

# Laurent Series and z-Transform Examples case 0.A

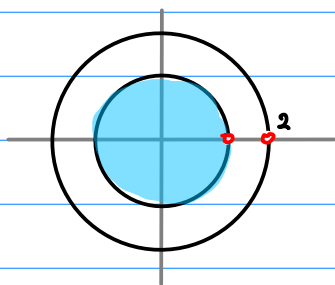
20171122

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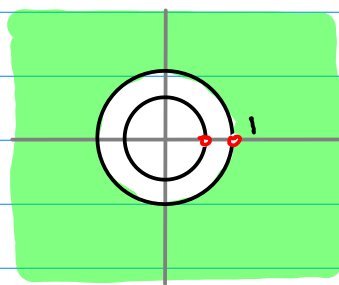
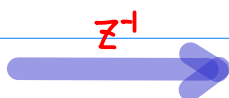
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1.A

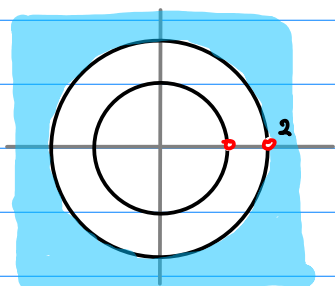
$$f(z) = \frac{-1}{(z-1)(z-2)} \xrightarrow{z^{-1}} X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$



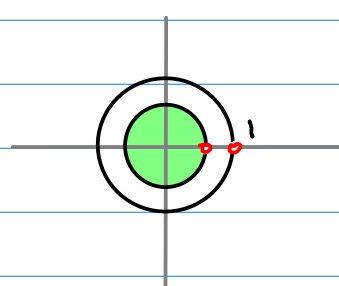
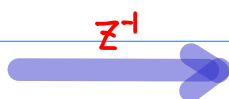
$$\sum_{n=0}^{\infty} \left[ \left(\frac{1}{2}\right)^{n+1} - 1 \right] z^n$$



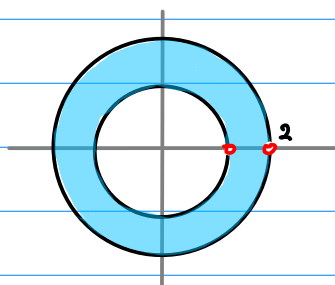
$$\sum_{n=0}^{\infty} \left[ \left(\frac{1}{2}\right)^{n+1} - 1 \right] z^{-n}$$



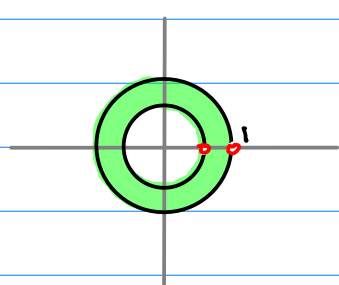
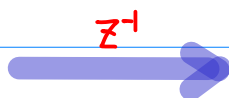
$$\sum_{n=-1}^{\infty} \left[ 1 - \left(\frac{1}{2}\right)^{n+1} \right] z^n$$



$$\sum_{n=-1}^{\infty} \left[ 1 - \left(\frac{1}{2}\right)^{n+1} \right] z^{-n}$$



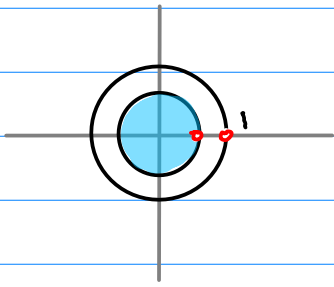
$$\sum_{n=-1}^{\infty} 1 \cdot z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \cdot z^n$$



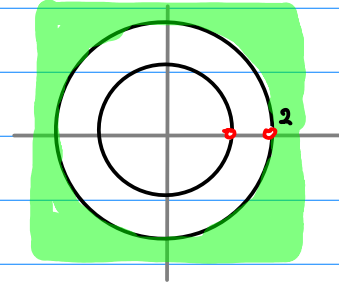
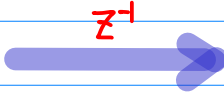
$$\sum_{n=-1}^{\infty} 1 \cdot z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} \cdot z^{-n}$$

2.A

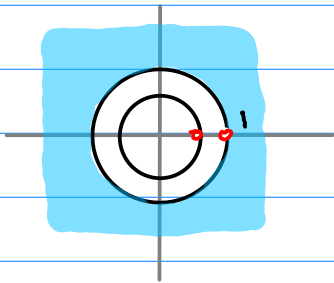
$$f(z) = \frac{-0.5 z^2}{(z-1)(z-0.5)} \xrightarrow{z^{-1}} X(z) = \frac{-1}{(z-1)(z-2)}$$



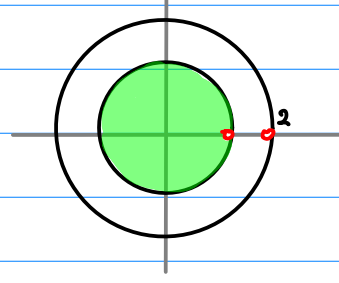
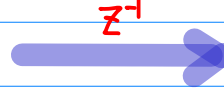
$$\sum_{n=-\infty}^{\infty} [1 - 2^{n-1}] z^n$$



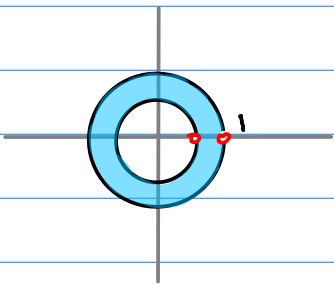
$$\sum_{n=-\infty}^{\infty} [1 - 2^{n-1}] z^{-n}$$



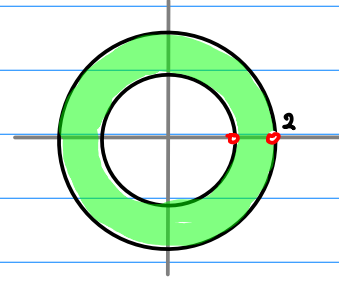
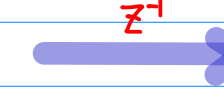
$$\sum_{n=0}^{\infty} [2^{n-1} - 1] z^n$$



$$\sum_{n=0}^{\infty} [2^{n-1} - 1] z^{-n}$$



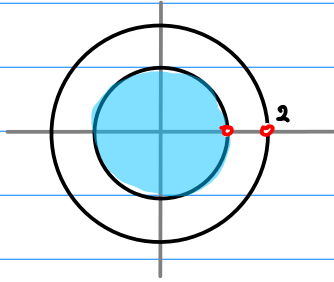
$$\sum_{n=-\infty}^{\infty} 1 \cdot z^n + \sum_{n=0}^{\infty} 2^{n-1} \cdot z^n$$



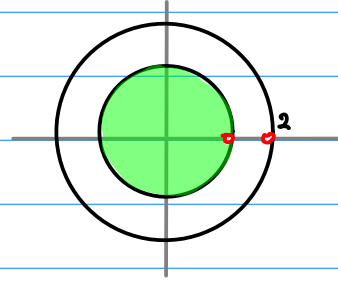
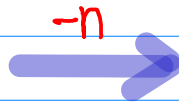
$$\sum_{n=-\infty}^{\infty} 1 \cdot z^{-n} + \sum_{n=0}^{\infty} 2^{n-1} \cdot z^{-n}$$

3. A

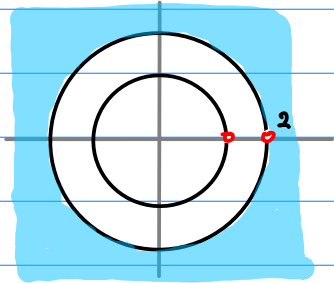
$$f(z) = \frac{-1}{(z-1)(z-2)} = X(z) = \frac{-1}{(z-1)(z-2)}$$



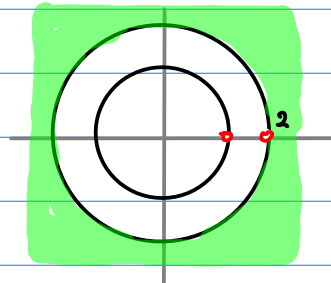
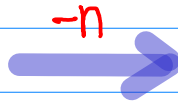
$$\sum_{n=0}^{\infty} \left[ \left(\frac{1}{2}\right)^{n+1} - 1 \right] z^n$$



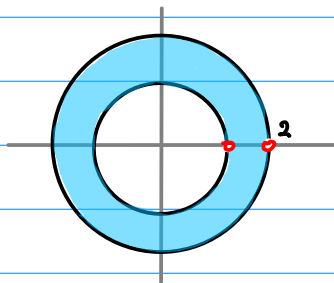
$$\sum_{n=0}^{-\infty} [2^{n-1} - 1] z^{-n}$$



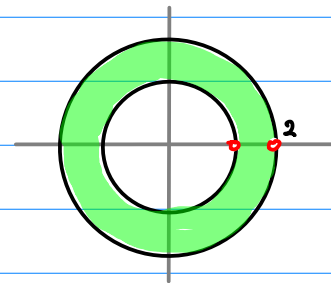
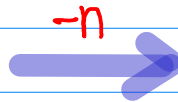
$$\sum_{n=-1}^{\infty} \left[ 1 - \left(\frac{1}{2}\right)^{n+1} \right] z^n$$



$$\sum_{n=-1}^{\infty} [1 - 2^{n-1}] z^{-n}$$



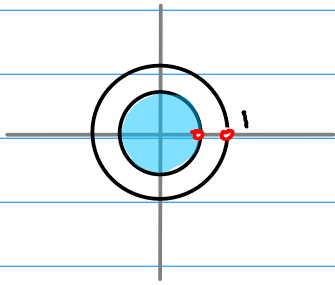
$$\sum_{n=-1}^{\infty} z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n$$



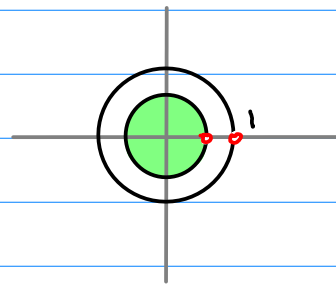
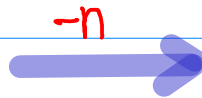
$$+ \sum_{n=-1}^{\infty} z^{-n} + \sum_{n=0}^{\infty} 2^{n-1} z^{-n}$$

4.A

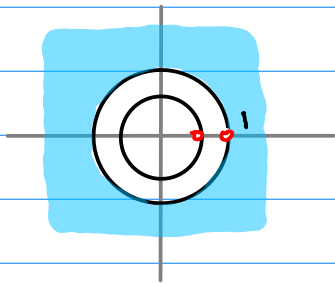
$$f(z) = \frac{-0.5 z^2}{(z-1)(z-0.5)} = X(z) = \frac{-0.5 z^2}{(z-1)(z-0.5)}$$



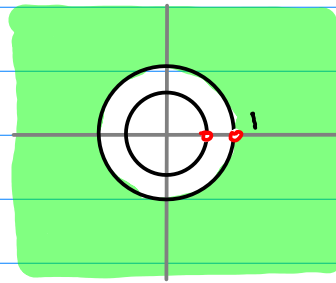
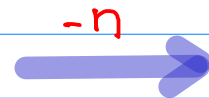
$$\sum_{n=-\infty}^{\infty} [1 - 2^{n-1}] z^n$$



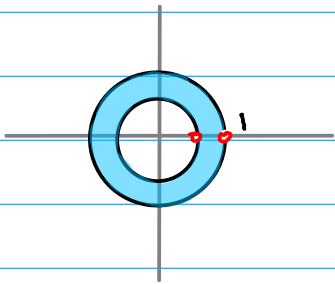
$$\sum_{n=-\infty}^{\infty} [1 - (\frac{1}{2})^{n+1}] z^{-n}$$



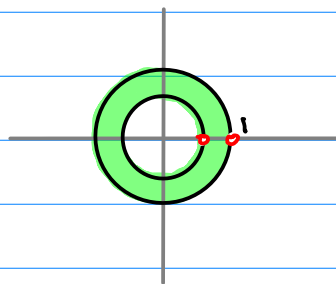
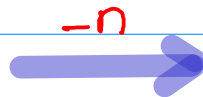
$$\sum_{n=-\infty}^{\infty} [2^{n-1} - 1] z^n$$



$$\sum_{n=-\infty}^{\infty} [(\frac{1}{2})^{n+1} - 1] z^{-n}$$



$$+\sum_{n=-\infty}^{\infty} z^n + \sum_{n=0}^{\infty} 2^{n-1} z^n$$

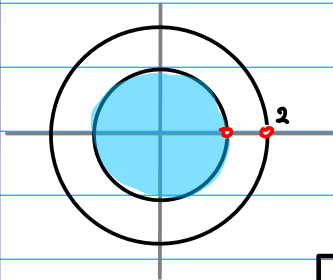


$$+\sum_{n=-\infty}^{\infty} z^{-n} + \sum_{n=0}^{\infty} (\frac{1}{2})^{n+1} z^{-n}$$

1. A

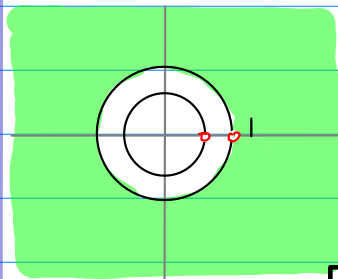
$$f(z) = \frac{-1}{(z-1)(z-2)} \xrightarrow{z^{-1}} X(z) = \frac{-0.5 z^2}{(z-1)(z-0.5)}$$

I



$$a_n = \begin{cases} (\frac{1}{2})^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

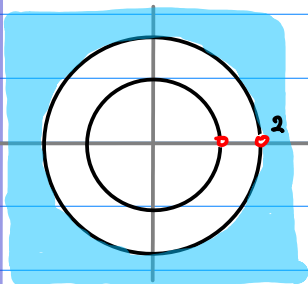
$$f(z) = \sum_{n=0}^{\infty} \left[ \left(\frac{1}{2}\right)^{n+1} - 1 \right] z^n$$



$$x_n = \begin{cases} (\frac{1}{2})^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

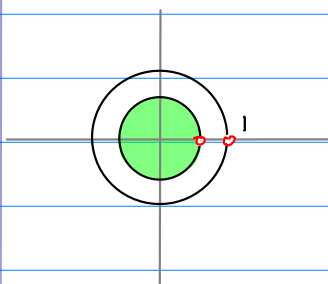
$$X(z) = \sum_{n=0}^{\infty} \left[ \left(\frac{1}{2}\right)^{n+1} - 1 \right] z^{-n}$$

II



$$a_n = \begin{cases} 0 & (n \geq 0) \\ 1 - (\frac{1}{2})^{n+1} & (n < 0) \end{cases}$$

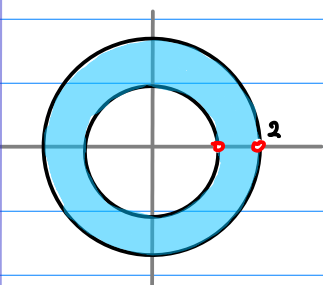
$$f(z) = \sum_{n=-1}^{-\infty} \left( 1 - \left(\frac{1}{2}\right)^{n+1} \right) z^n$$



$$x_n = \begin{cases} 0 & (n \geq 0) \\ 1 - (\frac{1}{2})^{n+1} & (n < 0) \end{cases}$$

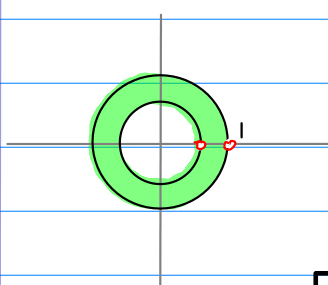
$$X(z) = \sum_{n=-1}^{-\infty} \left( 1 - \left(\frac{1}{2}\right)^{n+1} \right) z^{-n}$$

III



$$a_n = \begin{cases} (\frac{1}{2})^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{cases}$$

$$f(z) = \sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^n$$



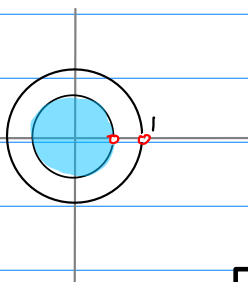
$$x_n = \begin{cases} (\frac{1}{2})^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{cases}$$

$$X(z) = \sum_{n=-1}^{-\infty} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^{-n}$$

2.A

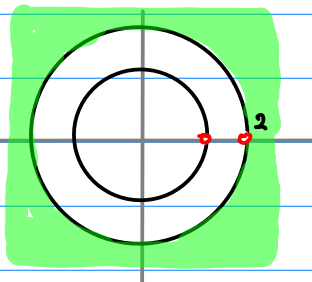
$$f(z) = \frac{-0.5 z^2}{(z-1)(z-0.5)} \xrightarrow{z^{-1}} X(z) = \frac{-1}{(z-1)(z-2)}$$

I



$$a_n = \begin{cases} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

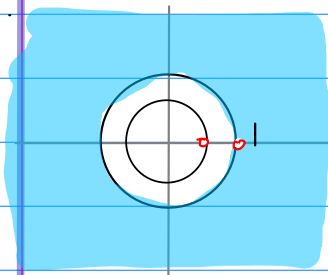
$$f(z) = \sum_{n=1}^{\infty} [1 - 2^{n-1}] z^n$$



$$x_n = \begin{cases} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

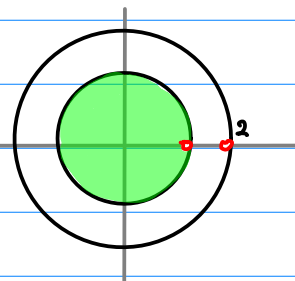
$$X(z) = \sum_{n=1}^{\infty} [1 - 2^{n-1}] z^{-n}$$

II



$$a_n = \begin{cases} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{cases}$$

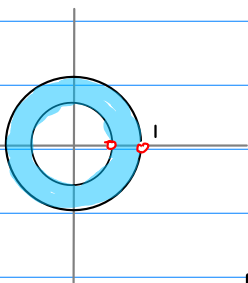
$$f(z) = \sum_{n=0}^{-\infty} [2^{n-1} - 1] z^n$$



$$x_n = \begin{cases} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{cases}$$

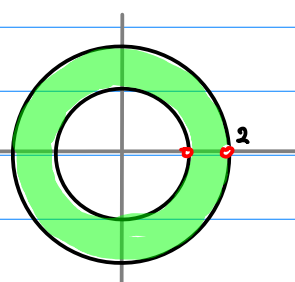
$$X(z) = \sum_{n=0}^{-\infty} [2^{n-1} - 1] z^{-n}$$

III



$$a_n = \begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

$$f(z) = \sum_{n=1}^{\infty} 1 \cdot z^n + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^n$$

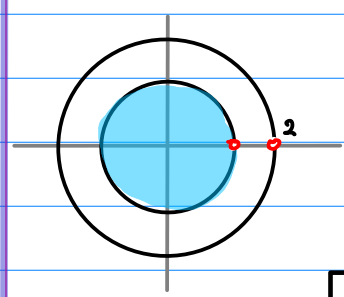


$$x_n = \begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

$$X(z) = \sum_{n=1}^{\infty} 1 \cdot z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n}$$

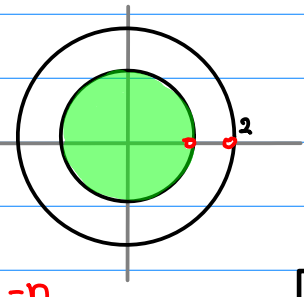
3.A  $f(z) = \frac{-1}{(z-1)(z-2)} = X(z) = \frac{-1}{(z-1)(z-2)}$

I



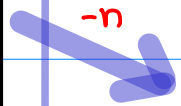
$$a_n = \begin{cases} (\frac{1}{2})^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

$$f(z) = \sum_{n=0}^{\infty} [(\frac{1}{2})^{n+1} - 1] z^n$$

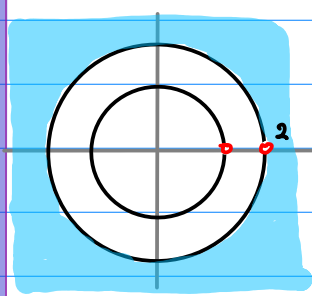


$$x_n = \begin{cases} 0 & (n > 0) \\ [2^{n-1} - 1] & (n \leq 0) \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} [2^{n-1} - 1] z^{-n}$$

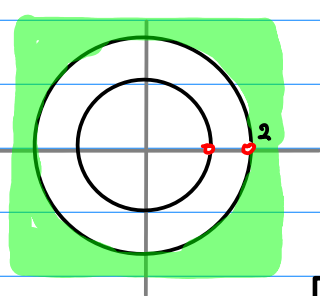


II



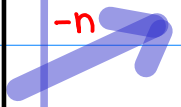
$$a_n = \begin{cases} 0 & (n \geq 0) \\ 1 - (\frac{1}{2})^{n+1} & (n < 0) \end{cases}$$

$$f(z) = \sum_{n=-1}^{\infty} [1 - (\frac{1}{2})^{n+1}] z^n$$

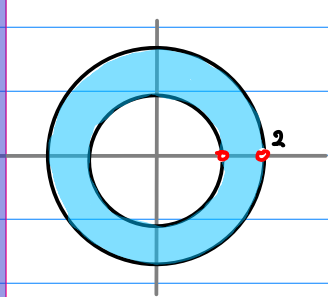


$$x_n = \begin{cases} [1 - 2^{n-1}] & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

$$X(z) = \sum_{n=1}^{\infty} [1 - 2^{n-1}] z^{-n}$$

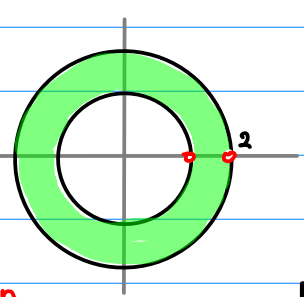


III



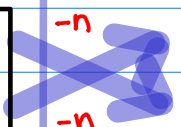
$$a_n = \begin{cases} (\frac{1}{2})^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{cases}$$

$$f(z) = \sum_{n=-1}^{\infty} z^n + \sum_{n=0}^{\infty} (\frac{1}{2})^{n+1} z^n$$



$$x_n = \begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

$$X(z) = \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty} 2^{n-1} z^{-n}$$

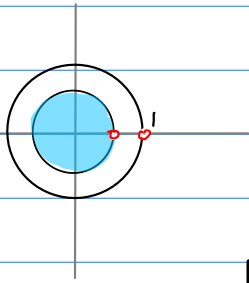




4.A

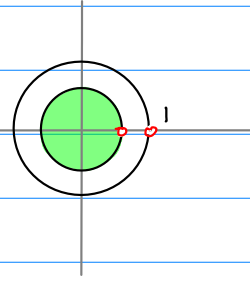
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I



$$a_n = \begin{cases} [1 - 2^{n-1}] & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

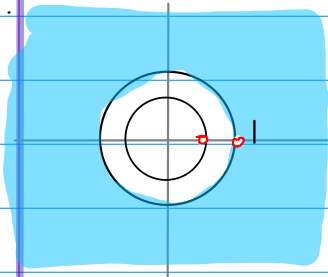
$$f(z) = \sum_{n=-\infty}^{\infty} [1 - 2^{n-1}] z^n$$



$$x_n = \begin{cases} 0 & (n \geq 0) \\ 1 - (\frac{1}{2})^{n+1} & (n < 0) \end{cases}$$

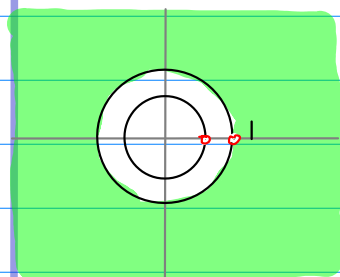
$$X(z) = \sum_{n=-\infty}^{\infty} [1 - (\frac{1}{2})^{n+1}] z^{-n}$$

II



$$a_n = \begin{cases} 0 & (n > 0) \\ [2^{n-1} - 1] & (n \leq 0) \end{cases}$$

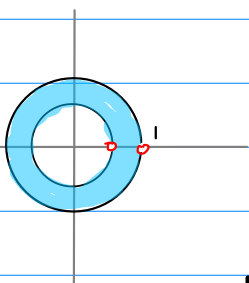
$$f(z) = \sum_{n=-\infty}^{\infty} [2^{n-1} - 1] z^n$$



$$x_n = \begin{cases} (\frac{1}{2})^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

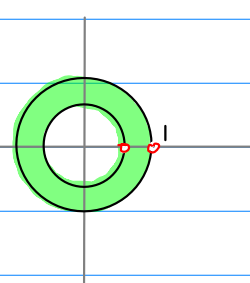
$$X(z) = \sum_{n=-\infty}^{\infty} [(\frac{1}{2})^{n+1} - 1] z^{-n}$$

III



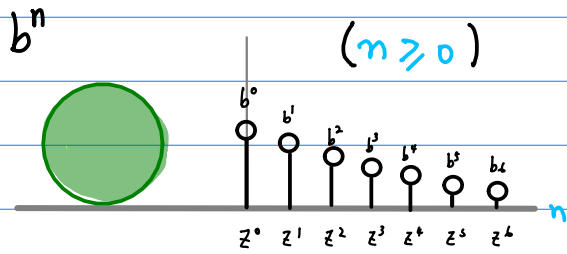
$$a_n = \begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

$$f(z) = + \sum_{n=1}^{\infty} z^n + \sum_{n=0}^{-\infty} 2^{n-1} z^n$$



$$x_n = \begin{cases} (\frac{1}{2})^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{cases}$$

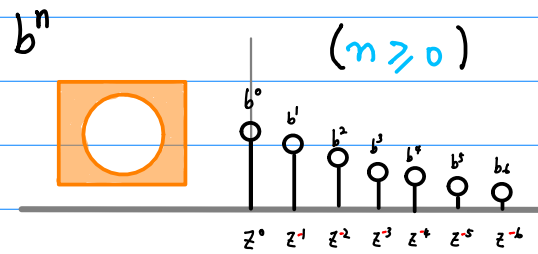
$$X(z) = + \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty} (\frac{1}{2})^{n+1} z^{-n}$$



$$X(z^{-1}) = \frac{z^{-1}}{z^{-1} - 0.5} \quad |z| < 2$$

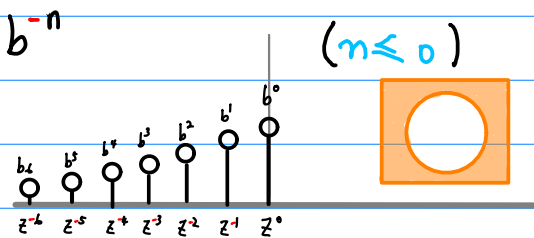
$$f(z) = \frac{z}{z - 0.5} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n$$

$$a_n = \left(\frac{1}{2}\right)^n = p^{-n} \quad p = 2$$



$$X(z) = \frac{z}{z - 0.5} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} \quad |z| > \frac{1}{2}$$

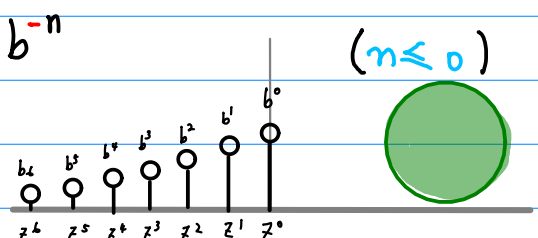
$$x_n = \left(\frac{1}{2}\right)^n = p^n \quad p = \frac{1}{2}$$



$$X(z^{-1}) = \frac{z}{z - 0.5} \quad |z| > \frac{1}{2}$$

$$f(z) = \frac{z}{z - 0.5} = \sum_{n=-\infty}^0 \left(\frac{1}{2}\right)^{-n} z^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$a_n = \left(\frac{1}{2}\right)^{-n} = p^{-n} \quad p = \frac{1}{2}$$

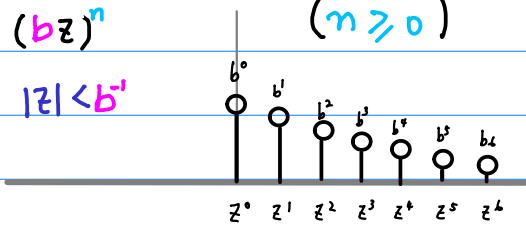


$$X(z) = \frac{z}{z - 0.5} = \sum_{n=-\infty}^0 \left(\frac{1}{2}\right)^{-n} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n \quad |z| < 2$$

$$x_n = \left(\frac{1}{2}\right)^{-n} = p^n \quad p = 2$$

$$(bz)^n \quad (n \geq 0)$$

$$|z| < b^{-1}$$

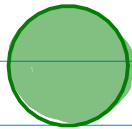


$$f(z) = \frac{1}{1-bz} = \frac{b^{-1}}{b^{-1}-z}$$

$$a_n = b^n$$

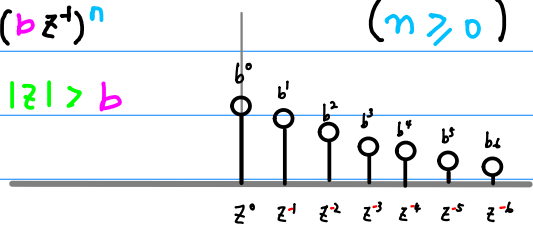
$$= p^{-n}$$

$$p = b^{-1}$$



$$(bz^{-1})^n \quad (n \geq 0)$$

$$|z| > b$$

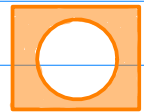


$$X(z) = \frac{1}{1-b/z} = \frac{z}{z-b}$$

$$x_n = b^n$$

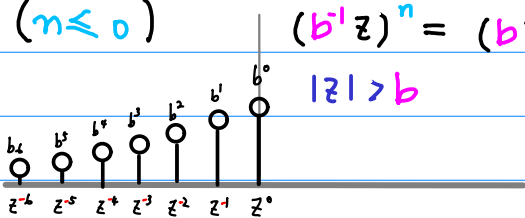
$$= p^n$$

$$p = b$$



$$(n \leq 0) \quad (b^{-1}z)^n = (bz^{-1})^{-n}$$

$$|z| > b$$

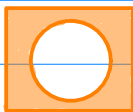


$$f(z) = \frac{1}{1-(bz^{-1})} = \frac{z}{z-b}$$

$$a_n = b^{-n}$$

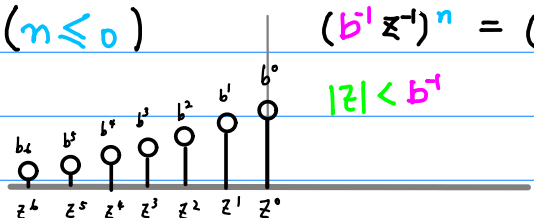
$$= p^{-n}$$

$$p = b$$



$$(n \leq 0) \quad (b^{-1}z^{-1})^n = (bz)^{-n}$$

$$|z| < b^{-1}$$

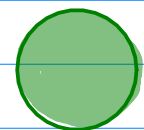


$$X(z) = \frac{1}{1-(bz)} = \frac{b^{-1}}{b^{-1}-z}$$

$$x_n = b^{-n}$$

$$= p^n$$

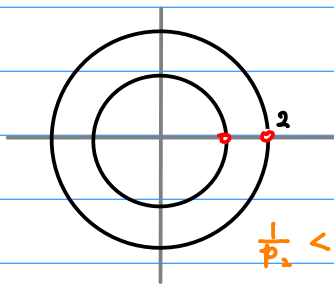
$$p = b^{-1}$$



$$f(z) \xrightarrow{z^{-1}} X(z)$$

1.A

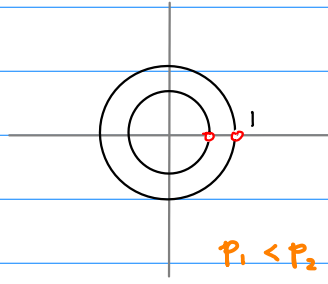
$$\frac{-1}{(z-1)(z-2)} \xrightarrow{z^{-1}} \frac{-0.5z^2}{(z-1)(z-0.5)}$$



$$\frac{1}{p_2} < \frac{1}{p_1}$$

$$\frac{1}{2} < \frac{1}{1}$$

$$\begin{matrix} p_1 = 1 \\ p_2 = 2 \end{matrix}$$



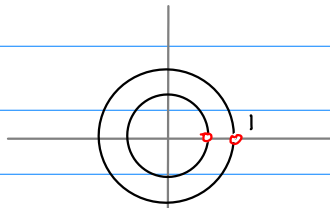
$$p_1 < p_2$$

$$0.5 < 1.0$$

$$\begin{matrix} p_1 = 0.5 \\ p_2 = 1 \end{matrix}$$

2.A

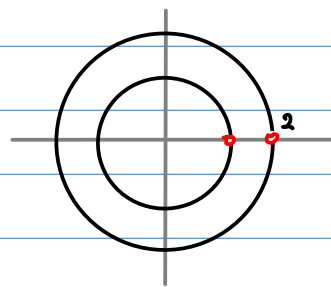
$$\frac{-0.5z^2}{(z-1)(z-0.5)} \xrightarrow{z^{-1}} \frac{-1}{(z-1)(z-2)}$$



$$\frac{1}{p_2} < \frac{1}{p_1}$$

$$\frac{1}{1} < \frac{1}{0.5}$$

$$\begin{matrix} p_1 = 0.5 \\ p_2 = 1 \end{matrix}$$



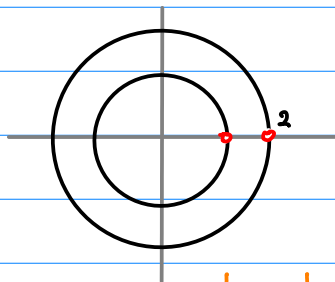
$$p_1 < p_2$$

$$1 < 2$$

$$\begin{matrix} p_1 = 1 \\ p_2 = 2 \end{matrix}$$

$$f(z) = X(z)$$

3.A  $\frac{-1}{(z-1)(z-2)} = \frac{-1}{(z-1)(z-2)}$

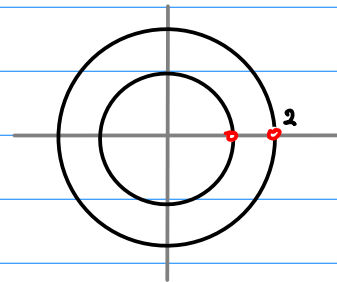


$$\frac{1}{p_2} < \frac{1}{p_1}$$

$$\frac{1}{2} < \frac{1}{1}$$

$$p_1 = 1$$

$$p_2 = 2$$



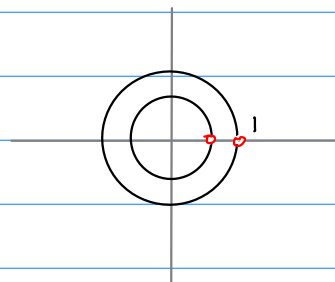
$$p_1 < p_2$$

$$1 < 2$$

$$p_1 = 1$$

$$p_2 = 2$$

4.A  $\frac{-0.5z^2}{(z-1)(z-0.5)} = \frac{-0.5z^2}{(z-1)(z-0.5)}$

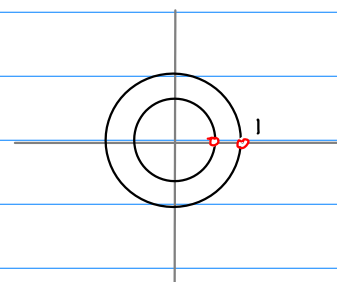


$$\frac{1}{p_2} < \frac{1}{p_1}$$

$$\frac{1}{1} < \frac{1}{0.5}$$

$$p_1 = 0.5$$

$$p_2 = 1$$



$$p_1 < p_2$$

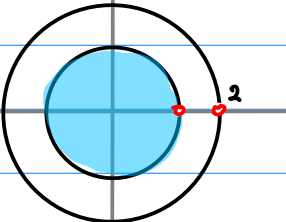
$$0.5 < 1$$

$$p_1 = 0.5$$

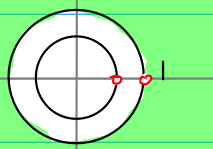
$$p_2 = 1$$

$$f(z) \xrightarrow{z^{-1}} X(z)$$

**I**

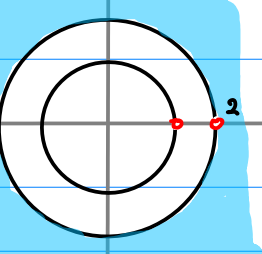


$(\frac{1}{2})^{n+1} - 1$	$(n \geq 0)$
0	$(n < 0)$

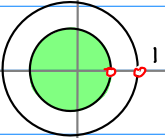


$(\frac{1}{2})^{n+1} - 1$	$(n \geq 0)$
0	$(n < 0)$

**II**

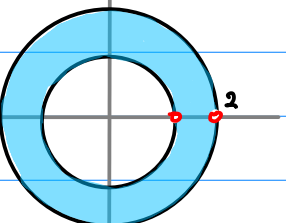


0	$(n \geq 0)$
$1 - (\frac{1}{2})^{n+1}$	$(n < 0)$

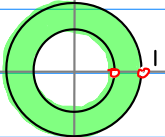


0	$(n \geq 0)$
$1 - (\frac{1}{2})^{n+1}$	$(n < 0)$

**III**

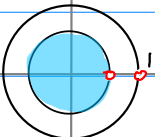


$(\frac{1}{2})^{n+1}$	$(n \geq 0)$
1	$(n < 0)$

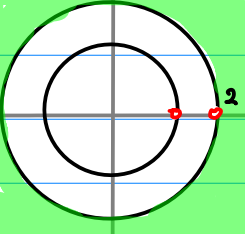


$(\frac{1}{2})^{n+1}$	$(n \geq 0)$
1	$(n < 0)$

**I**

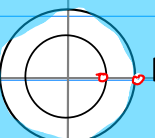


$1 - 2^{n-1}$	$(n > 0)$
0	$(n \leq 0)$

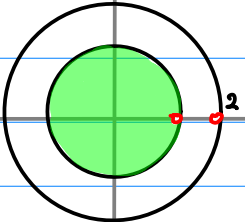


$1 - 2^{n-1}$	$(n > 0)$
0	$(n \leq 0)$

**II**

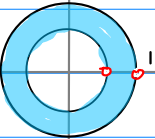


0	$(n > 0)$
$2^{n-1} - 1$	$(n \leq 0)$

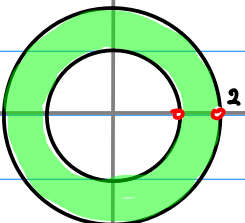


0	$(n > 0)$
$2^{n-1} - 1$	$(n \leq 0)$

**III**



1	$(n > 0)$
$2^{n-1}$	$(n \leq 0)$



1	$(n > 0)$
$2^{n-1}$	$(n \leq 0)$

$$f(z) = X(z)$$

I		$\begin{matrix} (\frac{1}{2})^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{matrix}$		$\begin{matrix} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{matrix}$
II		$\begin{matrix} 0 & (n \geq 0) \\ 1 - (\frac{1}{2})^{n+1} & (n < 0) \end{matrix}$		$\begin{matrix} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{matrix}$
III		$\begin{matrix} (\frac{1}{2})^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{matrix}$		$\begin{matrix} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{matrix}$
I		$\begin{matrix} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{matrix}$		$\begin{matrix} 0 & (n \geq 0) \\ 1 - (\frac{1}{2})^{n+1} & (n < 0) \end{matrix}$
II		$\begin{matrix} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{matrix}$		$\begin{matrix} (\frac{1}{2})^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{matrix}$
III		$\begin{matrix} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{matrix}$		$\begin{matrix} (\frac{1}{2})^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{matrix}$

$f(z) \xrightarrow{z^{-1}} X(z)$

Ⓘ

$P_1 = 1$   
 $P_2 = 2$

$$\begin{cases} (\frac{1}{2})^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

$$\begin{cases} (\frac{1}{2})^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

$P_1 = 0.5$   
 $P_2 = 1$

Ⓙ

$P_1 = 1$   
 $P_2 = 2$

$$\begin{cases} 0 & (n \geq 0) \\ 1 - (\frac{1}{2})^{n+1} & (n < 0) \end{cases}$$

$$\begin{cases} 0 & (n \geq 0) \\ 1 - (\frac{1}{2})^{n+1} & (n < 0) \end{cases}$$

$P_1 = 0.5$   
 $P_2 = 1$

Ⓚ

$P_1 = 1$   
 $P_2 = 2$

$$\begin{cases} (\frac{1}{2})^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{cases}$$

$$\begin{cases} (\frac{1}{2})^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{cases}$$

$P_1 = 0.5$   
 $P_2 = 1$

Ⓛ

$P_1 = 0.5$   
 $P_2 = 1$

$$\begin{cases} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

$$\begin{cases} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

$P_1 = 1$   
 $P_2 = 2$

Ⓜ

$P_1 = 0.5$   
 $P_2 = 1$

$$\begin{cases} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{cases}$$

$$\begin{cases} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{cases}$$

$P_1 = 1$   
 $P_2 = 2$

Ⓨ

$P_1 = 0.5$   
 $P_2 = 1$

$$\begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

$$\begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

$P_1 = 1$   
 $P_2 = 2$

\* -1

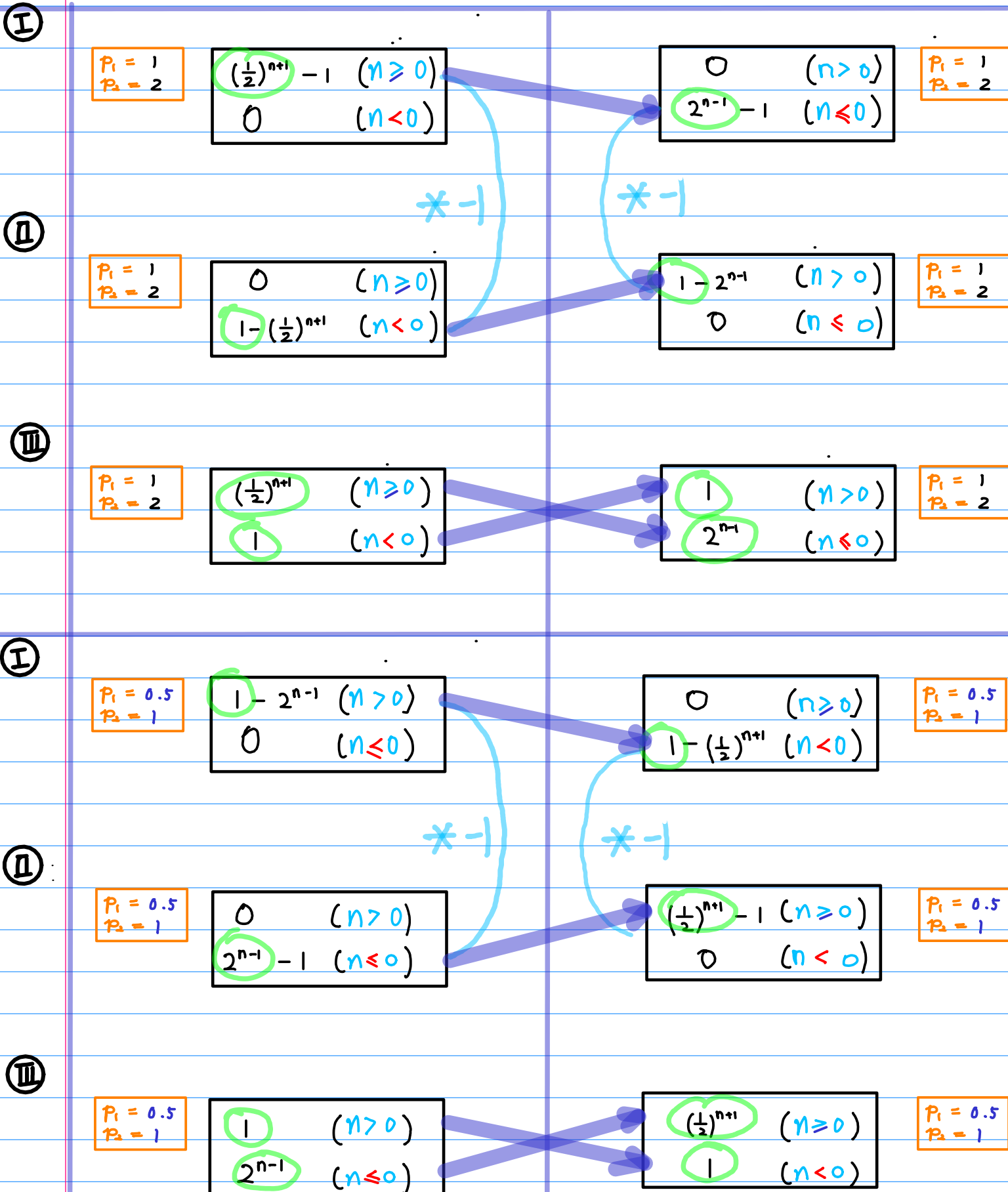
\* -1

\* -1

\* -1



$$f(z) = X(z)$$



$$\begin{array}{l} \left(\frac{1}{2}\right)^{n+1} - 1 \quad (n \geq 0) \\ 0 \quad (n < 0) \end{array}$$

$$\begin{array}{l} 0 \quad (n > 0) \\ 2^{n-1} - 1 \quad (n \leq 0) \end{array}$$

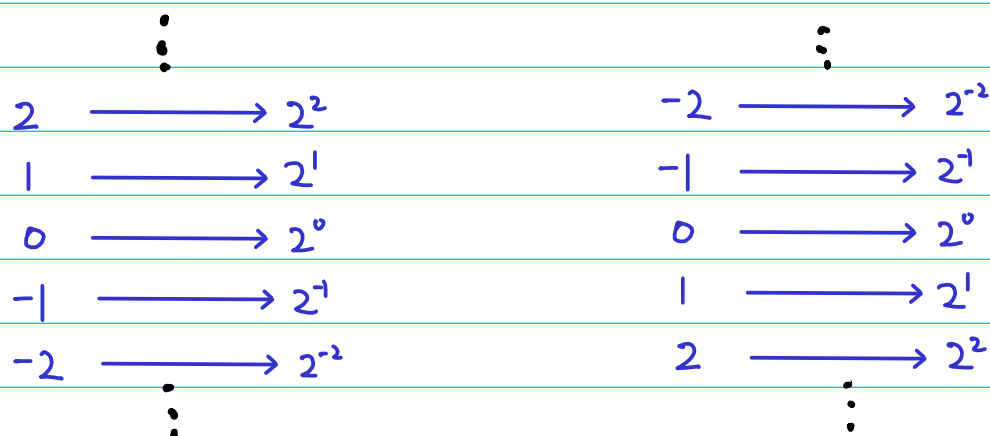
$n$	-3	-2	-1	0	1	2	3	4
$\left(\frac{1}{2}\right)^{n+1}$	$\left(\frac{1}{2}\right)^{-3+1}$	$\left(\frac{1}{2}\right)^{-2+1}$	$\left(\frac{1}{2}\right)^{-1+1}$	$\left(\frac{1}{2}\right)^{0+1}$	$\left(\frac{1}{2}\right)^{1+1}$	$\left(\frac{1}{2}\right)^{2+1}$	$\left(\frac{1}{2}\right)^{3+1}$	$\left(\frac{1}{2}\right)^{4+1}$
	$\left(\frac{1}{2}\right)^{-2}$	$\left(\frac{1}{2}\right)^{-1}$	$\left(\frac{1}{2}\right)^0$	$\left(\frac{1}{2}\right)^1$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^5$
$2^{-n-1}$	$2^2$	$2^1$	$2^0$	$2^{-1}$	$2^{-2}$	$2^{-3}$	$2^{-4}$	$2^{-5}$
$-n$	+3	+2	+1	0	-1	-2	-3	-4
$2^{-n-1}$	$2^{3-1}$	$2^{2-1}$	$2^{1-1}$	$2^{0-1}$	$2^{-1-1}$	$2^{-2-1}$	$2^{-3-1}$	$2^{-4-1}$

$n'$	$m$	-2	-1	0	+1	+2	+3
$2^{n'-1}$	$2^{m-1}$	$2^{-2-1}$	$2^{-1-1}$	$2^{0-1}$	$2^{1-1}$	$2^{2-1}$	$2^{3-1}$

$n$	-3	-2	-1	0	1	2	3
$2^n$	$2^{-3}$	$2^{-2}$	$2^{-1}$	$2^0$	$2^1$	$2^2$	$2^3$
$-n$	3	2	1	0	-1	-2	-3
$2^{-n}$	$2^3$	$2^2$	$2^1$	$2^0$	$2^{-1}$	$2^{-2}$	$2^{-3}$

rearrange

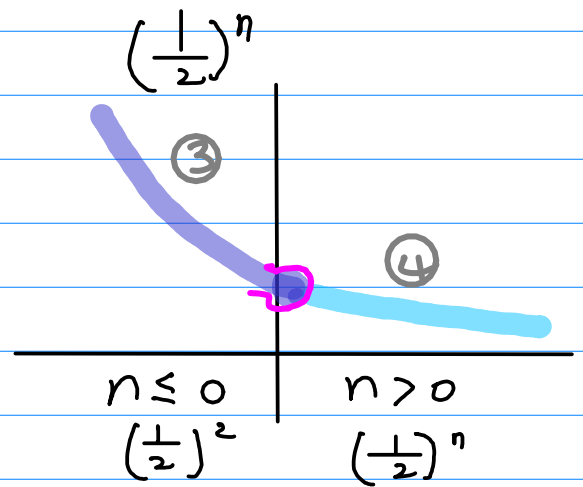
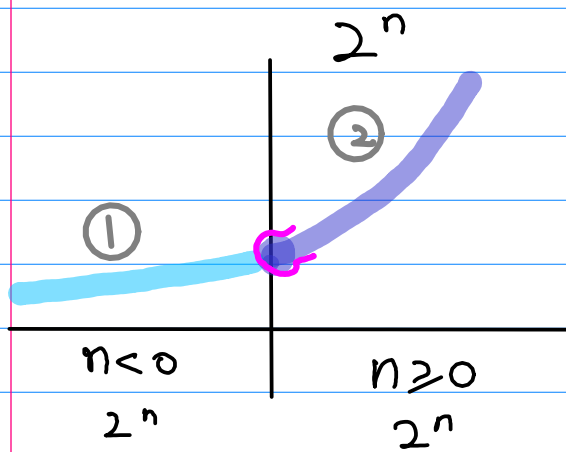
$n'$	$m$	$-n$	-3	-2	-1	0	1	2	3
$2^{n'}$	$2^m$	$2^{-n}$	$2^{-3}$	$2^{-2}$	$2^{-1}$	$2^0$	$2^1$	$2^2$	$2^3$



the same function

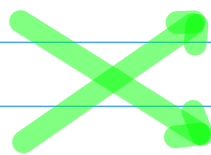
$n$	-3	-2	-1	0	1	2	3
$2^n$	$2^{-3}$	$2^{-2}$	$2^{-1}$	$2^0$	$2^1$	$2^2$	$2^3$
$n$	-3	-2	-1	0	1	2	3
$2^{-n}$	$2^3$	$2^2$	$2^1$	$2^0$	$2^{-1}$	$2^{-2}$	$2^{-3}$

different functions



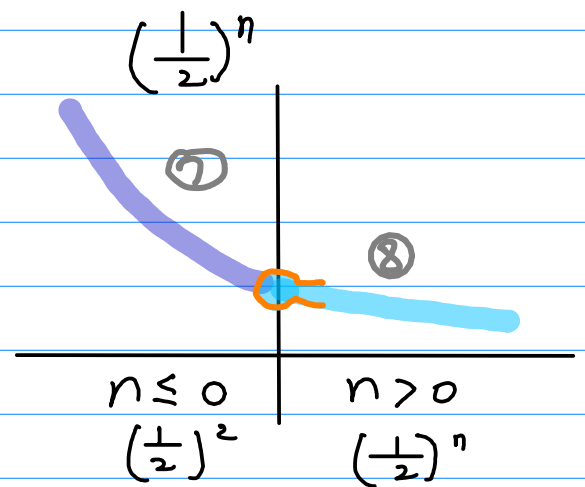
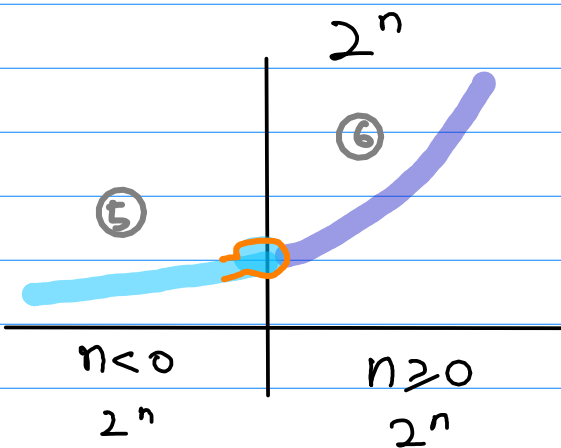
①  $2^n$  ( $n < 0$ )

②  $2^n$  ( $n > 0$ )



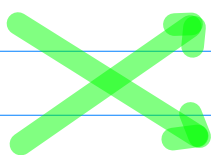
③  $(\frac{1}{2})^n$  ( $n < 0$ )

④  $(\frac{1}{2})^n$  ( $n > 0$ )



⑤  $2^n$  ( $n < 0$ )

⑥  $2^n$  ( $n > 0$ )



⑦  $(\frac{1}{2})^n$  ( $n < 0$ )

⑧  $(\frac{1}{2})^n$  ( $n > 0$ )

$$2^{n-1} \xleftrightarrow{-n} 2^{-n-1} = \left(\frac{1}{2}\right)^{n+1}$$

$$2^{n-1} \quad (n < 0) \quad \times \quad \left(\frac{1}{2}\right)^{n+1} \quad (n \geq 0)$$

$$2^{n-1} \quad (n \geq 0) \quad \times \quad \left(\frac{1}{2}\right)^{n+1} \quad (n < 0)$$

$$2^{n-1} \quad (n \leq 0) \quad \times \quad \left(\frac{1}{2}\right)^{n+1} \quad (n > 0)$$

$$2^{n-1} \quad (n > 0) \quad \times \quad \left(\frac{1}{2}\right)^{n+1} \quad (n \leq 0)$$

$f(z)$  $p_1 < p_2$  $p_1 \leq p_2 \leq 1$  $0.5 \quad 1.0$ 

$$\begin{cases} \left(\frac{1}{p_2}\right)^{n-1} - \left(\frac{1}{p_1}\right)^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

 $p_1 = 0.5 \quad p_2 = 1.0$  $1 \leq p_1 \leq p_2$  $1 \quad 2$ 

$$\begin{cases} \left(\frac{1}{p_2}\right)^{n+1} - \left(\frac{1}{p_1}\right)^{n+1} & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

 $p_1 = 1 \quad p_2 = 2$  $|z| < p_1$ 

$$\begin{cases} 0 & (n > 0) \\ \left(\frac{1}{p_1}\right)^{n-1} - \left(\frac{1}{p_2}\right)^{n-1} & (n \leq 0) \end{cases}$$

 $p_1 = 0.5 \quad p_2 = 1.0$ 

$$\begin{cases} 0 & (n \geq 0) \\ \left(\frac{1}{p_1}\right)^{n+1} - \left(\frac{1}{p_2}\right)^{n+1} & (n < 0) \end{cases}$$

 $p_1 = 1 \quad p_2 = 2$  $|z| > p_2$ 

$$\begin{cases} \left(\frac{1}{p_2}\right)^{n-1} & (n > 0) \\ \left(\frac{1}{p_1}\right)^{n-1} & (n \leq 0) \end{cases}$$

 $p_1 = 0.5 \quad p_2 = 1.0$ 

$$\begin{cases} \left(\frac{1}{p_2}\right)^{n+1} & (n \geq 0) \\ \left(\frac{1}{p_1}\right)^{n+1} & (n < 0) \end{cases}$$

 $p_1 = 1 \quad p_2 = 2$  $p_1 < |z| < p_2$

$X(z)$  $p_1 < p_2$ 

$$p_1 \leq p_2 \leq 1$$

0.5   1.0

$$1 \leq p_1 \leq p_2$$

1   2

$$\begin{array}{l} 0 \\ (p_1)^{n+1} - (p_2)^{n+1} \\ p_1 = 0.5 \quad p_2 = 1.0 \end{array} \quad \begin{array}{l} (n \geq 0) \\ (n < 0) \end{array}$$

$$\begin{array}{l} 0 \\ (p_1)^{n-1} - (p_2)^{n-1} \\ p_1 = 1 \quad p_2 = 2 \end{array} \quad \begin{array}{l} (n > 0) \\ (n \leq 0) \end{array}$$

$|z| < p_1$

$$\begin{array}{l} (p_2)^{n+1} - (p_1)^{n+1} \\ 0 \\ p_1 = 0.5 \quad p_2 = 1.0 \end{array} \quad \begin{array}{l} (n \geq 0) \\ (n < 0) \end{array}$$

$$\begin{array}{l} (p_2)^{n-1} - (p_1)^{n-1} \\ 0 \\ p_1 = 1 \quad p_2 = 2 \end{array} \quad \begin{array}{l} (n > 0) \\ (n \leq 0) \end{array}$$

$|z| > p_2$

$$\begin{array}{l} (p_2)^{n+1} \\ (p_1)^{n+1} \\ p_1 = 0.5 \quad p_2 = 1.0 \end{array} \quad \begin{array}{l} (n \geq 0) \\ (n < 0) \end{array}$$

$$\begin{array}{l} (p_2)^{n-1} \\ (p_1)^{n-1} \\ p_1 = 1 \quad p_2 = 2 \end{array} \quad \begin{array}{l} (n > 0) \\ (n \leq 0) \end{array}$$

$p_1 < |z| < p_2$

I

$(\frac{1}{2})^{n+1} - 1$	$(n \geq 0)$	$p_1 = 1$ $p_2 = 2$	$(\frac{1}{p_2})^{n+1} - (\frac{1}{p_1})^{n+1}$	$(n \geq 0)$
0	$(n < 0)$		0	$(n < 0)$

II

0	$(n \geq 0)$	$p_1 = 1$ $p_2 = 2$	0	$(n \geq 0)$
$1 - (\frac{1}{2})^{n+1}$	$(n < 0)$		$(\frac{1}{p_1})^{n+1} - (\frac{1}{p_2})^{n+1}$	$(n < 0)$

III

$(\frac{1}{2})^{n+1}$	$(n \geq 0)$	$p_1 = 1$ $p_2 = 2$	$(\frac{1}{p_2})^{n+1}$	$(n \geq 0)$
1	$(n < 0)$		$(\frac{1}{p_1})^{n+1}$	$(n < 0)$

I

$1 - 2^{n-1}$	$(n > 0)$	$p_1 = 0.5$ $p_2 = 1$	$(\frac{1}{p_2})^{n-1} - (\frac{1}{p_1})^{n-1}$	$(n > 0)$
0	$(n \leq 0)$		0	$(n \leq 0)$

II

0	$(n > 0)$	$p_1 = 0.5$ $p_2 = 1$	0	$(n > 0)$
$2^{n-1} - 1$	$(n \leq 0)$		$(\frac{1}{p_1})^{n-1} - (\frac{1}{p_2})^{n-1}$	$(n \leq 0)$

III

1	$(n > 0)$	$p_1 = 0.5$ $p_2 = 1$	$(\frac{1}{p_2})^{n-1}$	$(n > 0)$
$2^{n-1}$	$(n \leq 0)$		$(\frac{1}{p_1})^{n-1}$	$(n \leq 0)$



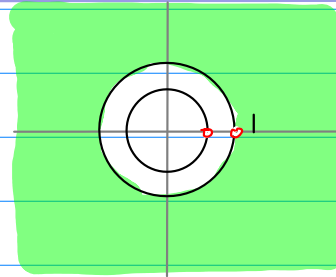
I

$$p_1 = 0.5$$

$$p_2 = 1$$

$$0$$

$$(p_1)^{n+1} - (p_2)^{n+1}$$



$$0 \quad (n > 0)$$

$$\left(\frac{1}{2}\right)^{n+1} - 1 \quad (n \leq 0)$$

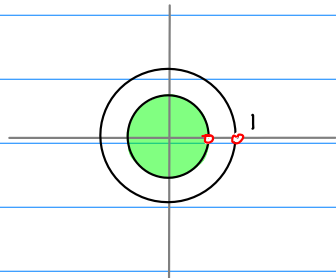
II

$$p_1 = 0.5$$

$$p_2 = 1$$

$$(p_2)^{n+1} - (p_1)^{n+1}$$

$$0$$



$$1 - \left(\frac{1}{2}\right)^{n+1} \quad (n > 0)$$

$$0 \quad (n \leq 0)$$

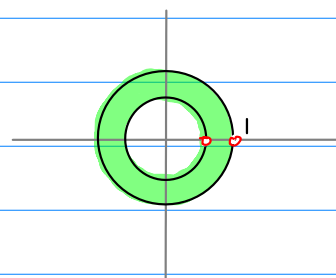
III

$$p_1 = 0.5$$

$$p_2 = 1$$

$$(p_2)^{n+1}$$

$$(p_1)^{n+1}$$



$$1 \quad (n > 0)$$

$$\left(\frac{1}{2}\right)^{n+1} \quad (n \leq 0)$$

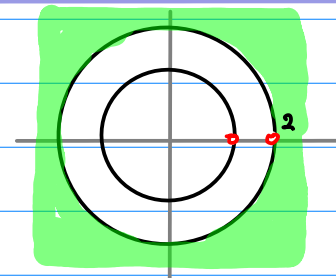
I

$$p_1 = 1$$

$$p_2 = 2$$

$$0$$

$$(p_1)^{n-1} - (p_2)^{n-1}$$



$$0 \quad (n \geq 0)$$

$$1 - 2^{n-1} \quad (n < 0)$$

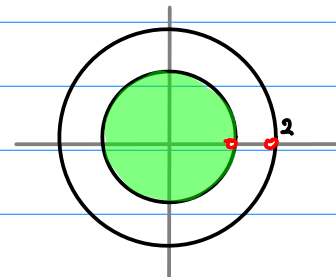
II

$$p_1 = 1$$

$$p_2 = 2$$

$$(p_2)^{n-1} - (p_1)^{n-1}$$

$$0$$



$$2^{n-1} - 1 \quad (n \geq 0)$$

$$0 \quad (n < 0)$$

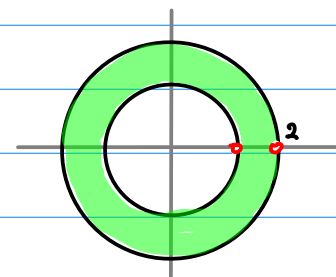
III

$$p_1 = 1$$

$$p_2 = 2$$

$$(p_2)^{n-1}$$

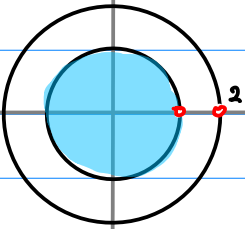
$$(p_1)^{n-1}$$



$$2^{n-1} \quad (n \geq 0)$$

$$1 \quad (n < 0)$$

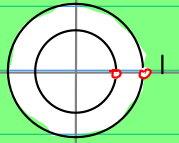
I



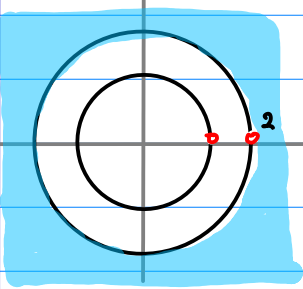
$$\begin{matrix} \left(\frac{1}{p_2}\right)^{n+1} - \left(\frac{1}{p_1}\right)^{n+1} & (n \geq 0) & (n > 0) & 0 \\ 0 & (n < 0) & (n \leq 0) & (p_1)^{n+1} - (p_2)^{n+1} \end{matrix}$$

$$\begin{matrix} p_1 = 1 \\ p_2 = 2 \end{matrix}$$

$$\begin{matrix} p_1 = 0.5 \\ p_2 = 1 \end{matrix}$$



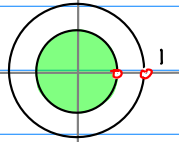
II



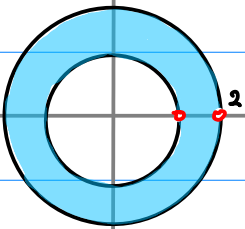
$$\begin{matrix} 0 & (n \geq 0) & (n > 0) & (p_2)^{n+1} - (p_1)^{n+1} \\ \left(\frac{1}{p_1}\right)^{n+1} - \left(\frac{1}{p_2}\right)^{n+1} & (n < 0) & (n \leq 0) & 0 \end{matrix}$$

$$\begin{matrix} p_1 = 1 \\ p_2 = 2 \end{matrix}$$

$$\begin{matrix} p_1 = 0.5 \\ p_2 = 1 \end{matrix}$$



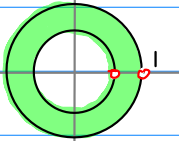
III



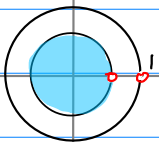
$$\begin{matrix} \left(\frac{1}{p_2}\right)^{n+1} & (n \geq 0) & (n > 0) & (p_1)^{n+1} \\ \left(\frac{1}{p_1}\right)^{n+1} & (n < 0) & (n \leq 0) & (p_2)^{n+1} \end{matrix}$$

$$\begin{matrix} p_1 = 1 \\ p_2 = 2 \end{matrix}$$

$$\begin{matrix} p_1 = 0.5 \\ p_2 = 1 \end{matrix}$$



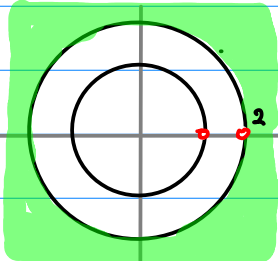
I



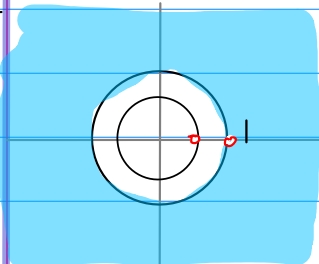
$$\begin{matrix} \left(\frac{1}{p_2}\right)^{n-1} - \left(\frac{1}{p_1}\right)^{n-1} & (n > 0) & (n \geq 0) & 0 \\ 0 & (n \leq 0) & (n < 0) & (p_1)^{n-1} - (p_2)^{n-1} \end{matrix}$$

$$\begin{matrix} p_1 = 0.5 \\ p_2 = 1 \end{matrix}$$

$$\begin{matrix} p_1 = 1 \\ p_2 = 2 \end{matrix}$$



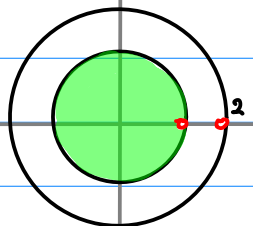
II



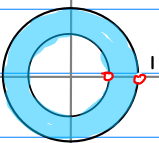
$$\begin{matrix} 0 & (n > 0) & (n \geq 0) & (p_2)^{n-1} - (p_1)^{n-1} \\ \left(\frac{1}{p_1}\right)^{n-1} - \left(\frac{1}{p_2}\right)^{n-1} & (n \leq 0) & (n < 0) & 0 \end{matrix}$$

$$\begin{matrix} p_1 = 0.5 \\ p_2 = 1 \end{matrix}$$

$$\begin{matrix} p_1 = 1 \\ p_2 = 2 \end{matrix}$$



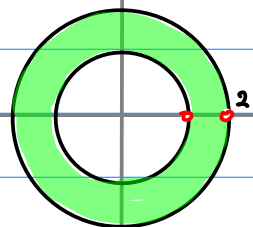
III



$$\begin{matrix} \left(\frac{1}{p_2}\right)^{n-1} & (n > 0) & (n \geq 0) & (p_2)^{n-1} \\ \left(\frac{1}{p_1}\right)^{n-1} & (n \leq 0) & (n < 0) & (p_1)^{n-1} \end{matrix}$$

$$\begin{matrix} p_1 = 0.5 \\ p_2 = 1 \end{matrix}$$

$$\begin{matrix} p_1 = 1 \\ p_2 = 2 \end{matrix}$$



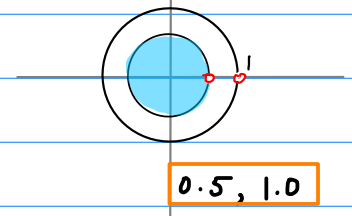
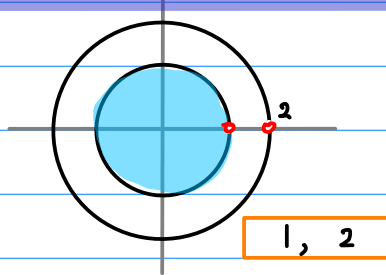
$$n > 0$$

$$n < 0$$

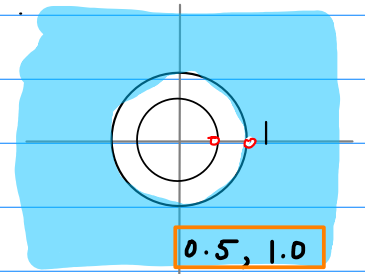
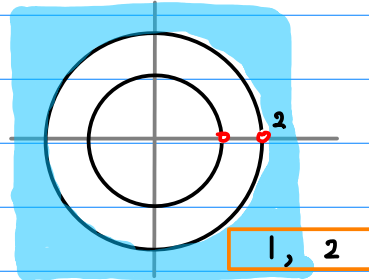
$$n > 0$$

$$n < 0$$

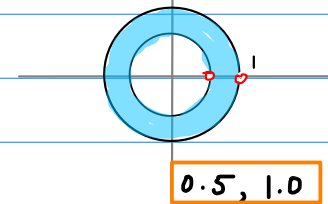
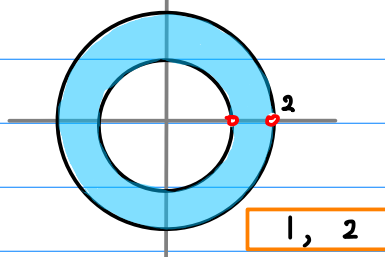
$$\left(\frac{1}{p_2}\right)^{n+1} - \left(\frac{1}{p_1}\right)^{n+1}$$
$$0$$



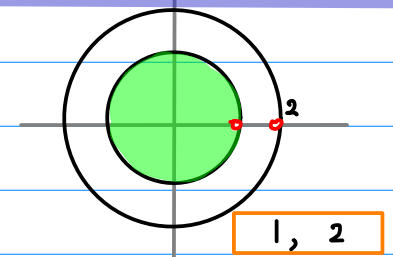
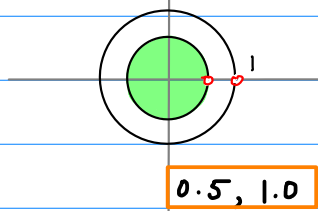
$$0$$
$$\left(\frac{1}{p_1}\right)^{n+1} - \left(\frac{1}{p_2}\right)^{n+1}$$



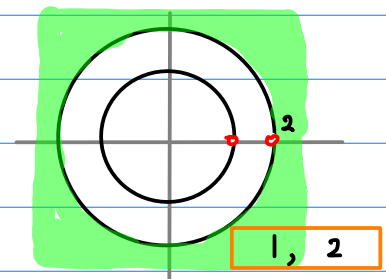
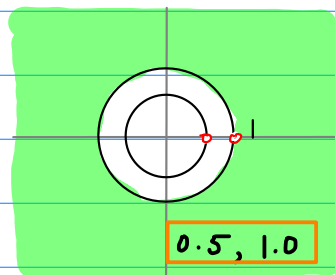
$$\left(\frac{1}{p_2}\right)^{n+1}$$
$$\left(\frac{1}{p_1}\right)^{n+1}$$



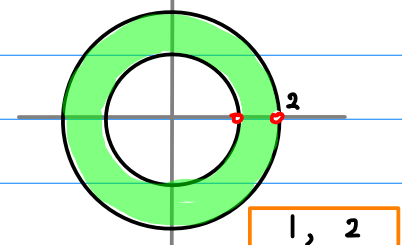
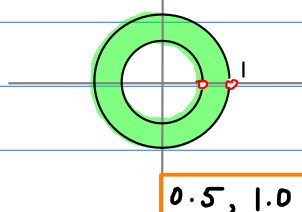
$$(p_2)^{n+1} - (p_1)^{n+1}$$
$$0$$

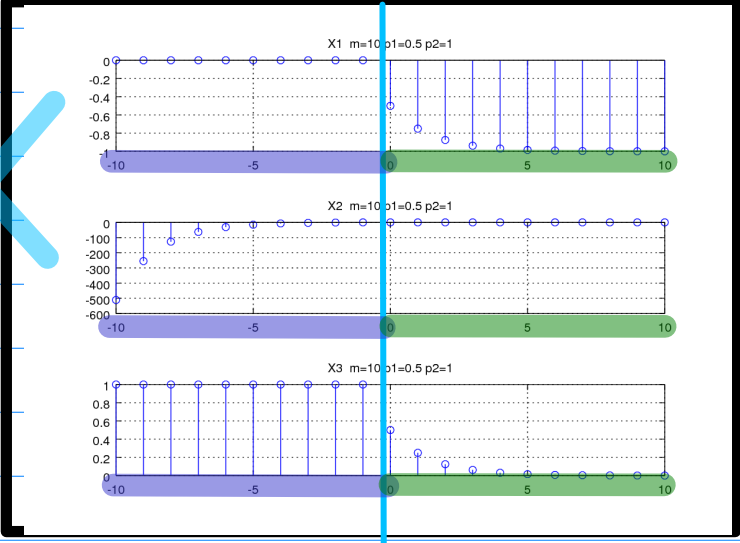
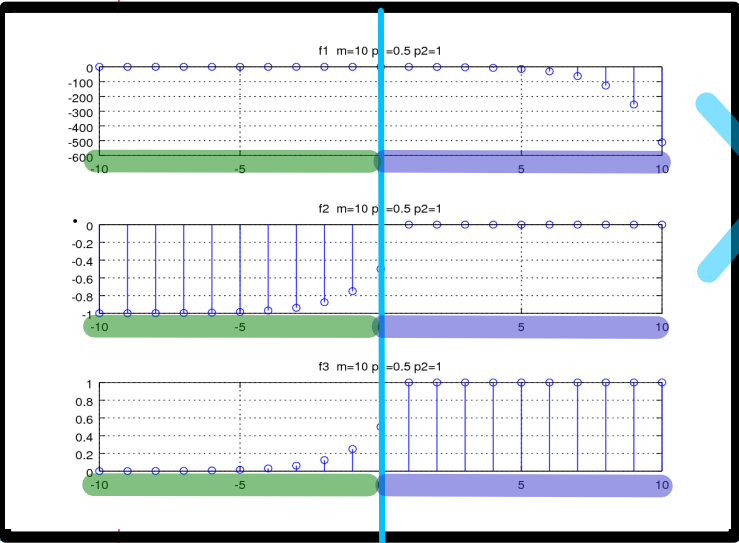
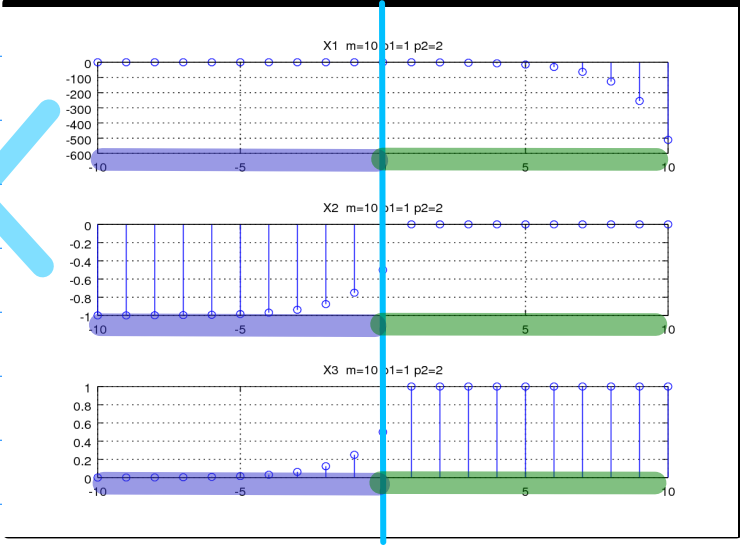
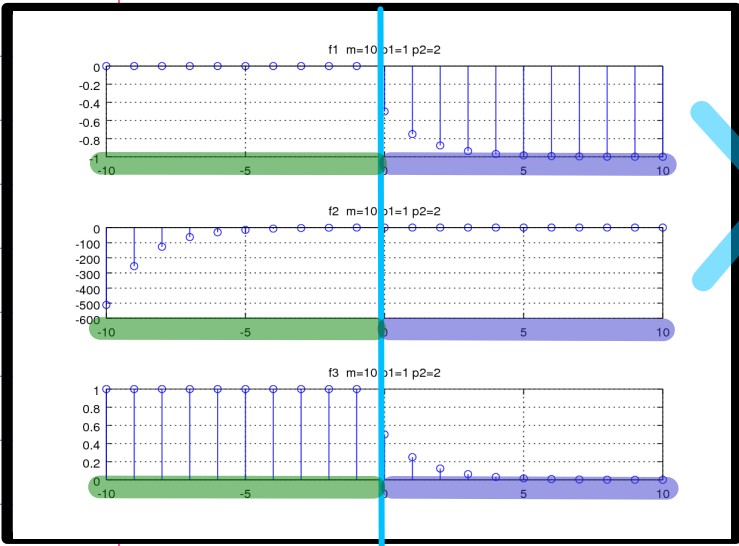


$$0$$
$$(p_1)^{n+1} - (p_2)^{n+1}$$



$$(p_2)^{n+1}$$
$$(p_1)^{n+1}$$





```

% Laurent Series and sequences
function plotseq1(m=1, p1=2, p2=2.1)

if (p1 >= 1 && p2 >= 1 && p1 < p2)
    t1n = -m: -1;
    t1p = 0: m;
    t1 = [t1n, t1p];
    f1 = [zeros(1,m), ((1/p2).^(t1p+1) - (1/p1).^(t1p+1))];
    f2 = [((1/p1).^(t1n+1) - (1/p2).^(t1n+1)), zeros(1,m+1)];
    f3 = [(1/p1).^(t1n+1), (1/p2).^(t1p+1)];
endif

if (p1 <= 1 && p2 <= 1 && p1 < p2)
    t1n = -m: 0;
    t1p = +1: m;
    t1 = [t1n, t1p];
    f1 = [zeros(1,m+1), ((1/p2).^(t1p-1) - (1/p1).^(t1p-1))];
    f2 = [((1/p1).^(t1n-1) - (1/p2).^(t1n-1)), zeros(1,m)];
    f3 = [(1/p1).^(t1n-1), (1/p2).^(t1p-1)];
endif

subplot(3, 1, 1);
stem(t1, f1);
grid on
%axis([0, m])
title(sprintf("f1 m=%d p1=%g p2=%g", m, p1, p2))

subplot(3, 1, 2);
stem(t1, f2);
grid on
%axis([0, m])
title(sprintf("f2 m=%d p1=%g p2=%g", m, p1, p2))

subplot(3, 1, 3);
stem(t1, f3);
grid on
%axis([0, m])
title(sprintf("f3 m=%d p1=%g p2=%g", m, p1, p2))

endfunction

```

```

% z-Transform and sequences
function plotseq2(m=1, p1=2, p2=2.1)

cla;

if (p1 <= 1 && p2 <= 1 && p1 < p2)
    t1n = -m: -1;
    t1p = 0: m;
    t1 = [t1n, t1p];
    X1 = [zeros(1,m), ((p1).^(t1p+1) - (p2).^(t1p+1))];
    X2 = [((p2).^(t1n+1) - (p1).^(t1n+1)), zeros(1,m+1)];
    X3 = [(p2).^(t1n+1), (p1).^(t1p+1)];
endif

if (p1 >= 1 && p2 >= 1 && p1 < p2)
    t1n = -m: 0;
    t1p = +1: m;
    t1 = [t1n, t1p];
    X1 = [zeros(1,m+1), ((p1).^(t1p-1) - (p2).^(t1p-1))];
    X2 = [((p2).^(t1n-1) - (p1).^(t1n-1)), zeros(1,m)];
    X3 = [(p2).^(t1n-1), (p1).^(t1p-1)];
endif

subplot(3, 1, 1);
stem(t1, X1);
grid on
%axis([0, m])
title(sprintf("X1 m=%d p1=%g p2=%g", m, p1, p2))

subplot(3, 1, 2);
stem(t1, X2);
grid on
%axis([0, m])
title(sprintf("X2 m=%d p1=%g p2=%g", m, p1, p2))

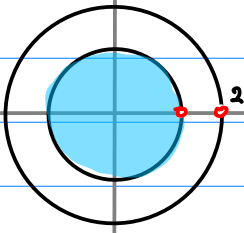
subplot(3, 1, 3);
stem(t1, X3);
grid on
%axis([0, m])
title(sprintf("X3 m=%d p1=%g p2=%g", m, p1, p2))

endfunction

```

$$p = 1 \quad q = 2$$

I



$$\left(\frac{1}{q}\right)^{n+1} - \left(\frac{1}{p}\right)^{n+1} \quad (n \geq 0) \quad (n > 0) \quad 0$$

$$0 \quad (n < 0) \quad (n \leq 0) \quad \left(\frac{1}{q}\right)^{n+1} - \left(\frac{1}{p}\right)^{n+1}$$

$$p_1 = 1 = p$$

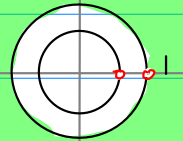
$$p_2 = 2 = q$$

$$0$$

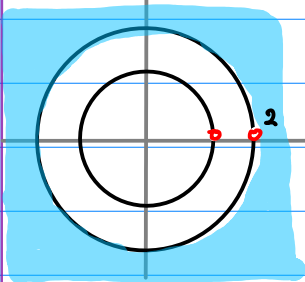
$$\left(\frac{1}{q}\right)^{n+1} - \left(\frac{1}{p}\right)^{n+1}$$

$$p_1 = 0.5 = \frac{1}{q}$$

$$p_2 = 1 = \frac{1}{p}$$



II



$$0 \quad (n \geq 0) \quad (n > 0)$$

$$\left(\frac{1}{p}\right)^{n+1} - \left(\frac{1}{q}\right)^{n+1} \quad (n < 0) \quad (n \leq 0) \quad 0$$

$$p_1 = 1 = p$$

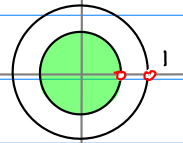
$$p_2 = 2 = q$$

$$\left(\frac{1}{p}\right)^{n+1} - \left(\frac{1}{q}\right)^{n+1}$$

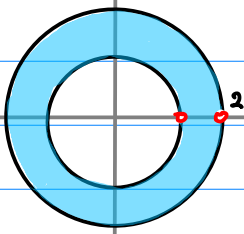
$$0$$

$$p_1 = 0.5 = \frac{1}{q}$$

$$p_2 = 1 = \frac{1}{p}$$



III



$$\left(\frac{1}{q}\right)^{n+1} \quad (n \geq 0) \quad (n > 0) \quad \left(\frac{1}{q}\right)^{n+1}$$

$$\left(\frac{1}{p}\right)^{n+1} \quad (n < 0) \quad (n \leq 0) \quad \left(\frac{1}{p}\right)^{n+1}$$

$$p_1 = 1 = p$$

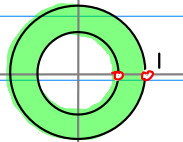
$$p_2 = 2 = q$$

$$\left(\frac{1}{q}\right)^{n+1}$$

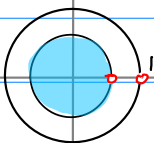
$$\left(\frac{1}{p}\right)^{n+1}$$

$$p_1 = 0.5 = \frac{1}{q}$$

$$p_2 = 1 = \frac{1}{p}$$



I



$$(p)^{n-1} - (q)^{n-1} \quad (n > 0) \quad (n \geq 0) \quad 0$$

$$0 \quad (n \leq 0) \quad (n < 0) \quad (p)^{n-1} - (q)^{n-1}$$

$$p_1 = 0.5 = \frac{1}{q}$$

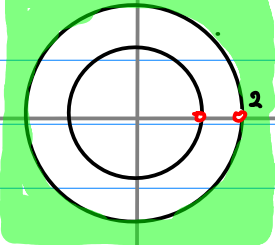
$$p_2 = 1 = \frac{1}{p}$$

$$0$$

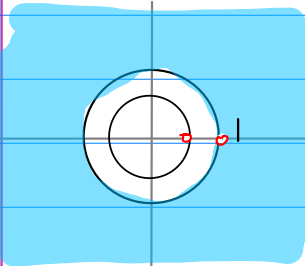
$$(p)^{n-1} - (q)^{n-1}$$

$$p_1 = 1 = p$$

$$p_2 = 2 = q$$



II



$$0 \quad (n > 0) \quad (n \geq 0) \quad (q)^{n-1} - (p)^{n-1}$$

$$(q)^{n-1} - (p)^{n-1} \quad (n \leq 0) \quad (n < 0) \quad 0$$

$$p_1 = 0.5 = \frac{1}{q}$$

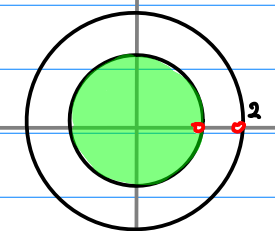
$$p_2 = 1 = \frac{1}{p}$$

$$(q)^{n-1} - (p)^{n-1}$$

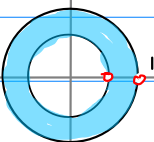
$$0$$

$$p_1 = 1 = p$$

$$p_2 = 2 = q$$



III



$$(p)^{n-1} \quad (n > 0) \quad (n \geq 0) \quad (p)^{n-1}$$

$$(q)^{n-1} \quad (n \leq 0) \quad (n < 0) \quad (q)^{n-1}$$

$$p_1 = 0.5 = \frac{1}{q}$$

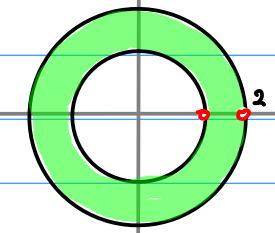
$$p_2 = 1 = \frac{1}{p}$$

$$(p)^{n-1}$$

$$(q)^{n-1}$$

$$p_1 = 1 = p$$

$$p_2 = 2 = q$$



$$n \geq 0$$

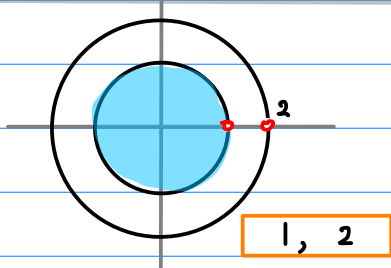
$$n < 0$$

$$n > 0$$

$$n \leq 0$$

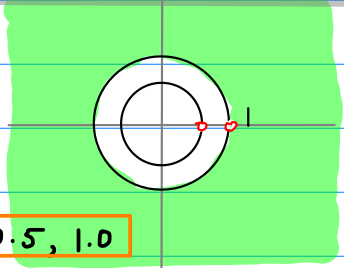
$$\left(\frac{1}{q}\right)^{n+1} - \left(\frac{1}{p}\right)^{n+1}$$

$$0$$


 $0.5, 1.0$ 

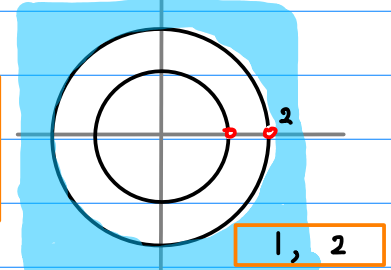
$$0$$

$$\left(\frac{1}{q}\right)^{n+1} - \left(\frac{1}{p}\right)^{n+1}$$



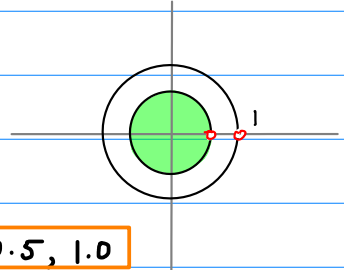
$$0$$

$$\left(\frac{1}{p}\right)^{n+1} - \left(\frac{1}{q}\right)^{n+1}$$


 $0.5, 1.0$ 

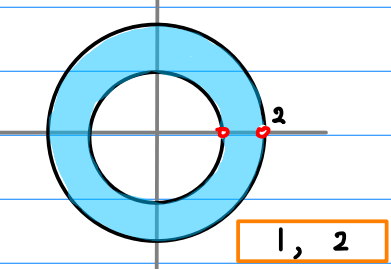
$$\left(\frac{1}{p}\right)^{n+1} - \left(\frac{1}{q}\right)^{n+1}$$

$$0$$



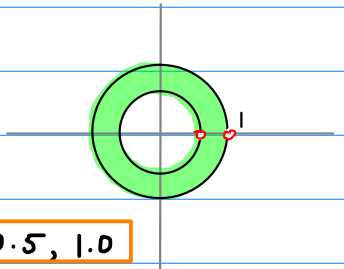
$$\left(\frac{1}{q}\right)^{n+1}$$

$$\left(\frac{1}{p}\right)^{n+1}$$


 $0.5, 1.0$ 

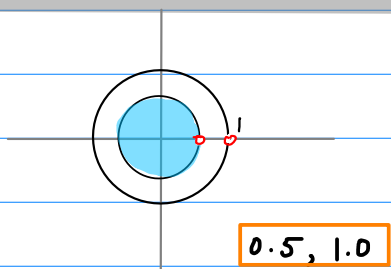
$$\left(\frac{1}{p}\right)^{n+1}$$

$$\left(\frac{1}{q}\right)^{n+1}$$



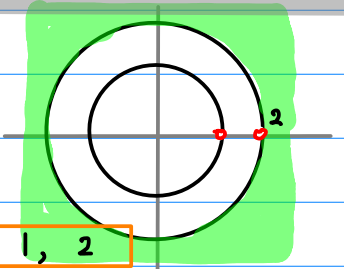
$$(p)^{n-1} - (q)^{n-1}$$

$$0$$


 $1, 2$ 

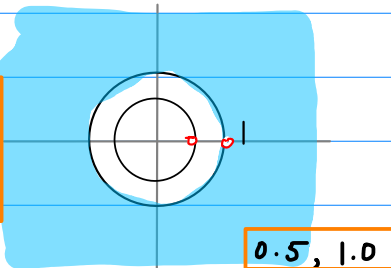
$$0$$

$$(p)^{n-1} - (q)^{n-1}$$



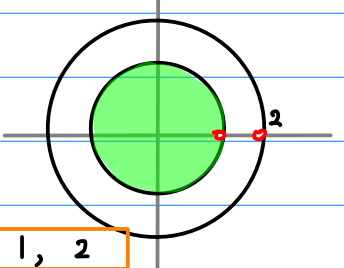
$$0$$

$$(q)^{n-1} - (p)^{n-1}$$


 $1, 2$ 

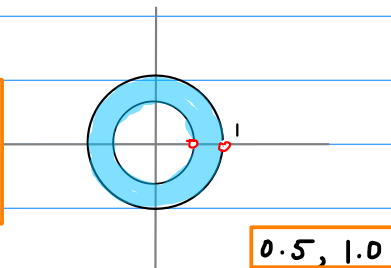
$$(q)^{n-1} - (p)^{n-1}$$

$$0$$



$$(p)^{n-1}$$

$$(q)^{n-1}$$


 $1, 2$ 

$$(q)^{n-1}$$

$$(p)^{n-1}$$

