

# Temporal Characteristics of Random Processes

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles, Jr. and B. Shi

# Outline

## 1 Joint Distributions, Independence, and Moments

# First Order Distribution Function

For one particular time  $t_1$ , the distribution function associated with the random variable  $X_1 = X(t_1)$

$$F_X(x_1; t_1) = P\{X(t_1) \leq x_1\}$$

the density function

$$f_X(x_1; t_1) = \frac{dF_X(x_1; t_1)}{dx_1}$$

# Second Order Distribution Function

For one particular time  $t_1, t_2$ , the distribution function associated with the random variables  $X_1 = X(t_1)$  and  $X_2 = X(t_2)$

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

the density function

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}$$

# $N$ -th Order Distribution Function

For one particular time  $t_1, t_2, \dots, t_N$ ,

the distribution function

associated with the random variables

$$X_1 = X(t_1), X_2 = X(t_2), \dots, X_N = X(t_N)$$

$$F_X(x_1, \dots, x_N; t_1, \dots, t_N) = P\{X(t_1) \leq x_1, \dots, X(t_N) \leq x_N\}$$

the density function

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) = \frac{\partial^N F_X(x_1, \dots, x_N; t_1, \dots, t_N)}{\partial x_1 \cdots \partial x_N}$$

# Statistical Independence

Two processes  $X(t)$ ,  $Y(t)$  are **statistically independent** if the random variable group  $X(t_1), X(t_2), \dots, X(t_N)$  is independent of the group  $Y(t'_1), Y(t'_2), \dots, Y(t'_M)$  for any choice of time  $t_1, t_2, \dots, t_N, t'_1, t'_2, \dots, t'_M$

**Independence** requires that the joint density be **factorable** by group

$$f_{X,Y}(x_1, \dots, x_N, y_1, \dots, y_M; t_1, \dots, t_N, t'_1, \dots, t'_M) \\ = f_X(x_1, \dots, x_N; t_1, \dots, t_N) f_Y(y_1, \dots, y_M; t_1, \dots, t'_M)$$

# The 1st order moment

The mean of a random process

$$m_X(t) = E[X(t)]$$

$$m_X(t) = \int_{-\infty}^{\infty} x f_X(x; t) dx$$

$$m_X[n] = E[X[n]]$$



# The autocorrelation function

The **correlation** of a random process at two instants of time  $X(t_1)$  and  $X(t_2)$ , in general varies with  $t_1$  and  $t_2$

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)]$$

$$R_{XX}[n, n + k] = E[X[n]X[n + k]]$$

# The autocovariance function

$$\begin{aligned}C_{XX}(t, t + \tau) &= E\{\{X(t) - m_X(t)\} \{X(t + \tau) - m_X(t + \tau)\}\} \\ &= R_{XX}(t, t + \tau) - m_X(t)m_X(t + \tau)\end{aligned}$$

$$\begin{aligned}C_{XX}[n, n + k] &= E\{\{X[n] - m_X[n]\} \{X[n + k] - m_X[n + k]\}\} \\ &= R_{XX}[n, n + k] - m_X[n]m_X[n + k]\end{aligned}$$

# The variance of a random process

$$\begin{aligned}C_{XX}(t, t + \tau) &= E[\{X(t) - m_X(t)\} \{X(t + \tau) - m_X(t + \tau)\}] \\ &= R_{XX}(t, t + \tau) - m_X(t)m_X(t + \tau)\end{aligned}$$

$$C_{XX}(t, t) = R_{XX}(t, t) - m_X^2(t) = \sigma_X^2(t) \quad (\tau = 0)$$

# The cross-correlation function

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)]$$

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

$$R_{XY}(t, t + \tau) = E[X(t)Y(t + \tau)]$$

## The cross-covariance function (1)

$$\begin{aligned}C_{XX}(t, t + \tau) &= E[\{X(t) - m_X(t)\} \{X(t + \tau) - m_X(t + \tau)\}] \\ &= R_{XX}(t, t + \tau) - m_X(t)m_X(t + \tau)\end{aligned}$$

$$\begin{aligned}C_{XY}(t, t + \tau) &= E[\{X(t) - m_X(t)\} \{Y(t + \tau) - m_Y(t + \tau)\}] \\ &= R_{XY}(t, t + \tau) - m_X(t)m_Y(t + \tau)\end{aligned}$$

## The cross-covariance function (2)

$$\begin{aligned}C_{XX}[n, n+k] &= E[\{X[n] - m_X[n]\} \{X[n+k] - m_X[n+k]\}] \\ &= R_{XX}[n, n+k] - m_X[n]m_X[n+k]\end{aligned}$$

$$\begin{aligned}C_{XY}[n, n+k] &= E[\{X[n] - m_X[n]\} \{Y[n+k] - m_Y[n+k]\}] \\ &= R_{XY}[n, n+k] - m_X[n]m_Y[n+k]\end{aligned}$$

## DT and CT relations

$$m_X[n] = m_Y(nT_s)$$

$$R_{XX}[n, n+k] = R_{YY}(nT_s, (n+k)T_s)$$

$$C_{XX}[n, n+k] = C_{YY}(nT_s, (n+k)T_s)$$

Random variables  $X$ 

- the moments, the mean value, the central moments, the variance over a random variable  $X$

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

$$\mu_X = m_1 = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$E[(X - \mu_X)^n] = \int_{-\infty}^{\infty} (x - \mu_X)^n f_X(x) dx$$

$$\sigma_X^2 = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$$

<https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf>



# Random Process $X(t)$

- the moments, the mean value, the central moments, the variance: over a random process  $X(t)$

$$m_n(t_1) = E[X^n(t_1)] = \int_{-\infty}^{\infty} x^n(t_1) f_X(x; t_1) dx$$

$$\mu_X(t_1) = E[X(t_1)] = \int_{-\infty}^{\infty} x^1(t_1) f_X(x; t_1) dx$$

$$E[(X(t_1) - \mu_X(t_1))^n] = \int_{-\infty}^{\infty} (x(t_1) - \mu_X(t_1))^n f_X(x; t_1) dx$$

$$\sigma_X^2(t_1) = E[(X(t_1) - \mu_X(t_1))^2] = \int_{-\infty}^{\infty} (x(t_1) - \mu_X(t_1))^2 f_X(x; t_1) dx$$

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Random variables  $X$  and  $Y$ 

- the covariance :

$$\begin{aligned} \text{cov}(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E[XY] - \mu_X \mu_Y \end{aligned}$$

- the correlation coefficients :

$$\text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sqrt{\sigma_X^2 \sigma_Y^2}}$$

<https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf>

Random process  $X(t)$ 

- the auto-covariance :

$$\begin{aligned}C_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] - \mu_X(t_1)\mu_X(t_2) \\ &= R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)\end{aligned}$$

- the auto-correlation :

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

- the correlation coefficients :

$$\rho_{XX}(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sqrt{\sigma_X^2(t_1)\sigma_X^2(t_2)}}$$

<https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf>

Random processes  $X(t)$  and  $Y(t)$ 

- the cross-covariance :

$$\begin{aligned} C_{XY}(t_1, t_2) &= E[X(t_1)Y(t_2)] - \mu_X(t_1)\mu_Y(t_2) \\ &= R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2) \end{aligned}$$

- the cross-correlation :

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

- the correlation coefficients :

$$\rho_{XY}(t_1, t_2) = \frac{C_{XY}(t_1, t_2)}{\sqrt{\sigma_X^2(t_1)\sigma_Y^2(t_2)}}$$

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