Temporal Characteristics of Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

First Order Distribution Function

For one particular time t_1, the distribution function associated with the random variable $X_1 = X(t_1)$

$$F_X(x_1; t_1) = P\{X(t_1) \leq x_1\}$$

the density function

$$f_X(x_1; t_1) = \frac{dF_X(x_1; t_1)}{dx_1}$$

Second Order Distribution Function

For one particular time t_1, t_2, the distribution function associated with the random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

the density function

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}$$

N-th Order Distribution Function

For one particular time t_1, t_2, ... ,t_N,
the distribution function
associated with the random variables
$X_1 = X(t_1)$, $X_2 = X(t_2)$, ..., $X_N = X(t_N)$

$$F_X(x_1, \ldots, x_N; t_1, \ldots, t_N) = P\{X(t_1) \leq x_1, \ldots, X(t_N) \leq x_N\}$$

the density function

$$f_X(x_1, \ldots, x_N; t_1, \ldots, t_N) = \frac{\partial^N F_X(x_1, \ldots, x_N; t_1, \ldots, t_N)}{\partial x_1 \cdots \partial x_N}$$

Statistical Independence

Two processes $X(t)$, $Y(t)$ are **statistically independent**
if the random variable group $X(t_1), X(t_2), \cdots X(t_N)$
is independent of the group $Y(t_1'), Y(t_2'), \cdots Y(t_M')$
for any choice of time $t_1, t_2, \cdots, t_N, t_1', t_2', \cdots, t_M'$
Independence requires that
the joint density be **factorable** by group

$$f_{X,Y}(x_1, \ldots, x_N, y_1, \ldots, y_M; t_1, \ldots, t_N, t_1', \ldots, t_M')$$

$$= f_X(x_1, \ldots, x_N; t_1, \ldots, t_N) f_Y(y_1, \ldots, y_M; t, \ldots, t_M')$$

The 1st order moment

The mean of a random process

$$m_X(t) = E[X(t)]$$

$$m_X(t) = \int_{-\infty}^{\infty} x f_X(x;t) dx$$

$$m_X[n] = E[X[n]]$$

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The autocorrelation function

The **correlation** of a random process
at two instants of time $X(t_1)$ and $X(t_2)$,
in general varies with t_1 and t_2

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$R_{XX}[n, n+k] = E[X[n]X[n+k]]$$

–

The autocovariance function

$$C_{XX}(t, t+\tau) = E[\{X(t) - m_X(t)\}\{X(t+\tau) - m_X(t+\tau)\}]$$
$$= R_{XX}(t, t+\tau) - m_X(t)m_X(t+\tau)$$

$$C_{XX}[n, n+k] = E[\{X[n] - m_X[n]\}\{X[n+k] - m_X[n+k]\}]$$
$$= R_{XX}[n, n+k] - m_X[n]m_X[n+k]$$

The variance of a random process

$$C_{XX}(t, t+\tau) = E\left[\{X(t) - m_X(t)\}\{X(t+\tau) - m_X(t+\tau)\}\right]$$
$$= R_{XX}(t, t+\tau) - m_X(t)m_X(t+\tau)$$

$$C_{XX}(t, t) = R_{XX}(t, t) - m_X^2(t) = \sigma_X^2(t) \qquad (\tau = 0)$$

The cross-correlation function

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$
$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

The cross-covariance function (1)

$$C_{XX}(t, t+\tau) = E\left[\{X(t) - m_X(t)\}\{X(t+\tau) - m_X(t+\tau)\}\right]$$
$$= R_{XX}(t, t+\tau) - m_X(t)m_X(t+\tau)$$

$$C_{XY}(t, t+\tau) = E\left[\{X(t) - m_X(t)\}\{Y(t+\tau) - m_Y(t+\tau)\}\right]$$
$$= R_{XY}(t, t+\tau) - m_X(t)m_Y(t+\tau)$$

The cross-covariance function (2)

$$C_{XX}[n, n+k] = E\left[\{X[n] - m_X[n]\}\{X[n+k] - m_X[n+k]\}\right]$$
$$= R_{XX}[n, n+k] - m_X[n]m_X[n+k]$$

$$C_{XY}[n, n+k] = E\left[\{X[n] - m_X[n]\}\{Y[n+k] - m_Y[n+k]\}\right]$$
$$= R_{XY}[n, n+k] - m_X[n]m_Y[n+k]$$

DT and CT relations

$$m_X[n] = m_Y(nT_s)$$

$$R_{XX}[n, n+k] = R_{YY}(nT_s, (n+k)T_s)$$

$$C_{XX}[n, n+k] = C_{YY}(nT_s, (n+k)T_s)$$

Random variables X

- the moments, the mean value, the central moments, the variance over a random variable X

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

$$\mu_X = m_1 = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$E[(X - \mu_X)^n] = \int_{-\infty}^{\infty} (x - \mu_X)^n f_X(x) dx$$

$$\sigma_X^2 = E\left[(X - \mu_X)^2\right] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$$

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf

Random Process $X(t)$

- the moments, the mean value, the central moments, the variance: over a random process $X(t)$

$$m_n(t_1) = E\left[X^n(t_1)\right] = \int_{-\infty}^{\infty} x^n(t_1) f_X(x; t_1)dx$$

$$\mu_X(t_1) = E\left[X(t_1)\right] = \int_{-\infty}^{\infty} x^1(t_1) f_X(x; t_1)dx$$

$$E\left[(X(t_1) - \mu_X(t_1))^n\right] = \int_{-\infty}^{\infty} (x(t_1) - \mu_X(t_1))^n f_X(x; t_1)dx$$

$$\sigma_X^2(t_1) = E\left[(X(t_1) - \mu_X(t_1))^2\right] = \int_{-\infty}^{\infty} (x(t_1) - \mu_X(t_1))^2 f_X(x; t_1)dx$$

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Random variables X and Y

- the covariance :

$$
\begin{aligned}
cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\
&= E[XY] - \mu_X \mu_Y
\end{aligned}
$$

- the correlation coefficients :

$$
corr(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sqrt{\sigma_X^2 \sigma_Y^2}}
$$

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Random process $X(t)$

- the auto-covariance :

$$C_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] - \mu_X(t_1)\mu_X(t_2)$$
$$= R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

- the auto-correlation :

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

- the correlation coefficients :

$$\rho_{XX}(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{\sigma_X^2(t_1)\sigma_X^2(t_2)}}$$

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Random processes $X(t)$ and $Y(t)$

- the cross-covariance :

$$\begin{aligned} C_{XY}(t_1,t_2) &= E[X(t_1)Y(t_2)] - \mu_X(t_1)\mu_Y(t_2) \\ &= R_{XY}(t_1,t_2) - \mu_X(t_1)\mu_Y(t_2) \end{aligned}$$

- the cross-correlation :

$$R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)]$$

- the correlation coefficients :

$$\rho_{XY}(t_1,t_2) = \frac{C_{XY}(t_1,t_2)}{\sqrt{\sigma_X^2(t_1)\sigma_Y^2(t_2)}}$$

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