **Aperiodic x[n]**

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \qquad \hat{\omega} \leftarrow k\hat{\omega}_0$$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{N T_s}\right) \quad rad/sec$$

Computations using DFT

CTFS — **Periodic x(t)**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k \approx \frac{1}{N} \mathbf{DFT}\{x(nT_s)\}$$

$k\omega_0$

$$@ \quad k\omega_0 = k\left(\frac{2\pi}{T}\right) \quad rad/sec$$

CTFT — **Aperiodic x(t)**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(jk\omega_0) \approx T_s \mathbf{DFT}\{x(nT_s)\}$$

$\omega \leftarrow k\omega_0$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{NT_s}\right) \quad rad/sec$$

DTFS — **Periodic x[n]**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n}$$

$$\gamma[k] = \frac{1}{N} \mathbf{DFT}\{x[n]\}$$

$k\hat{\omega}_0$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{NT_s}\right) \quad rad/sec$$

DTFT — **Aperiodic x[n]**

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

$$X(jk\hat{\omega}_0) \approx \mathbf{DFT}\{x[n]\}$$

$\hat{\omega} \leftarrow k\hat{\omega}_0$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{NT_s}\right) \quad rad/sec$$

Computations using DFT

CTFS **Periodic x(t)**

$$C_k \approx \frac{1}{N} \, \boldsymbol{DFT}\{x(nT_s)\}$$

$$x(nT_s) \approx N \, \boldsymbol{IDFT}\{C_k\}$$

CTFT **Aperiodic x(t)**

$$X(jk\omega_0) \approx T_s \, \boldsymbol{DFT}\{x(nT_s)\}$$

$$x(nT_s) \approx \frac{1}{T_s} \, \boldsymbol{IDFT}\{X(jk\omega_0)\}$$

DTFS **Periodic x[n]**

$$\gamma[k] = \frac{1}{N} \, \boldsymbol{DFT}\{x[n]\}$$

$$x[n] = N \, \boldsymbol{IDFT}\{\gamma_k\}$$

DTFT **Aperiodic x[n]**

$$X(jk\hat{\omega}_0) \approx \boldsymbol{DFT}\{x[n]\}$$

$$x[n] \approx \boldsymbol{IDFT}\{X(jk\hat{\omega}_0)\}$$

FFT Amplitude and Power Spectrum

Two-Sided Amplitude Spectrum

$$A_k = \frac{1}{N}|X[k]| \qquad (V/Hz^{-1/2})$$

$$= \frac{1}{N}\sqrt{\Re^2\{X[k]\} + \Im^2\{X[k]\}}$$

$$k = 0,1,2,\cdots,N/2,N/2+1,\cdots,N-1$$

Two-Sided Power Spectrum

$$P_k = \frac{1}{N^2}|X[k]|^2 \qquad (V^2/Hz^{-1})$$

$$= \frac{1}{N^2}\left(\Re^2\{X[k]\} + \Im^2\{X[k]\}\right)$$

$$k = 0,1,2,\cdots,N/2,N/2+1,\cdots,N-1$$

One-Sided Amplitude Spectrum

$$\bar{A}_0 = \frac{1}{N}|X[0]| \qquad k=0$$

$$\bar{A}_k = \frac{2}{N}|X[k]| \qquad k=1,2,\cdots,N/2$$

One-Sided Power Spectrum

$$\bar{P}_0 = \frac{1}{N^2}|X[0]|^2 \qquad k=0$$

$$\bar{P}_k = \frac{2}{N^2}|X[k]|^2 \qquad k=1,2,\cdots,N/2$$

Frequency Bin

$$f = k\frac{f_s}{N}$$

Frequency Bin

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FFT Amplitude and Phase Spectrum

Two-Sided Amplitude Spectrum

$$A_k = \frac{1}{N}|X[k]|$$

$$= \frac{1}{N}\sqrt{\Re^2\{X[k]\}+\Im^2\{X[k]\}}$$

$$k = 0,1,2,\cdots,N/2,N/2+1,\cdots,N-1$$

Two-Sided Phase Spectrum

$$\phi_k = \tan^{-1}\left(\frac{\Im\{X[k]\}}{\Re\{X[k]\}}\right)$$

$$k = 0,1,2,\cdots,N/2,N/2+1,\cdots,N-1$$

Frequency Bin

$$f = k\frac{f_s}{N}$$

CTFS and Power Spectrum

Two-Sided Power Spectrum

$$\frac{1}{N^2}|X[k]|^2 = |C_k|^2 = \frac{1}{4}(a_k^2 + b_k^2) = \frac{1}{4}|g_k|^2$$

Single-Sided Power Spectrum

$$\frac{2}{N^2}|X[k]|^2 = 2|C_k|^2 = \frac{1}{2}|g_k|^2 = |g_{k,rms}|^2$$

$$C_k = \frac{1}{2} g_{+k}\, e^{+j\phi_k} \quad (k > 0)$$

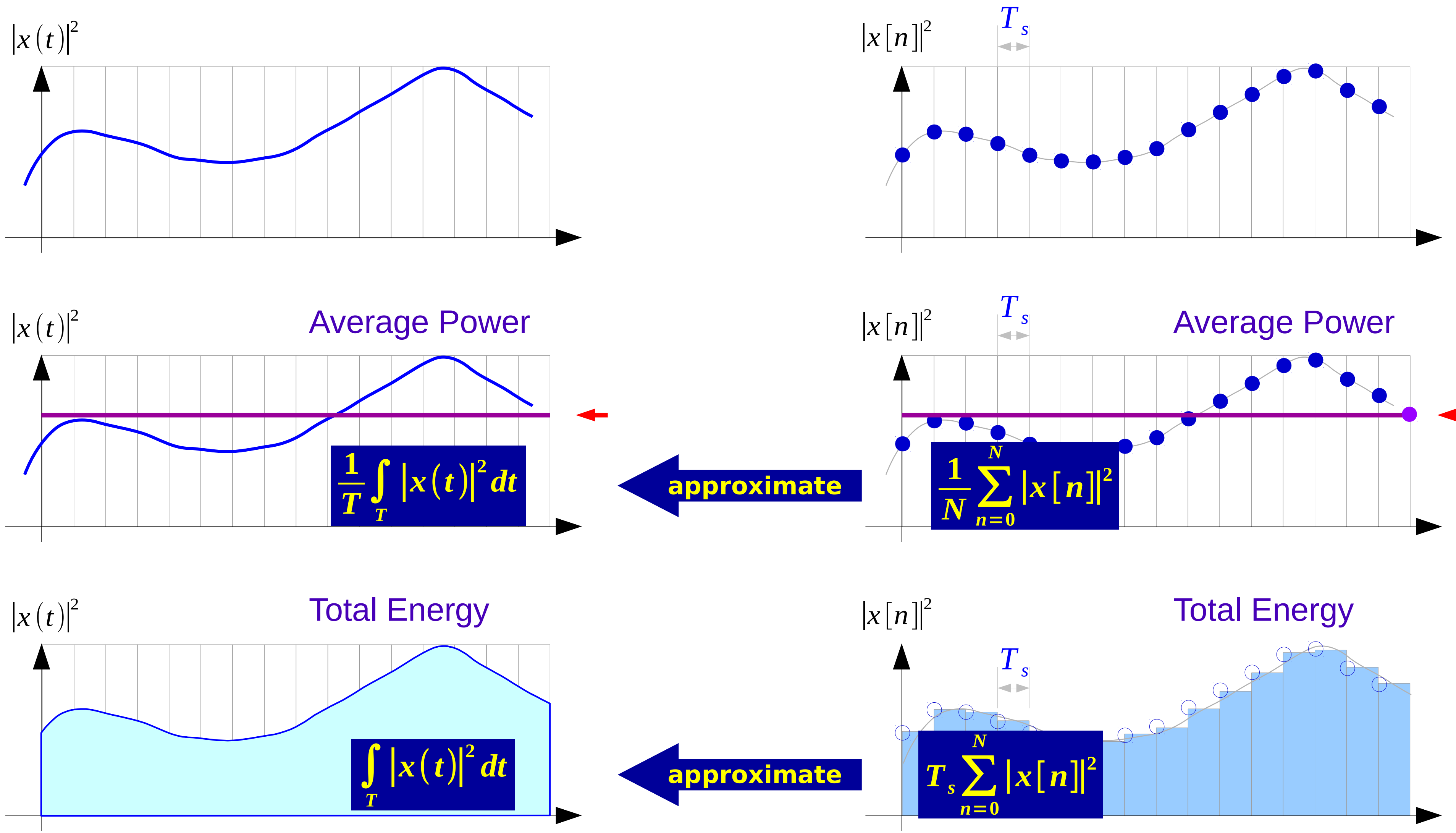
$$C_k = \frac{1}{2} g_{-k}\, e^{-j\phi_k} \quad (k < 0)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

g_k *each sinusoid's amplitude*

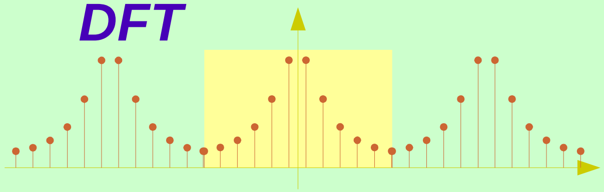
$g_{k,rms}$ *each sinusoid's amplitude rms value*

Average Power and Total Energy

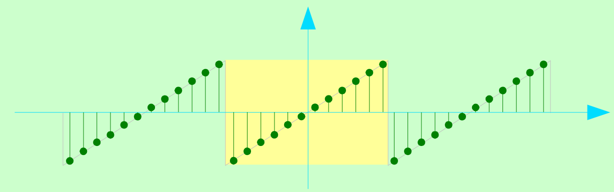


Parseval’s Theorem for DFT

DFT



$$\frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2$$



$$\sum_{n=0}^{N-1}|x[n]|^2$$

Average Power

$$\frac{1}{N}\sum_{k=0}^{N-1}P_{xx}[k] = \frac{1}{N^2}\sum_{k=0}^{N-1}|X[k]|^2$$

$$\frac{1}{N^2}\sum_{k=0}^{N-1}|X[k]|^2 = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2$$

Average Power

Periodogram

$$P_{xx}[k] = \frac{1}{N}|X[k]|^2 = \left|\frac{X[k]}{\sqrt{N}}\right|^2$$

$$\frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2 = \sum_{n=0}^{N-1}|x[n]|^2$$

Parseval’s Theorem

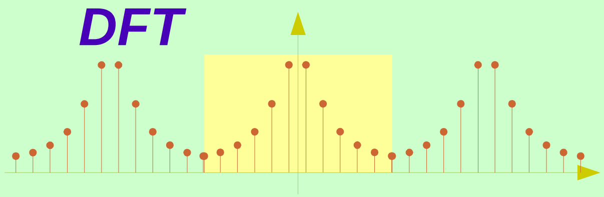
Total Energy

$$T_s\sum_{k=0}^{N-1}P_{xx}[k] = \frac{T_s}{N}\sum_{k=0}^{N-1}|X[k]|^2$$

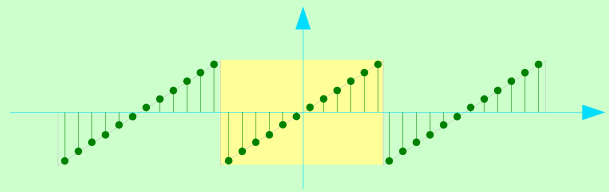
$$\frac{T_s}{N}\sum_{k=0}^{N-1}|X[k]|^2 = T_s\sum_{n=0}^{N-1}|x[n]|^2$$

Total Energy

Periodogram as a frequency domain samples



$$\frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2$$



$$\sum_{n=0}^{N-1}|x[n]|^2$$

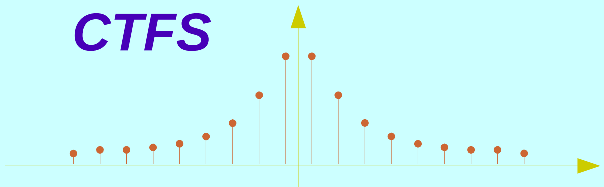
Average Power $\frac{1}{N}\sum_{k=0}^{N-1}P_{xx}[k] = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2$ Average Power

freq-domain samples | time-domain samples

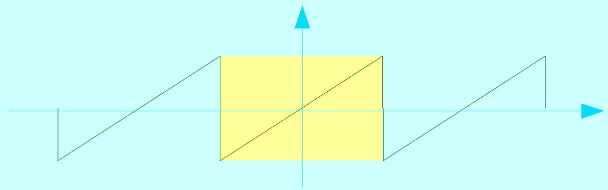
Total Energy $T_s\sum_{k=0}^{N-1}P_{xx}[k] = T_s\sum_{n=0}^{N-1}|x[n]|^2$ Total Energy

Parseval's Theorem

CTFS



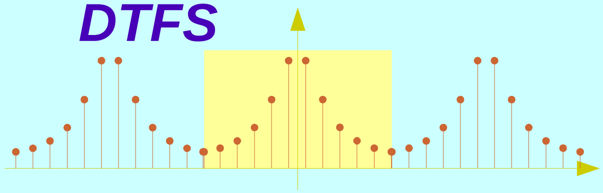
$$\sum_{k=-\infty}^{+\infty} |C_k|^2$$



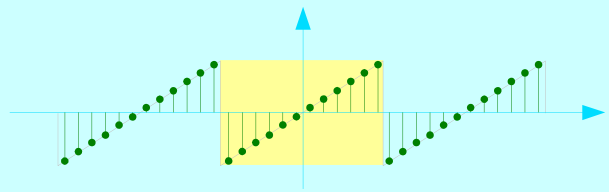
Average Power

$$\frac{1}{T}\int_T |x(t)|^2\, dt$$

DTFS



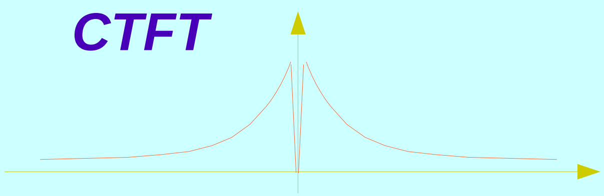
$$\sum_{k=0}^{N-1} |\gamma[k]|^2$$



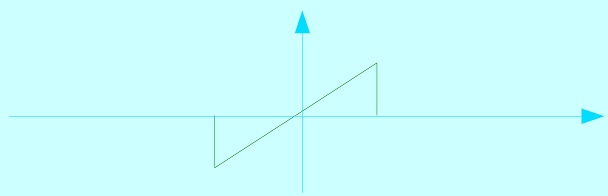
Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

CTFT



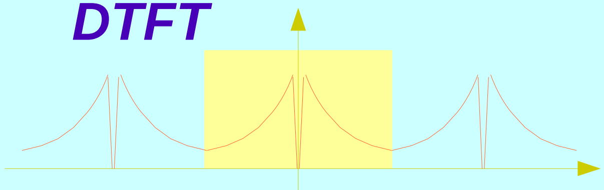
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\, d\omega$$



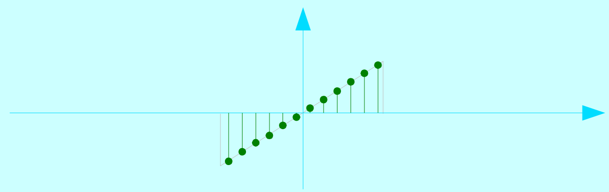
Total Energy

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt$$

DTFT



$$\frac{T_s}{2\pi}\int_{2\pi} |X(j\hat{\omega})|^2\, d\hat{\omega}$$

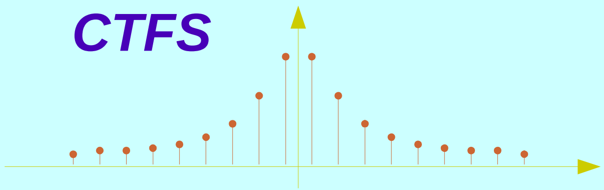


Total Energy

$$T_s \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

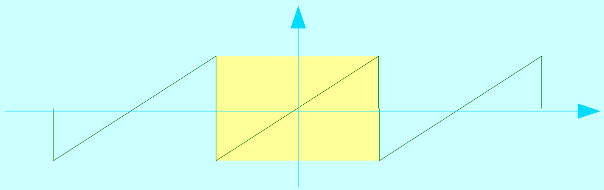
Approximate **CTFS** Parseval's Theorem

CTFS



Average Power

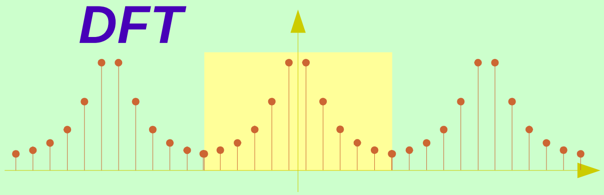
$$\sum_{k=-\infty}^{+\infty} |C_k|^2$$



Average Power

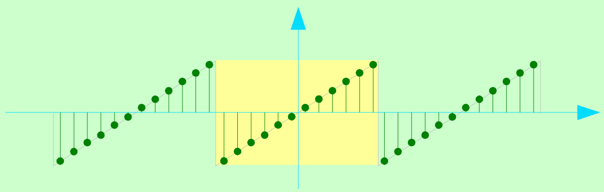
$$\frac{1}{T}\int_T |x(t)|^2 \, dt$$

DFT



Average Power

$$\frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2$$



Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

$$C_k \approx \frac{X[k]}{N}$$

$$\sum_{k=0}^{N-1} |C_k|^2 \quad \Rightarrow \quad \sum_{k=0}^{N-1} \left|\frac{X[k]}{N}\right|^2 = \frac{T_s}{T}\sum_{n=0}^{N-1} |x[n]|^2 \quad \Leftarrow \quad \frac{1}{T}\int_T |x(t)|^2 \, dt$$

1

$$C_k \approx \frac{1}{N}\, \boldsymbol{DFT}\{x(nT_s)\}$$

$$\frac{T_s}{T} = \frac{T_s}{N\,T_s}$$

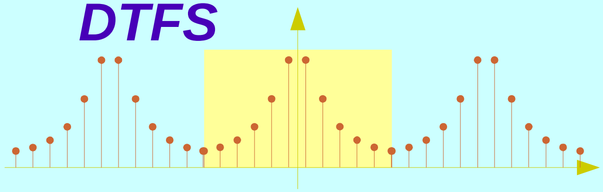
$$\frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

Average Power

$$\frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \sum_{n=0}^{N-1} |x[n]|^2$$

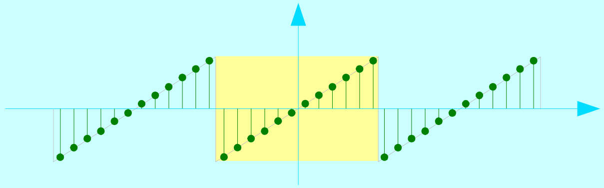
DTFS Parseval's Theorem

DTFS



Average Power

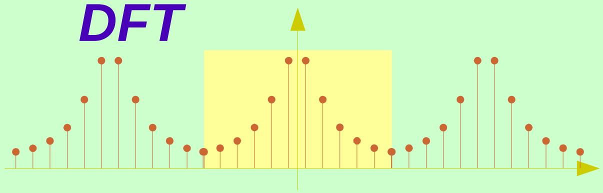
$$\sum_{k=0}^{N-1} |\gamma[k]|^2$$



Average Power

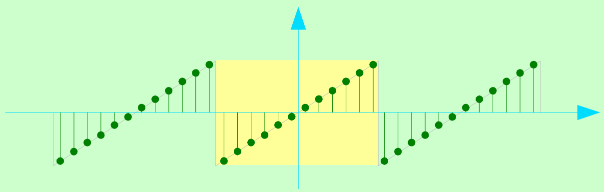
$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

DFT



Average Power

$$\frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2$$



Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

$$\gamma[k] = \frac{X[k]}{N}$$

$$\sum_{k=0}^{N-1} |\gamma[k]|^2 \Rightarrow \sum_{k=0}^{N-1} \left|\frac{X[k]}{N}\right|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 \Leftarrow \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

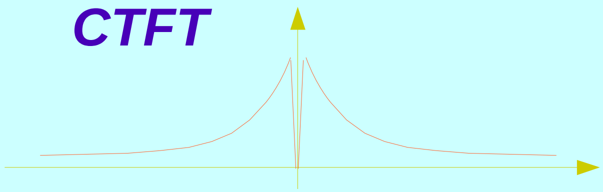
2 $$\gamma[k] = \frac{1}{N} \boldsymbol{DFT}\{x[n]\}$$

$$\frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$ Average Power

$$\frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \sum_{n=0}^{N-1} |x[n]|^2$$

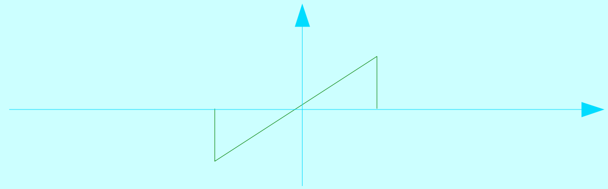
Approximate **CTFT** Parseval's Theorem

CTFT



Total Energy

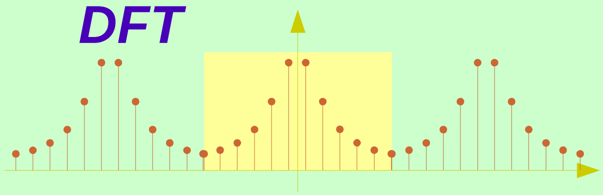
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 \, d\omega$$



Total Energy

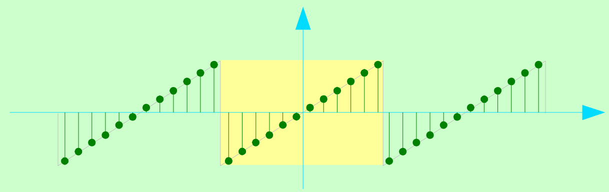
$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

DFT



Total Energy

$$\frac{T_s}{N}\sum_{k=0}^{N-1} |X[k]|^2$$



Total Energy

$$T_s \sum_{n=0}^{N-1} |x[n]|^2$$

$$X(jk\omega_0) \approx T_s X[k]$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 \, d\omega \;\Rightarrow\; \frac{1}{N T_s}\sum_{k=0}^{N-1} |T_s X[k]|^2 = T_s \sum_{n=0}^{N-1} |x[n]|^2 \;\Leftarrow\; \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

$$\frac{1}{2\pi}\omega_0 = \frac{1}{T_0}$$

3 $X(jk\omega_0) \approx T_s \, \boldsymbol{DFT}\{x(nT_s)\}$

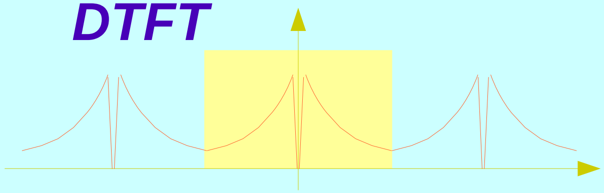
$$\frac{T_s}{N}\sum_{k=0}^{N-1} |X[k]|^2 = T_s \sum_{n=0}^{N-1} |x[n]|^2$$

Total Energy

$$\frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \sum_{n=0}^{N-1} |x[n]|^2$$

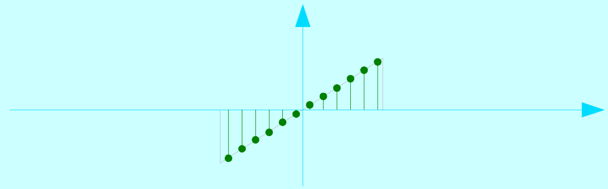
Approximate **DTFT** Parseval's Theorem

DTFT



Total Energy

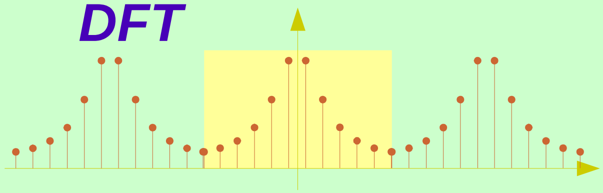
$$\frac{T_s}{2\pi}\int_{2\pi} |X(j\hat{\omega})|^2 \, d\hat{\omega}$$



Total Energy

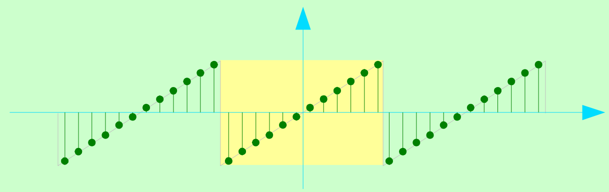
$$T_s \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

DFT



Total Energy

$$\frac{T_s}{N}\sum_{k=0}^{N-1} |X[k]|^2$$



Total Energy

$$T_s \sum_{n=0}^{N-1} |x[n]|^2$$

$$X(jk\hat{\omega}_0) \approx X[k]$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\hat{\omega})|^2 \, d\hat{\omega} \;\Rightarrow\; \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2 \;\Leftarrow\; \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

$$\frac{1}{2\pi}\hat{\omega}_0 = \frac{1}{N}$$

4 $X(jk\hat{\omega}_0) \approx \boldsymbol{DFT}\{x[n]\}$

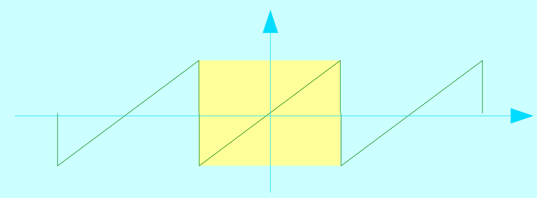
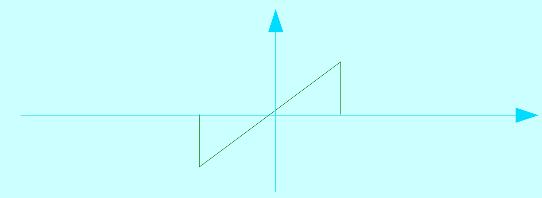
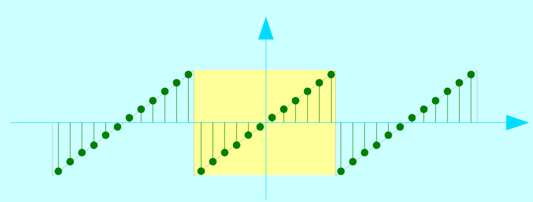
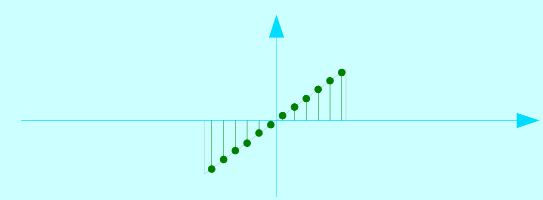
$$\frac{T_s}{N}\sum_{k=0}^{N-1} |X[k]|^2 = T_s \sum_{n=0}^{N-1} |x[n]|^2$$

Total Energy

$$\frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \sum_{n=0}^{N-1} |x[n]|^2$$

Average Power and Total Energy

Periodic Signals	Aperiodic Signals
Average Power	**Total Energy**

Continuous Time	 Average Power $\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt$	 Total Energy $\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt$
Discrete Time	 Average Power $\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2$	 Total Energy $T_s \sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2$

Parseval's Theorem

	Periodic Signals	Aperiodic Signals
	Average Power	**Total Energy**
Continuous Time	***CTFS*** Average Power $\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{+\infty} \lvert C_k\rvert^2$	***CTFT*** Total Energy $\int_T \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\omega)\rvert^2 d\omega$
Discrete Time	***DTFS*** Average Power $\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \sum_{k=0}^{N-1} \lvert \gamma[k]\rvert^2$	***DTFT*** Total Energy $T_s \sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{T_s}{2\pi}\int_{2\pi} \lvert X(j\hat{\omega})\rvert^2 d\hat{\omega}$

Average Power and Total Energy

Periodic Signals	Aperiodic Signals
Average Power	**Total Energy**

Continuous Time	***CTFS*** Average Power $\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt$	$\cdot T$ →	***CTFT*** Total Energy $\int_T \lvert x(t)\rvert^2 dt$
Discrete Time	***DTFS*** Average Power $\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2$	$\cdot T$ → $T = N \cdot T_s$	***DTFT*** Total Energy $T_s \sum_{n=0}^{N-1} \lvert x[n]\rvert^2$
	Average Power		Total Energy

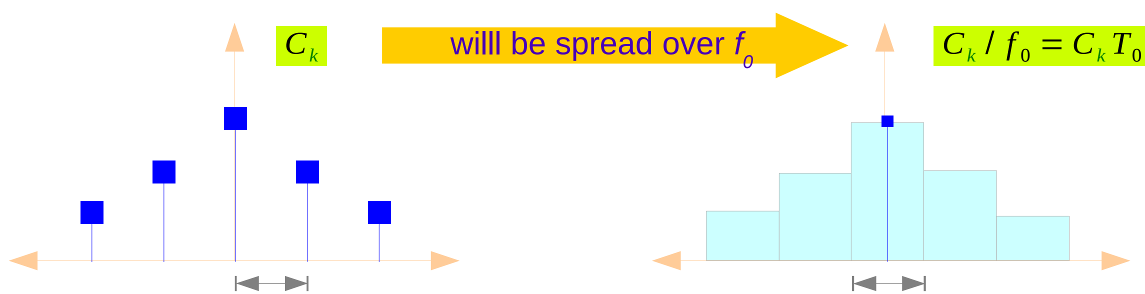
DFT Approximation

Periodic Signals	Aperiodic Signals
Average Power	Total Energy

Continuous Time	**CTFS** Average Power $$\frac{1}{N^2}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2 = \frac{1}{N}\sum_{n=0}^{N-1}\lvert x[n]\rvert^2$$	**CTFT** Total Energy $$\frac{T_s}{N}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2 = T_s\sum_{n=0}^{N-1}\lvert x[n]\rvert^2$$
Discrete Time	**DTFS** Average Power $$\frac{1}{N^2}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2 = \frac{1}{N}\sum_{n=0}^{N-1}\lvert x[n]\rvert^2$$	**DTFT** Total Energy $$\frac{T_s}{N}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2 = T_s\sum_{n=0}^{N-1}\lvert x[n]\rvert^2$$

Fourier Series Coefficients

	Periodic Signals	Aperiodic Signals
Frequency Spacing	$\omega_0 = \dfrac{2\pi}{N T_s}$	$\omega_0 = \dfrac{2\pi}{N T_s} \quad \left(\hat{\omega}_0 = \dfrac{2\pi}{N} \right)$
Two Sided F.S. Coefficient	$\dfrac{1}{N} X[k] = C_k$	$\dfrac{T_0}{N} X[k] = T_s X[k] = X(jk\omega_0)$
Frequency Bin	$k\omega_0 = k\left(\dfrac{2\pi}{N T_s}\right)$	$k\omega_0 = k\hat{\omega}_0 f_s = k\left(\dfrac{2\pi}{N T_s}\right)$



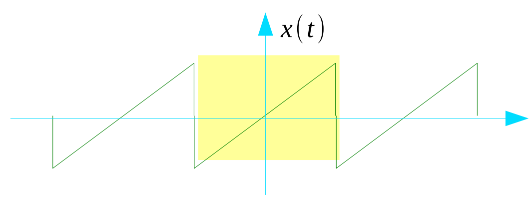
$f_0 = \dfrac{1}{T_0} = \dfrac{1}{N T_s}$ $\qquad$ $f_0 = \dfrac{1}{T_0} = \dfrac{1}{N T_s}$

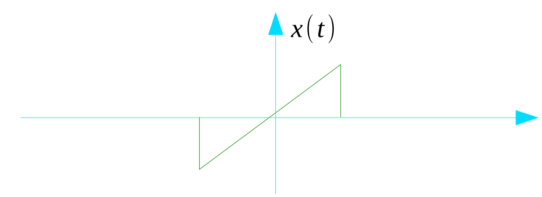
One-sided Fourier Series Coefficients

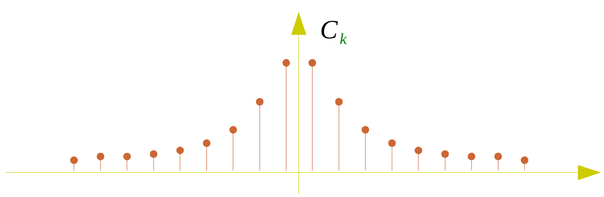
	Periodic Signals	Aperiodic Signals
Frequency Spacing	$\omega_0 = \frac{2\pi}{N T_s} = \frac{2\pi}{T_0}$	$\omega_0 = \frac{2\pi}{N T_s} \quad \left(\hat{\omega}_0 = \frac{2\pi}{N}\right)$
Two Sided F.S. Coefficient	$\frac{1}{N} X[k] = C_k$	$\frac{T_0}{N} X[k] = X(j k \omega_0)$
One Sided F.S. Coefficient	$\frac{1}{N} X[k] \quad k=0, \frac{N}{2}$ $\frac{2}{N} X[k] \quad k=1,\cdots,\frac{N}{2}-1$	$\frac{T_0}{N} X[k] \quad k=0, \frac{N}{2}$ $\frac{2T_0}{N} X[k] \quad k=1,\cdots,\frac{N}{2}-1$
Frequency Bin	$k\omega_0 = k\left(\frac{2\pi}{N T_s}\right)$	$k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{N T_s}\right)$
	Average Power	Total Energy

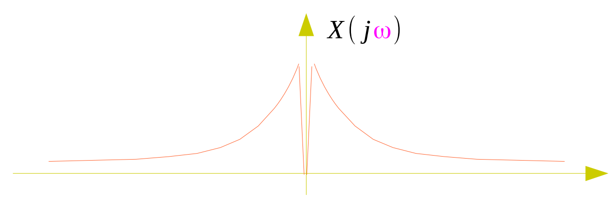
Spectral Density Functions

	Periodic Signals	Aperiodic Signals
Parseval's Theorem	$\sum_{k=0}^{N-1} \lvert C_k \rvert^2 = \frac{1}{T}\int_T \lvert x(t) \rvert^2 dt$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\omega) \rvert^2 d\omega = \int_{-\infty}^{+\infty} \lvert x(t) \rvert^2 dt$
	Average Power	Total Energy









Amplitude Spectral Density

C_k

Power Spectral Density

$$\sum_{k=-\infty}^{+\infty} \lvert C_k \rvert^2 \delta(f - k f_0)$$

Amplitude Spectral Density

$X(j\omega)$

Energy Spectral Density

$$\lvert X(f) \rvert^2$$

Using Periodograms

	Periodic Signals	**Aperiodic Signals**
Two Sided F.S. Coefficient	$\frac{1}{N} X[k] = C_k$	$\frac{T_0}{N} X[k] = X(j k \omega_0)$
Parseval's Theorem	$\sum_{k=0}^{N-1} \lvert C_k \rvert^2 = \frac{1}{T} \int_T \lvert x(t) \rvert^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\omega) \rvert^2 d\omega = \int_{-\infty}^{+\infty} \lvert x(t) \rvert^2 dt$
		$\cdot \omega_0 \left(= \frac{2\pi}{T_0} \right)$
Approximation By DFT's	$\sum_{k=0}^{N-1} \left\lvert \frac{X[k]}{N} \right\rvert^2 = \frac{T_s}{T} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$ $\frac{1}{N} \sum_{k=0}^{N-1} \left\{ \frac{1}{N} \lvert X[k] \rvert^2 \right\} = \frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$ Averaging Operation	$\frac{1}{N T_s} \sum_{k=0}^{N-1} \lvert T_s X[k] \rvert^2 = T_s \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$ $T_s \sum_{k=0}^{N-1} \left\{ \frac{1}{N} \lvert X[k] \rvert^2 \right\} = T_s \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$ Integrating Operation
Using Periodograms	$\frac{1}{N} \sum_{k=0}^{N-1} P_{xx}[k] = \frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$ Average Power	$T_s \sum_{k=0}^{N-1} P_{xx}[k] = T_s \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$ Total Energy

Using Periodograms

	Periodic Signals	Aperiodic Signals
Two Sided F.S. Coefficient	$\frac{1}{N} X[k] = C_k$	$\frac{T_0}{N} X[k] = X(jk\omega_0)$
Parseval's Theorem	$\sum_{k=0}^{N-1} \lvert C_k\rvert^2 = \frac{1}{T}\int_T \lvert x(t)\rvert^2 dt$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\omega)\rvert^2 d\omega = \int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt$
Using Periodograms	$\frac{1}{N}\sum_{k=0}^{N-1} P_{xx}[k] = \frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2$	$T_s\sum_{k=0}^{N-1} P_{xx}[k] = T_s\sum_{n=0}^{N-1} \lvert x[n]\rvert^2$
	Average Power	Total Energy
Periodograms	$P_{xx}[k] = \sum_{k=0}^{N-1} \frac{1}{N}\lvert X[k]\rvert^2$	$P_{xx}[k] = \sum_{k=0}^{N-1} \frac{1}{N}\lvert X[k]\rvert^2$
Approximated PSD & ESD	$PSD[k] = \frac{1}{N^2}\lvert X[k]\rvert^2$	$ESD[k] = \frac{T_s}{N}\lvert X[k]\rvert^2$
Average Power & Total Energy	$\sum_{k=0}^{N-1} PSD[k] = \sum_{k=0}^{N-1} \frac{1}{N^2}\lvert X[k]\rvert^2$	$\sum_{k=0}^{N-1} ESD[k] = \sum_{k=0}^{N-1} \frac{T_s}{N}\lvert X[k]\rvert^2$

Using Periodograms

	Periodic Signals	Aperiodic Signals
Two Sided F.S. Coefficient	$\frac{1}{N} X[k] = C_k$	$\frac{T_0}{N} X[k] = X(jk\omega_0)$
Approximated PSD & ESD	$\frac{\lvert X[k] \rvert^2}{N^2}$	$\frac{T_s}{N} \lvert X[k] \rvert^2$
	$\Vert$ Averaging the periodogram	$\Vert$ Integrating the periodogram
Using Periodograms	$\frac{1}{N} \sum_{k=0}^{N-1} P_{xx}[k] = \frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$T_s \sum_{k=0}^{N-1} P_{xx}[k] = T_s \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$
	Average Power	Total Energy
Approximated PSD & ESD	$\sum_{k=0}^{N-1} PSD[k] = \sum_{k=0}^{N-1} \frac{1}{N^2} \lvert X[k] \rvert^2$	$\sum_{k=0}^{N-1} ESD[k] = \sum_{k=0}^{N-1} \frac{T_s}{N} \lvert X[k] \rvert^2$

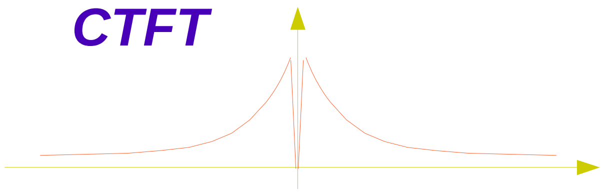
FS Coefficients of Periodic and Aperiodic Signals

Frequency Spacing	$\omega_0 = \frac{2\pi}{N T_s}$	$\omega_0 = \frac{2\pi}{N T_s} \quad \left(\hat{\omega}_0 = \frac{2\pi}{N} \right)$
Approximation of PSD & ESD	$\frac{1}{N} \lvert X[k] \rvert^2$	$\frac{T_s}{N} \lvert X[k] \rvert^2$
Frequency Bin	$k\,\omega_0 = k \left(\frac{2\pi}{N T_s} \right)$	$k\,\omega_0 = k\,\hat{\omega}_0 f_s = k \left(\frac{2\pi}{N T_s} \right)$
	Average Power (**CTFS**, **DTFS**)	Total Energy (**CTFT**, **DTFT**)
DFT Approximation	$\frac{1}{N^2} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2 = \frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$\frac{T_s}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2 = T_s \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$

FS Coefficients of Periodic and Aperiodic Signals

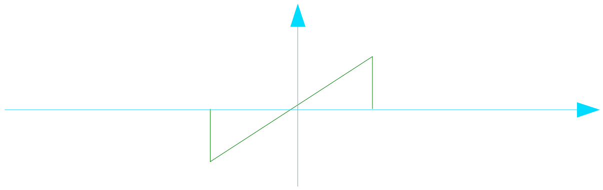
	Periodic Signals	Aperiodic Signals
Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$	$\Delta f = \frac{1}{N \Delta t}$
Two Sided F.S. Coefficient	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$
One Sided F.S. Coefficient	$\frac{1}{N} X(k) \quad k=0, \frac{N}{2}$ $\frac{2}{N} X(k) \quad k=1,\cdots,\frac{N}{2}-1$	$\frac{\Delta t}{N} X(k) \quad k=0, \frac{N}{2}$ $\frac{2\Delta t}{N} X(k) \quad k=1,\cdots,\frac{N}{2}-1$
Frequency Bin	$k \Delta f$	$k \Delta f$
	Average Power	Total Energy

CTFT



Total Energy

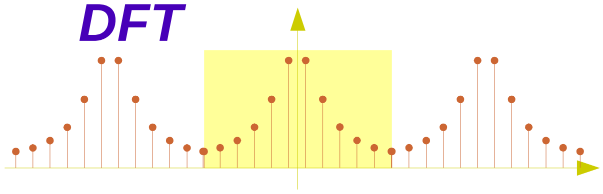
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 \, d\omega$$



Total Energy

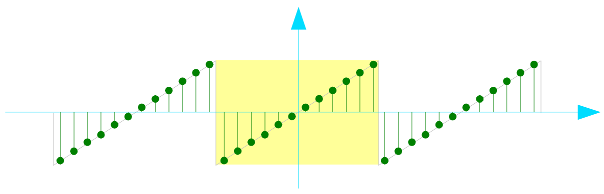
$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

DFT



Total Energy

$$\frac{T_s}{N}\sum_{k=0}^{N-1} |X[k]|^2$$



Total Energy

$$T_s \sum_{n=0}^{N-1} |x[n]|^2$$

$$X(jk\omega_0) \approx T_s X[k]$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 \, d\omega$$

$$\frac{1}{N T_s}\sum_{k=0}^{N-1} |T_s X[k]|^2 = T_s \sum_{n=0}^{N-1} |x[n]|^2$$

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

$$\frac{1}{2\pi}\omega_0 = \frac{1}{T_0}$$

3 $X(jk\omega_0) \approx T_s \, \boldsymbol{DFT}\{x(nT_s)\}$

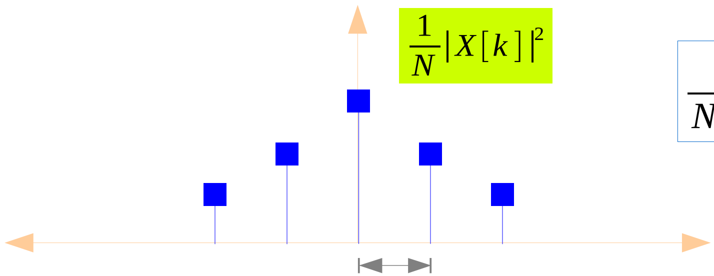
$$\frac{T_s}{N}\sum_{k=0}^{N-1} |X[k]|^2 = T_s \sum_{n=0}^{N-1} |x[n]|^2$$

Total Energy

$$\frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \sum_{n=0}^{N-1} |x[n]|^2$$

Power Spectrum and Power Spectral Density

Power Spectrum

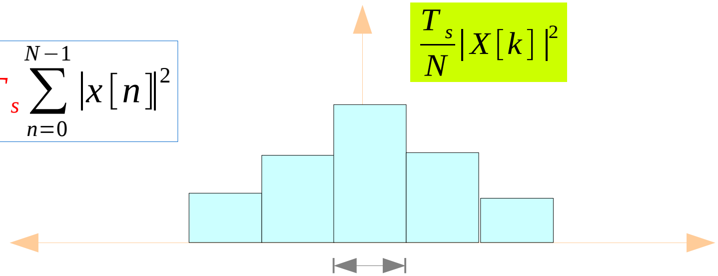


$$\Delta f = \frac{1}{N} f_s = \frac{1}{N T}$$

$$\frac{1}{N T_s} \sum_{k=0}^{N-1} |T_s X[k]|^2 = T_s \sum_{n=0}^{N-1} |x[n]|^2$$

Power Spectral Density

Hz



$$\Delta f = \frac{1}{N} f_s = \frac{1}{N T_s}$$

$$\frac{1}{N} \sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2} \sum_{k=0}^{N-1} |X[k]|^2$$

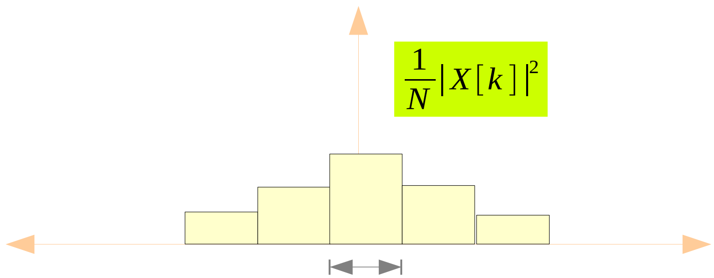
$$\Rightarrow \sum_{k=0}^{N-1} S[k] \Delta f$$

$$= \frac{1}{N T_s} \sum_{k=0}^{N-1} S[k] = \frac{1}{N^2} \sum_{k=0}^{N-1} |X[k]|^2$$

$$S[k] = \frac{T_s}{N} |X[k]|^2$$

Power Spectral Density

Hz ▪ sec



$$\frac{\Delta f}{f_s} = \frac{1}{N}$$

Normalized Frequency

Periodogram → Power Spectral Density

FS Coefficients of Random Signals

Random Signals

	Random Signals
Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$
Two Sided Power Spectral Density	$P = \sum_{k=0}^{N-1} S(k) \Delta f$
One Sided Power Spectral Density	$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$ $S_1(k) = 2S(k) \quad k = 0, \frac{N}{2}$ $S_1(k) = S(k) \quad k = 1, \cdots, \frac{N}{2} - 1$
Frequency Bin	$k \Delta f$

$$\frac{1}{N \Delta t} \sum x^2 \Delta t$$

$$\sum S \Delta f = \frac{1}{N \Delta t} \sum S$$

$$S(k) = \frac{\Delta t}{N} |X(k)|^2$$

Power Spectrum using FFT

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

X = fft(x)

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

x = ifft(X)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

CTFS

Approximated
Fourier Series Coefficients

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k\, e^{+jk\omega_0 t}$$

DTFS

fc = fft(x)/N = X/N

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$

Approximated
Fourier Coefficients

x = ifft(fc)*N

$$|C_k|^2 \approx \frac{|X[k]|^2}{N^2}$$

Approximated
Power Spectrum

Periodogram using FFT

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$ Approximated Fourier Coefficients

$$|C_k|^2 \approx \frac{|X[k]|^2}{N^2}$$ Approximated Power Spectrum

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2}\sum_{k=0}^{N-1}|X[k]|^2$$ Average Power

$$\left(\sqrt{\frac{\sum_{k=0}^{N-1}\frac{|X[k]|^2}{N}}{N}}\right)^2$$ RMS of sq root Periodogram

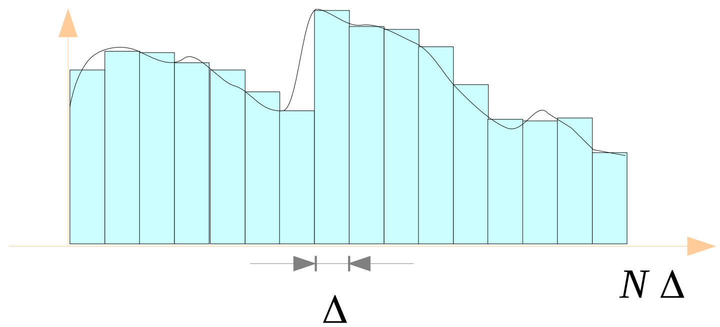
$$\frac{|X[k]|^2}{N} \quad k = 0, 1, \ldots, N-1$$ Approximated Periodogram

$$\frac{|X[k]|}{\sqrt{N}} \quad k = 0, 1, \ldots, N-1$$ Square root Periodogram

RMS in continuous time

$$\frac{1}{T}\int_0^T g^2(t)\, dt$$

RMS in discrete time



$$\frac{1}{N\,\Delta}\sum_{k=0}^{N-1}|g[k]|^2\Delta = \frac{1}{N}\sum_{k=0}^{N-1}|g[k]|^2$$

From CTFS to CTFT

Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad x_{T_0}(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{2\pi} \cdot \frac{T_0}{T_0}$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=0}^{\infty} C_k T_0\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

$$T_0 \rightarrow \infty \qquad \omega_0 = \frac{2\pi}{T_0} \rightarrow d\omega \qquad C_k T_0 \rightarrow X(j\omega) \qquad x_{T_0} \rightarrow x(t)$$

Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

CTFS and CTFT

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$x[\boldsymbol{n}] = \sum_{\boldsymbol{k}=-M}^{+M} \gamma_{\boldsymbol{k}}\, e^{+j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{n} = 0,\ 1,\ 2,\ \ldots\ ,\ N-1,$$

$$\gamma_{\boldsymbol{k}} = \frac{1}{N} \sum_{\boldsymbol{n}=0}^{N-1} x[\boldsymbol{n}]\, e^{-j\left(\frac{2\pi}{N}\right)\boldsymbol{nk}}$$

$$\boldsymbol{k} = -M,\ \ldots\ ,\ 0,\ \ldots\ ,\ +M$$

$$C_k \approx \gamma_{\boldsymbol{k}} = \frac{X[k]}{N}$$

Approximated Fourier Coefficients

Continuous Time Fourier Series

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=0}^{\infty} C_k T_0\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$T_0 \rightarrow \infty,\quad \omega_0 \rightarrow 0 \quad (\omega_0 \rightarrow d\omega)$$

Signals without discontinuity
Signals with discontinuity

Sampling frequency is not an integer
multiple of the FFT length

Leakage

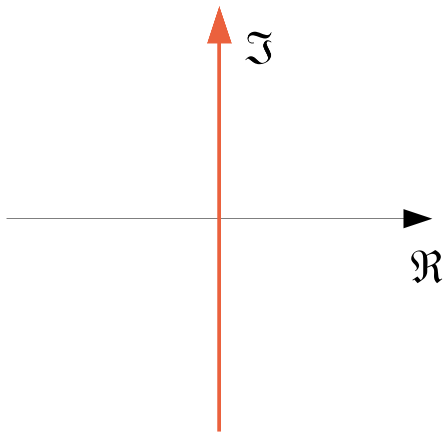
$$\left[0, \quad \frac{f_s}{2}\right]$$

Fourier Transform

$f(t)$ A continuous sum of weighted exponential functions :

$f(t)\, e^{-j\omega t}$

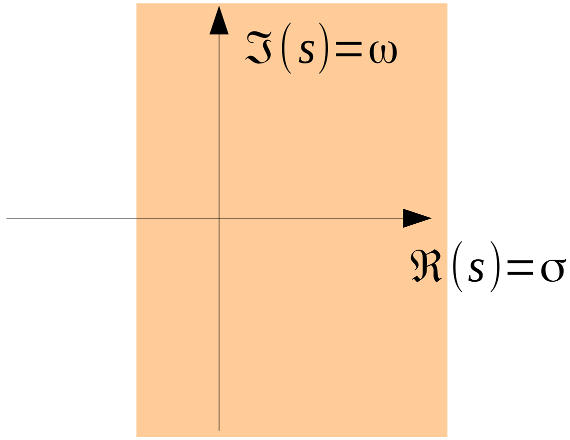
$-\infty < \omega < +\infty$



Not so useful in transient analysis

Laplace Transform

$f(t)\, e^{-st} = f(t) e^{-(\sigma + j\omega)t}$



Linear Time Domain Analysis

Initial Condition

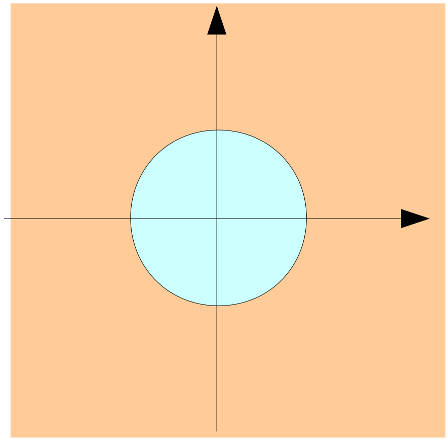
z Transform

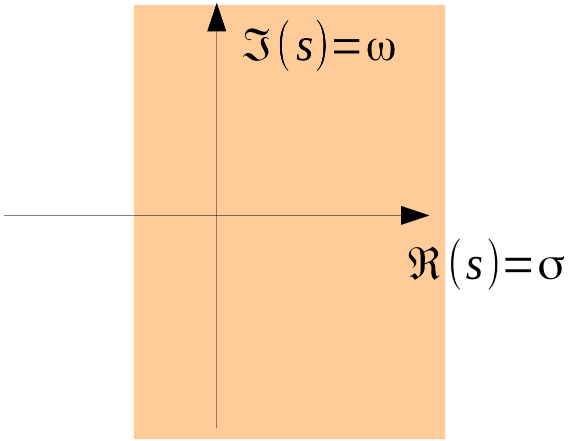
$$f[n]\, z^{-n}$$

Discrete Time System

Difference Equation

$$z = e^{sT} = e^{\sigma T}\, e^{j\omega T}$$





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