CLTI Time Response (7B)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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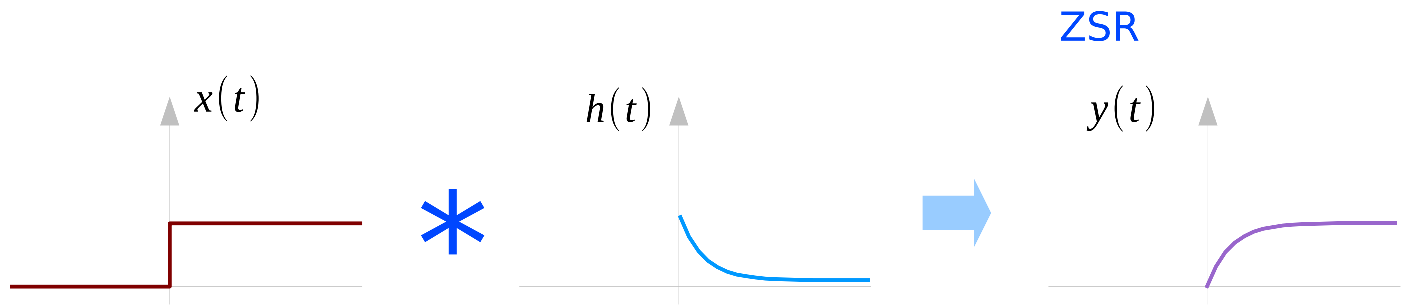
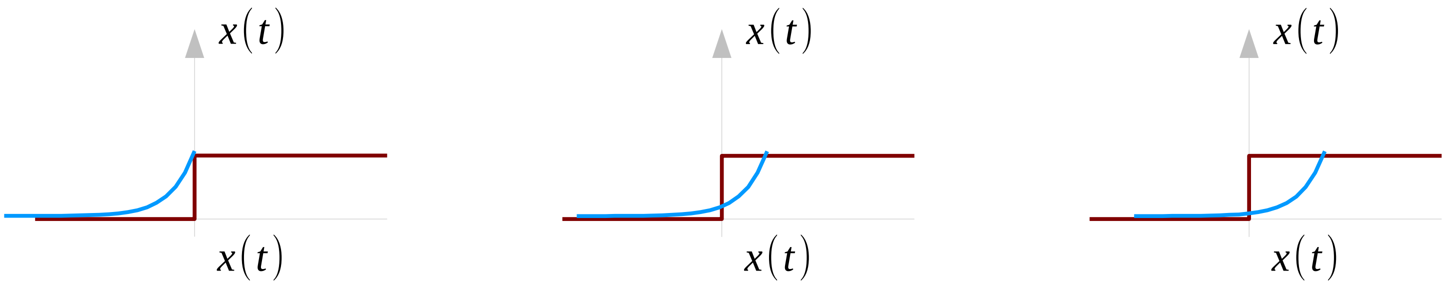
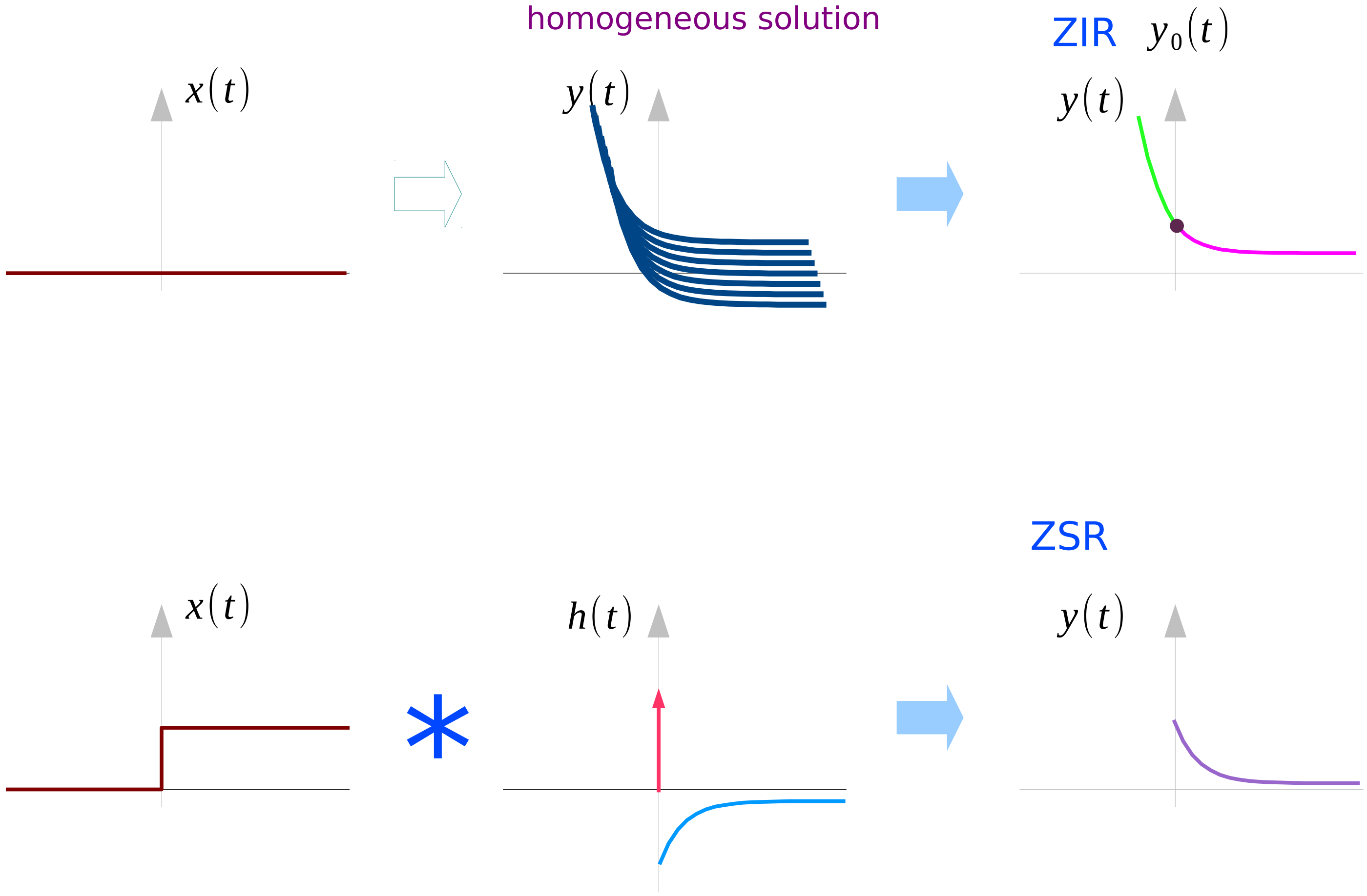
Total Response = ZIR + ZSR (Ex1)





Total Response = ZIR + ZSR (Ex2)

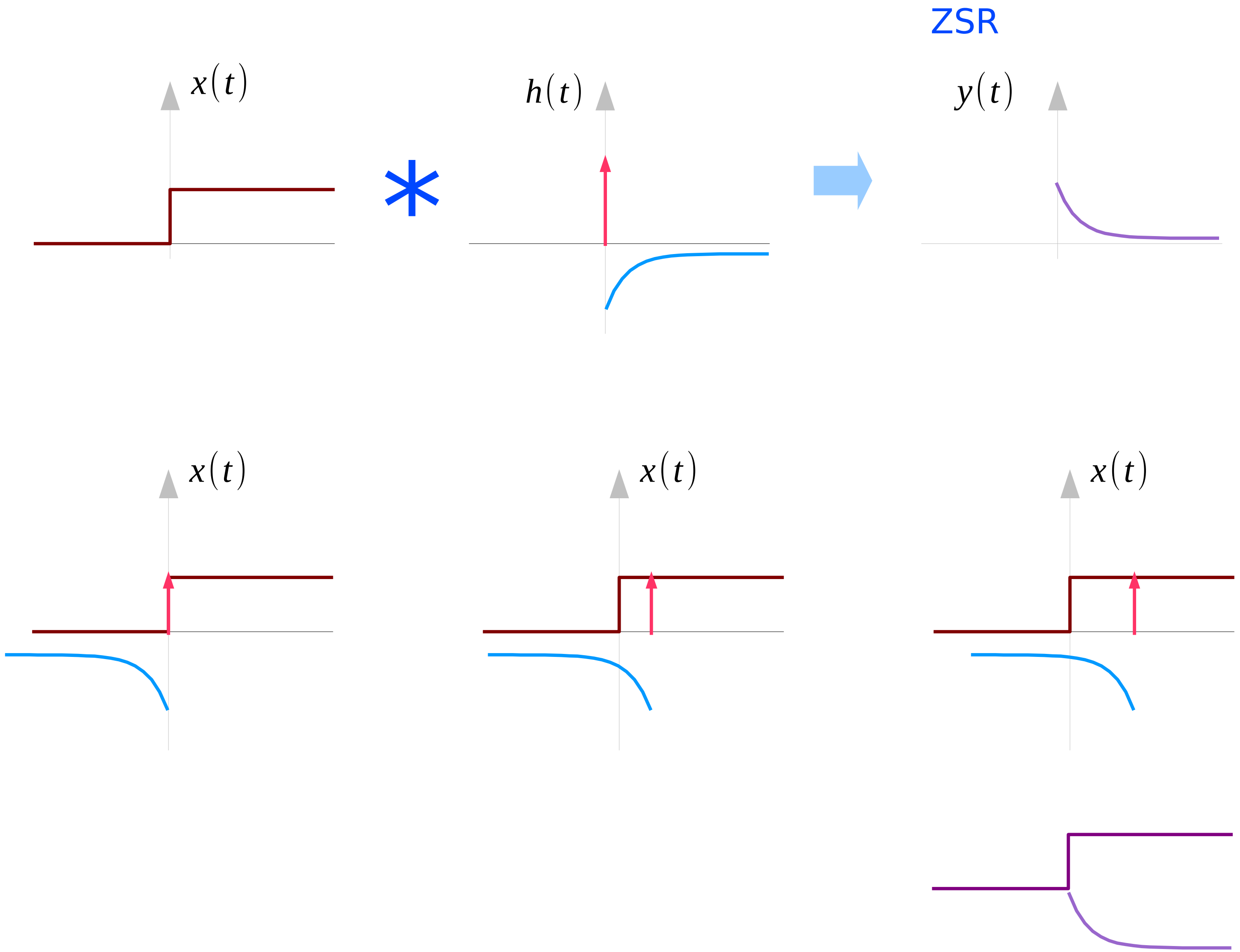


homogeneous solution

ZIR $y_0(t)$

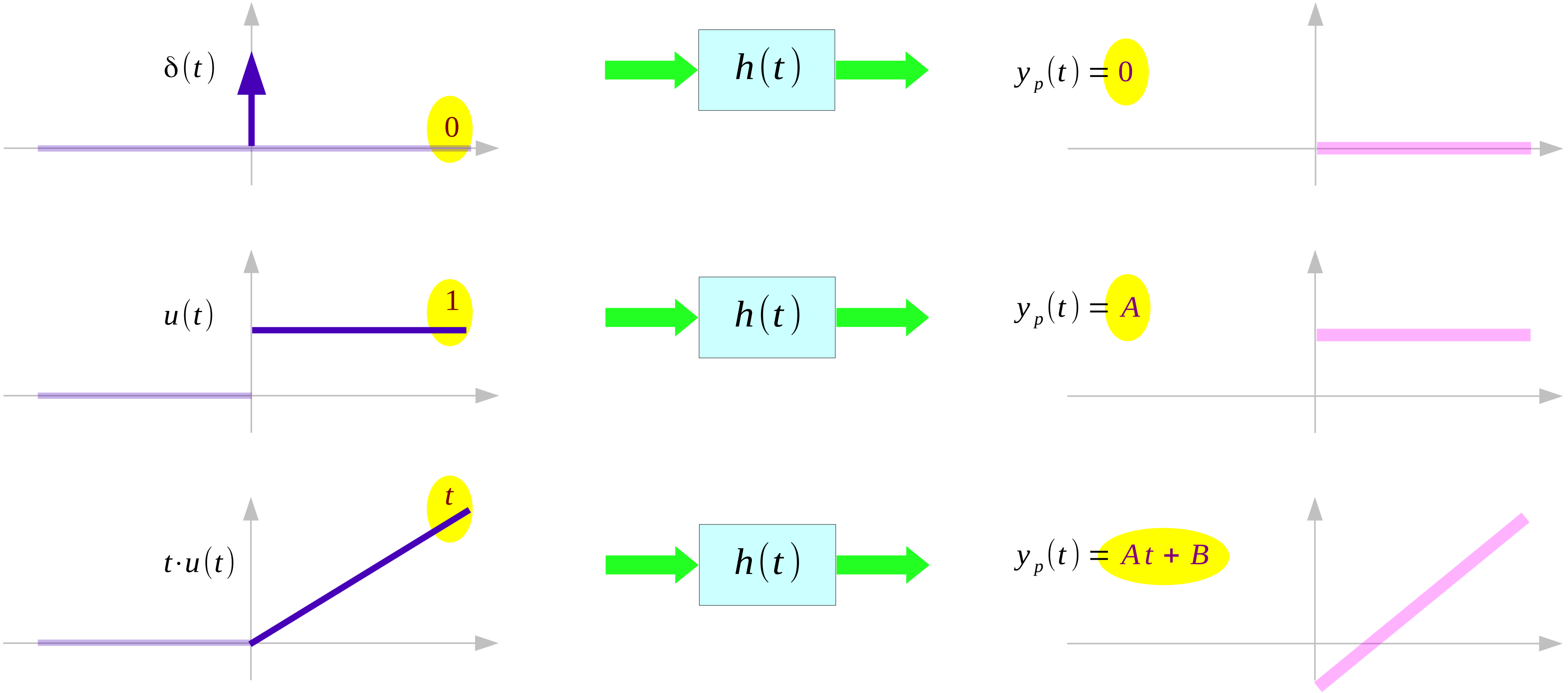
ZSR

Total Response = ZIR + ZSR (Ex2)



Forced Response Examples

$$\frac{d^N y}{dt^N} + \boldsymbol{a}_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots + \boldsymbol{a}_{N-1} \frac{dy}{dt} + \boldsymbol{a}_N y(t) = \boldsymbol{b}_0 \frac{d^N x}{dt^N} + \boldsymbol{b}_1 \frac{d^{N-1} x}{dt^{N-1}} + \cdots + \boldsymbol{b}_{N-1} \frac{dx}{dt} + \boldsymbol{b}_N x(t)$$



Exponential and Sinusoid Functions

exponential function

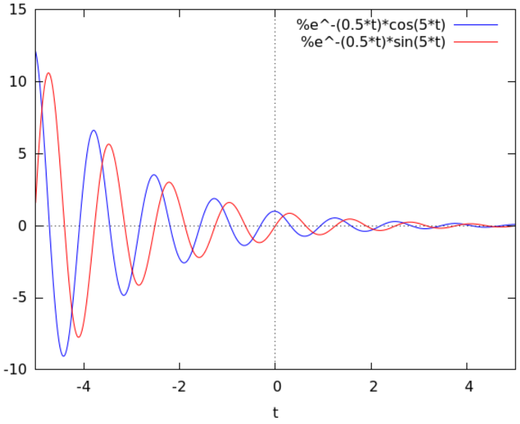
$$e^{st} = e^{\sigma t + i\omega t} \qquad (s = \sigma + i\omega)$$

$$= e^{\sigma t}(\cos(\omega t) + i\sin(\omega t))$$

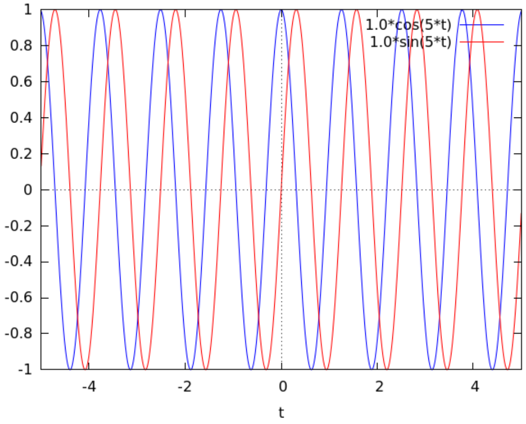
sinusoid function

$$e^{\zeta t} = e^{i\omega t} \qquad (\zeta = i\omega)$$

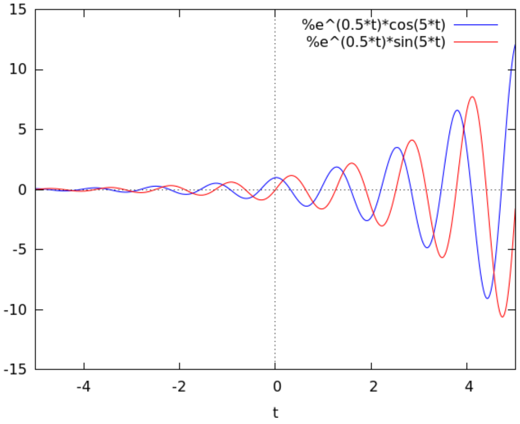
$\Re\{s\} < 0 \quad (\sigma < 0)$



$\Re\{s\} = 0 \quad (\sigma = 0)$



$\Re\{s\} > 0 \quad (\sigma > 0)$



Everlasting & Causal Function

$$\left(s = \sigma + i\omega\right)$$

$$\left(\zeta = i\omega\right)$$

- everlasting **exponential** function

applied at $t = -\infty$

$$e^{st}$$

- everlasting **sinusoid** function

applied at $t = -\infty$

$$e^{\zeta t}$$

- causal **exponential** function

applied at $t = 0$ applied at $t = 0$

$$e^{st}\,u(t)$$

- causal **sinusoid** function

$$e^{\zeta t}\,u(t)$$

Sinusoidal Functions and Initial Conditions

- **everlasting** sinusoid function

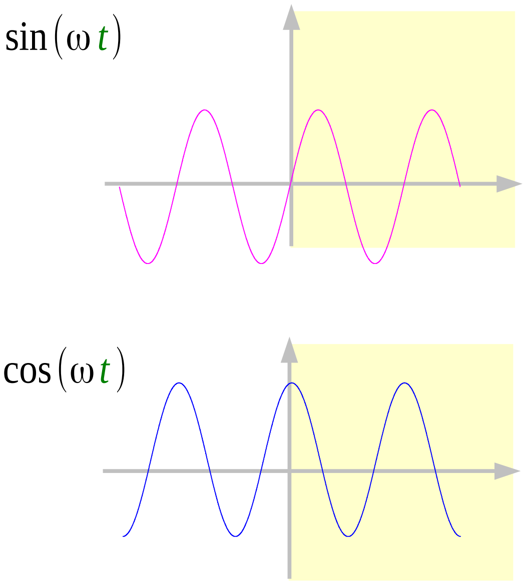
state at t = 0– = state at t = 0+

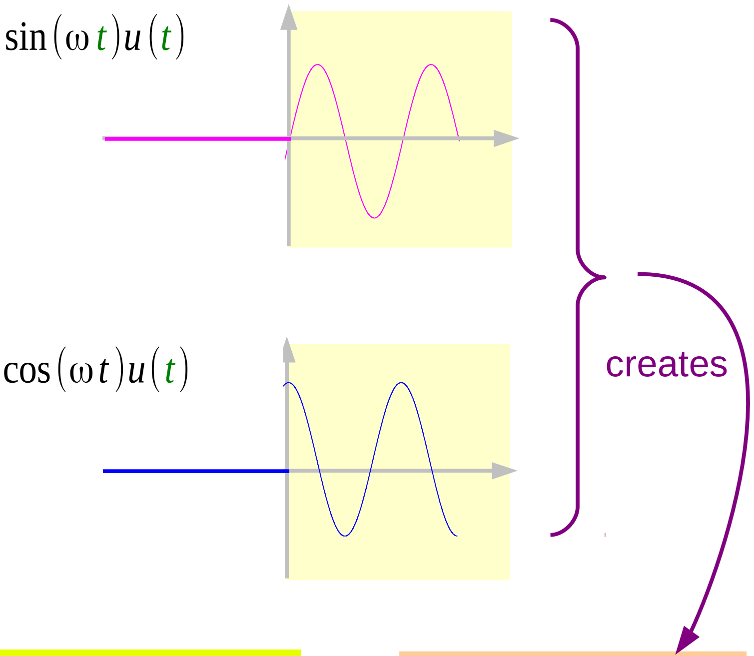
continuous state between t = 0– & 0+

- **causal** sinusoid function

zero state at t = 0–

non-zero state at t = 0+

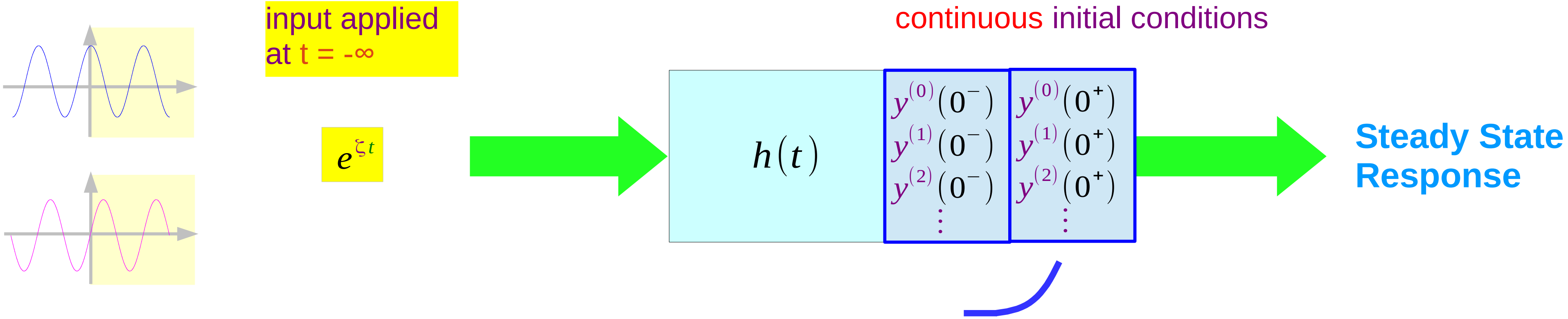




zero conditions
at time $t = 0^-$

non-zero conditions
at time $t = 0^+$

Steady State Sinusoidal Response



input applied at t = -∞

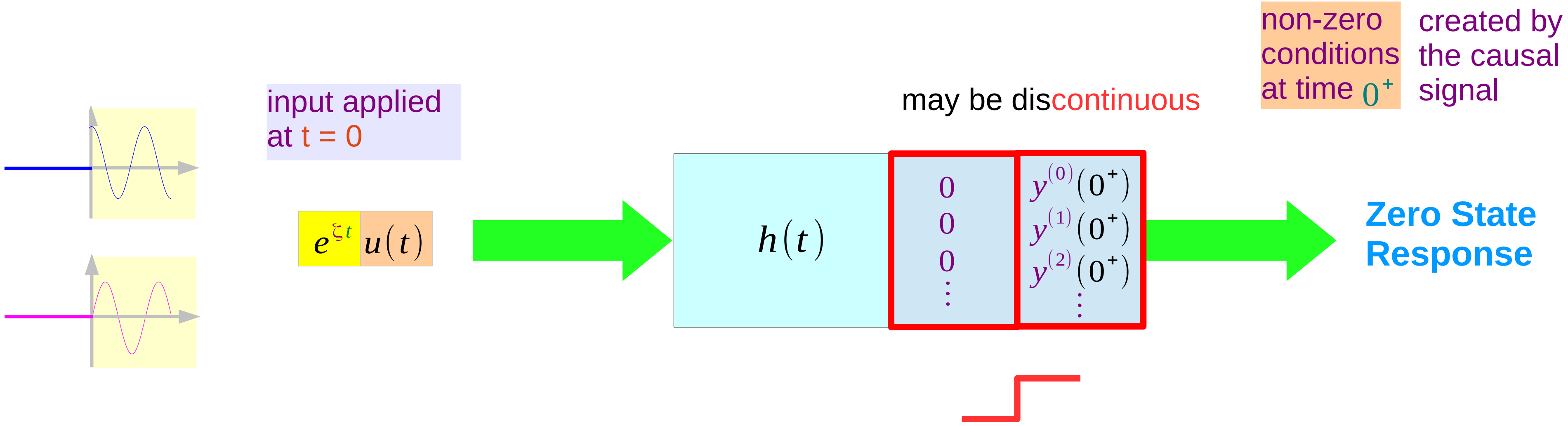
$e^{\zeta t}$

continuous initial conditions

$h(t)$

$$\begin{array}{c|c} y^{(0)}(0^-) & y^{(0)}(0^+) \\ y^{(1)}(0^-) & y^{(1)}(0^+) \\ y^{(2)}(0^-) & y^{(2)}(0^+) \\ \vdots & \vdots \end{array}$$

Steady State Response



input applied at t = 0

$e^{\zeta t}u(t)$

may be discontinuous

non-zero conditions at time 0^+

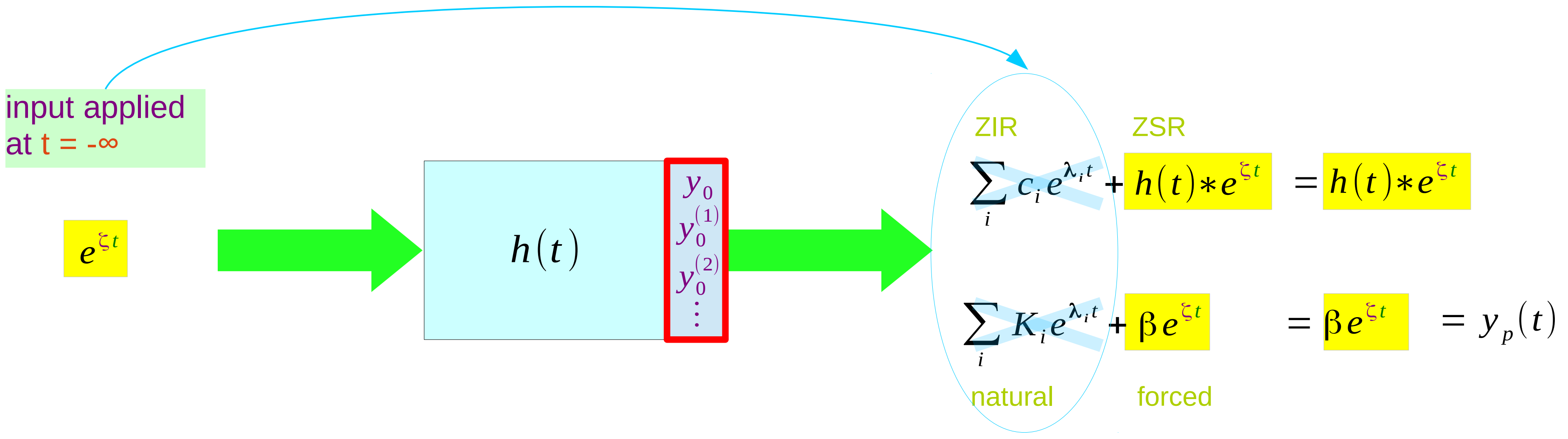
created by the causal signal

$h(t)$

$$\begin{array}{c|c} 0 & y^{(0)}(0^+) \\ 0 & y^{(1)}(0^+) \\ 0 & y^{(2)}(0^+) \\ \vdots & \vdots \end{array}$$

Zero State Response

Total Response to exponential inputs



input applied
at t = -∞

$e^{\zeta t}$

$h(t)$

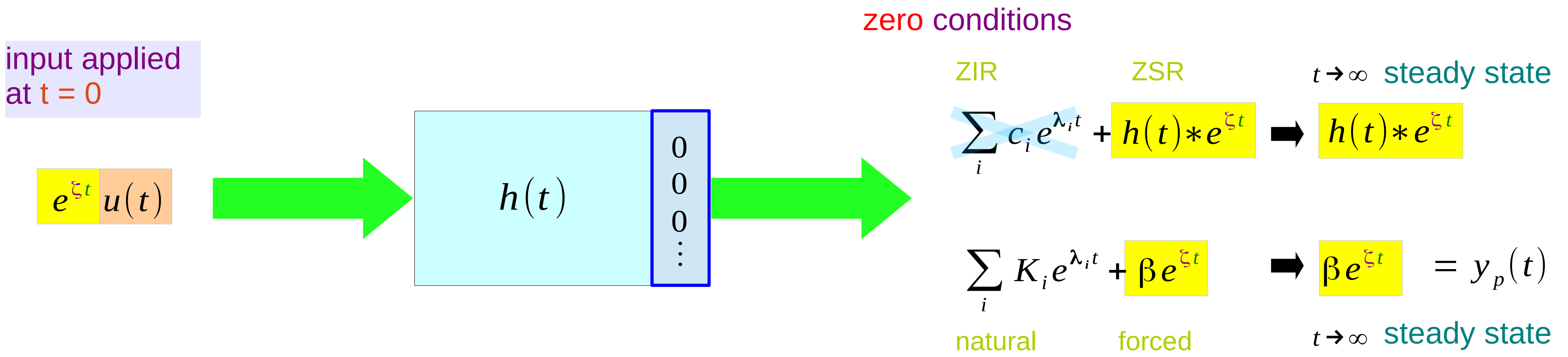
y_0
$y_0^{(1)}$
$y_0^{(2)}$
$\vdots$

ZIR ZSR

$$\cancel{\sum_i c_i e^{\lambda_i t}} + h(t)*e^{\zeta t} = h(t)*e^{\zeta t}$$

$$\cancel{\sum_i K_i e^{\lambda_i t}} + \beta e^{\zeta t} = \beta e^{\zeta t} = y_p(t)$$

natural forced



zero conditions

input applied
at t = 0

$e^{\zeta t} u(t)$

$h(t)$

0
0
0
$\vdots$

ZIR ZSR $t \to \infty$ steady state

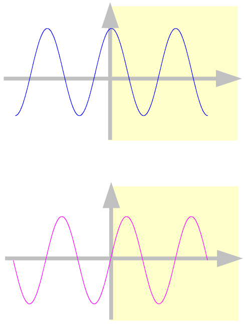
$$\cancel{\sum_i c_i e^{\lambda_i t}} + h(t)*e^{\zeta t} \Rightarrow h(t)*e^{\zeta t}$$

$$\sum_i K_i e^{\lambda_i t} + \beta e^{\zeta t} \Rightarrow \beta e^{\zeta t} = y_p(t)$$

natural forced $t \to \infty$ steady state

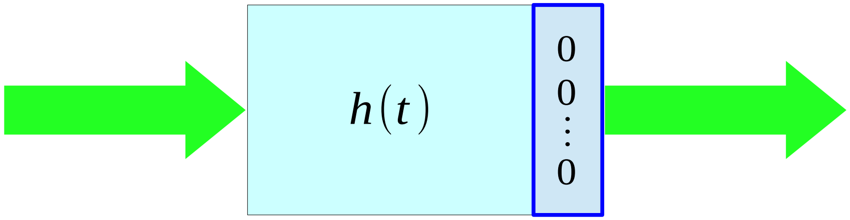
for a pure imaginary ζ

Steady State Response



input applied
at t = -∞

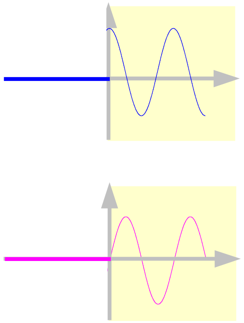
$e^{\zeta t}$



Total Response

$= h(t) * e^{\zeta t}$

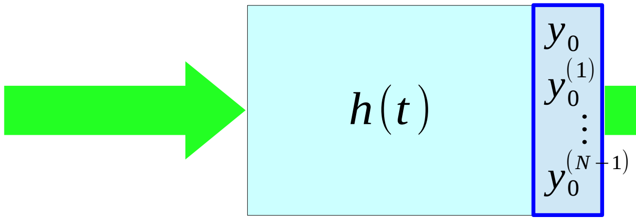
$= \beta e^{\zeta t} \quad = y_p(t)$



input applied
at t = 0

$e^{\zeta t} u(t)$

Steady State Response = ZSR $(t \to \infty)$

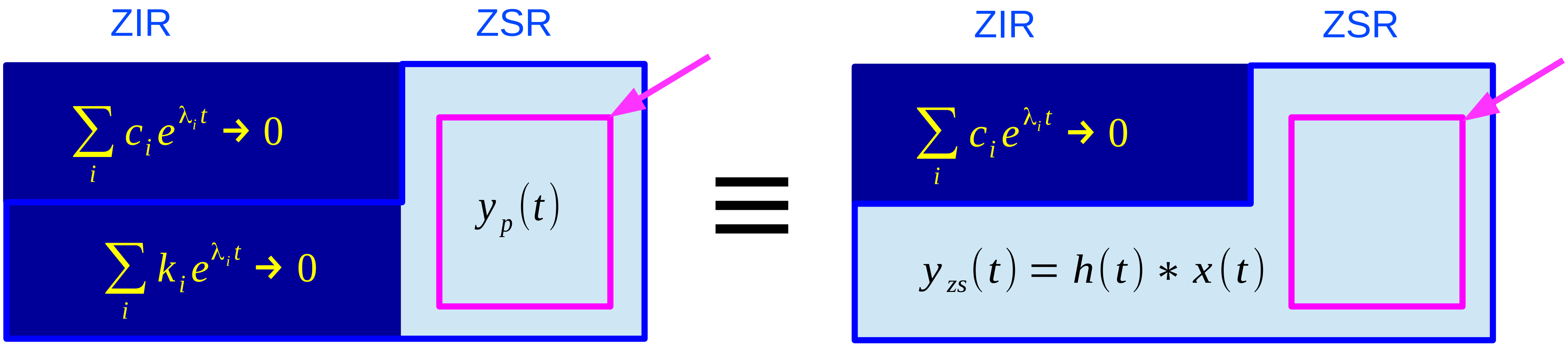


$= h(t) * e^{\zeta t}$

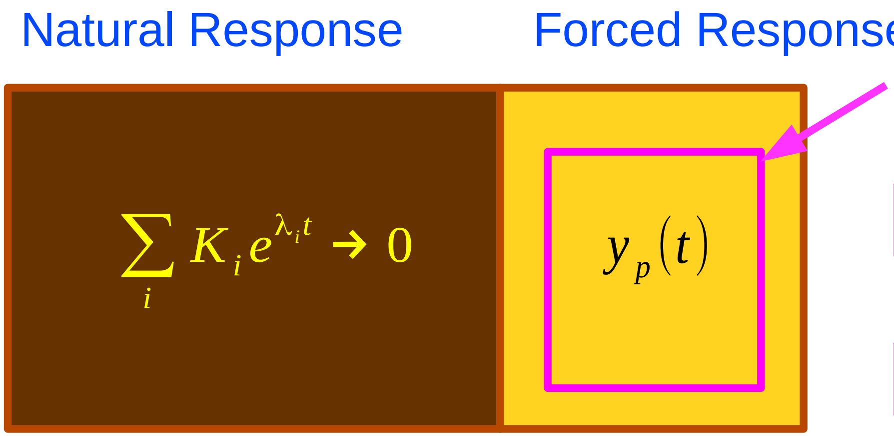
$= \beta e^{\zeta t} \quad = y_p(t)$

for a pure imaginary ζ

Steady State Responses $\quad (t \to \infty)$

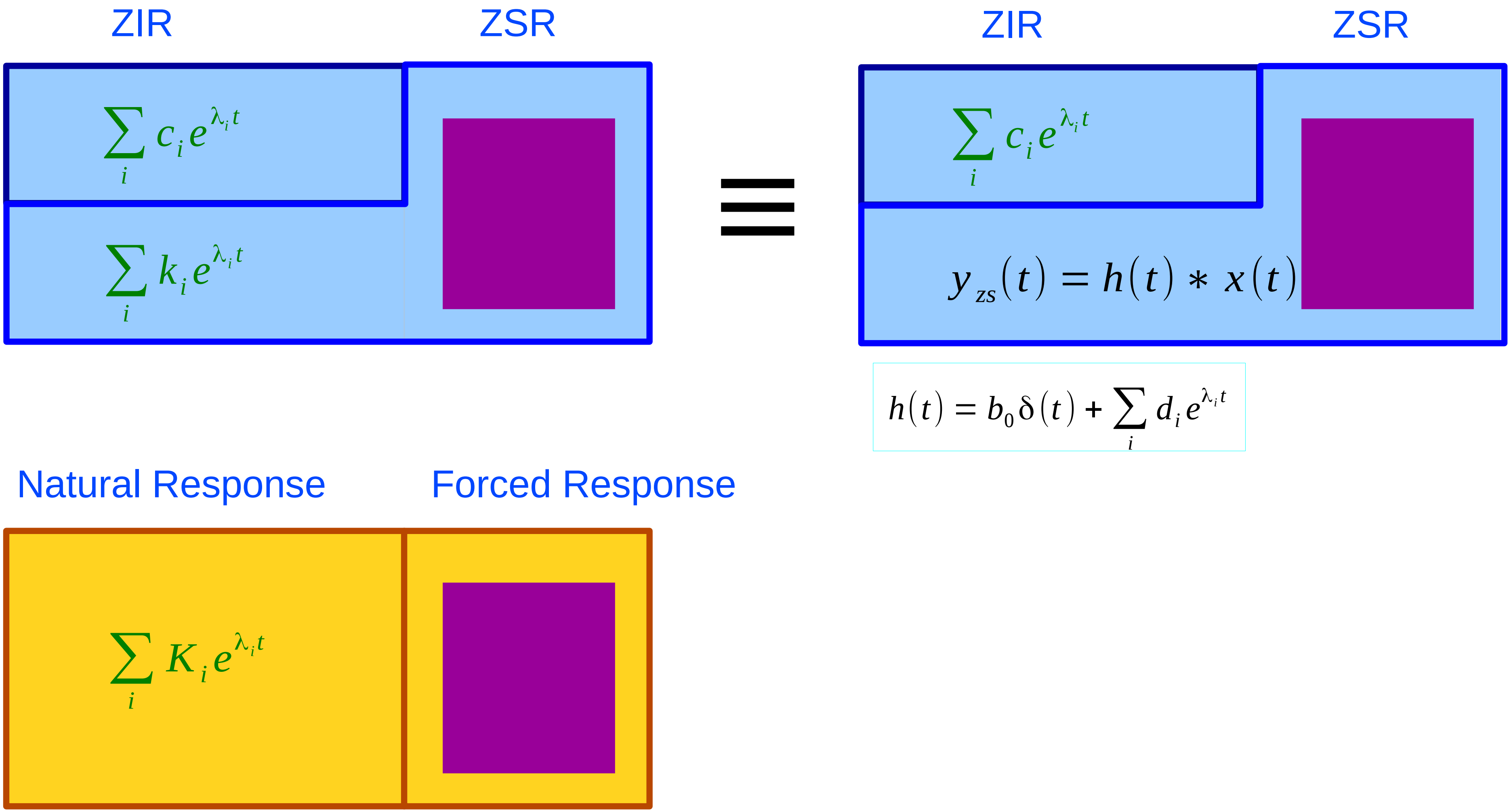


$$h(t) = b_0 \delta(t) + \sum_i d_i e^{\lambda_i t}$$



zero	$y_p(t) = 0$	⬅	$x(t) = \delta(t)$	
constant	$y_p(t) = \beta$	⬅	$x(t) = k$	
diverges	$y_p(t) = \beta_1 t + \beta_0$	⬅	$x(t) = t\,u(t)$	
oscillates	$y_p(t) = \beta e^{\zeta t}$	⬅	$x(t) = e^{\zeta t}$	$\zeta \neq \lambda_i$
zero	$y_p(t) = e^{\sigma t}(\alpha \cos(\omega t) + \beta \sin(\omega t))$			$\sigma < 0$
		⬅	$x(t) = e^{st}$	$s \neq \lambda_i$

Transient Responses



ZIR ZSR

$$\sum_i c_i e^{\lambda_i t}$$

$$\sum_i k_i e^{\lambda_i t}$$

$$\equiv$$

ZIR ZSR

$$\sum_i c_i e^{\lambda_i t}$$

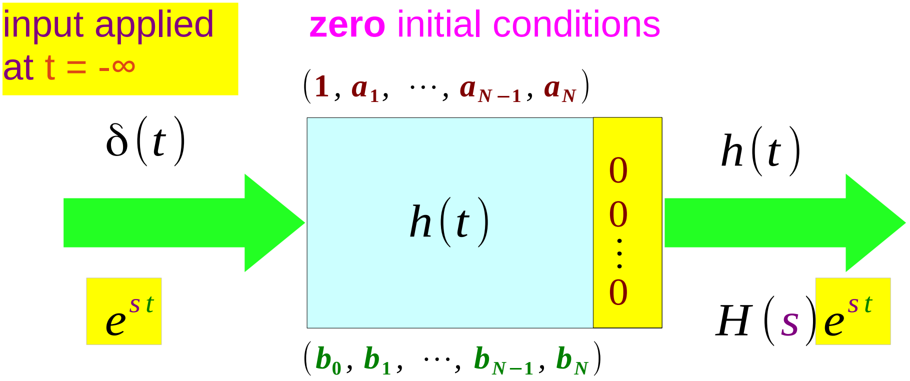
$$y_{zs}(t) = h(t) * x(t)$$

$$h(t) = b_0 \delta(t) + \sum_i d_i e^{\lambda_i t}$$

Natural Response Forced Response

$$\sum_i K_i e^{\lambda_i t}$$

ZSR to an everlasting exponential input



$$\begin{aligned} Q(D)y(t) &= P(D)x(t) \\ Q(D)[h(t) * x(t)] &= P(D)x(t) \\ Q(D)[e^{st}H(s)] &= P(D)e^{st} \\ H(s)Q(D)e^{st} &= P(D)e^{st} \end{aligned}$$

$$\begin{aligned} y(t) &= h(t) * e^{st} \\ &= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}\,d\tau \\ &= e^{st} \cdot \boxed{\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau} \\ &= e^{st} \cdot \boxed{H(s)} \end{aligned}$$

$h(t) = 0$
$(t < 0)$

$$\boxed{H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}\,d\tau}$$

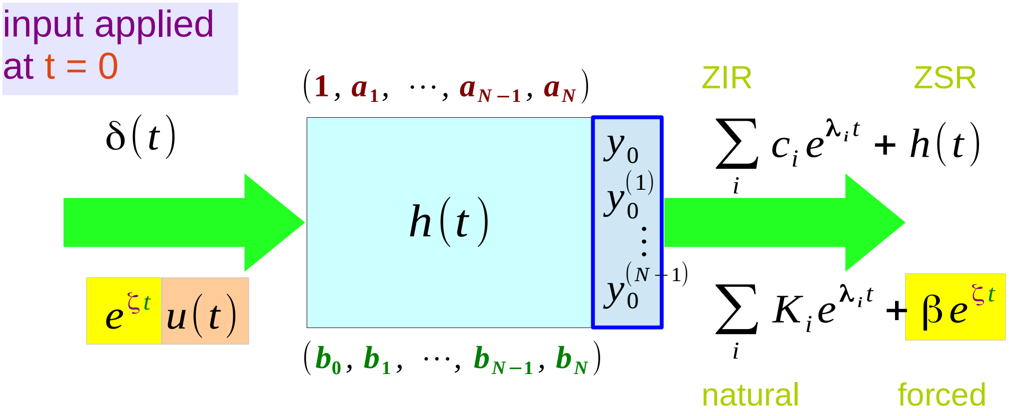
$$D^r e^{st} = \frac{d^r}{dt^r}e^{st} = s^r e^{st}$$

$$\begin{aligned} Q(D)e^{st} &= Q(s)e^{st} \\ P(D)e^{st} &= P(s)e^{st} \end{aligned}$$

$$H(s)Q(s)e^{st} = P(s)e^{st}$$

$$\boxed{H(s) = \frac{P(s)}{Q(s)}}$$

Forced Response to a causal exponential input



input applied at t = 0

$\delta(t)$

$e^{\zeta t} u(t)$

$(1, a_1, \cdots, a_{N-1}, a_N)$

$h(t)$

y_0, $y_0^{(1)}$, $\vdots$, $y_0^{(N-1)}$

$(b_0, b_1, \cdots, b_{N-1}, b_N)$

ZIR ZSR

$\sum_i c_i e^{\lambda_i t} + h(t)$

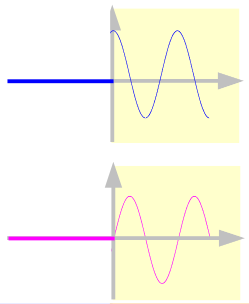
$\sum_i K_i e^{\lambda_i t} + \beta e^{\zeta t}$

natural forced

$$Q(D)\,y(t) = P(D)\,x(t)$$

$$Q(D)[\beta e^{\zeta t}] = P(D)e^{\zeta t}$$

$$\beta Q(D)e^{\zeta t} = P(D)e^{\zeta t}$$



initial conditions at time $t = 0^-$

initial conditions at time $t = 0^+$

$$D^r e^{\zeta t} = \frac{d^r}{dt^r} e^{\zeta t} = \zeta^r e^{\zeta t}$$

$$Q(D)e^{\zeta t} = Q(\zeta)e^{\zeta t}$$

$$P(D)e^{\zeta t} = P(\zeta)e^{\zeta t}$$

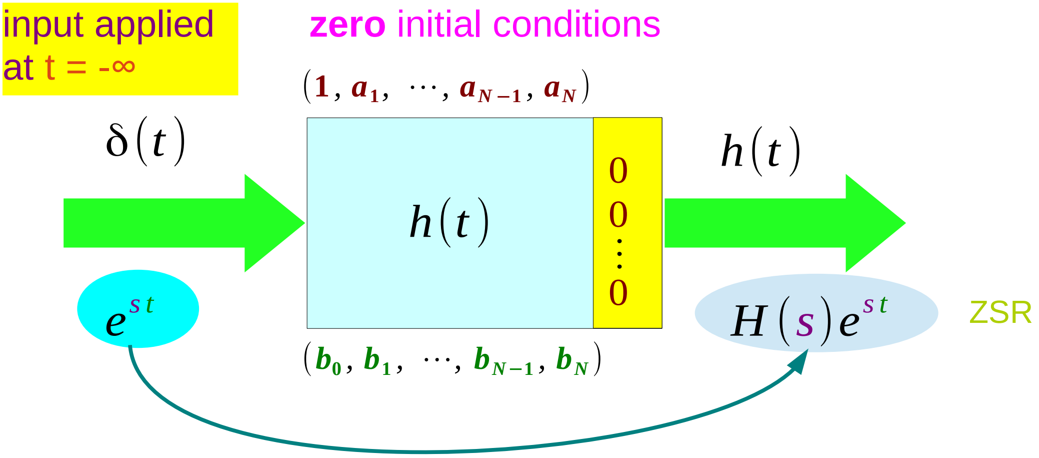
ζ : **NOT** a characteristic mode

$$y_n(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} + \cdots + K_N e^{\lambda_N t} = \sum_i K_i e^{\lambda_i t}$$

$y_n(t) + y_p(t)$ $\{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\}$ → K_i

$$\beta = \frac{P(\zeta)}{Q(\zeta)}$$

Everlasting Exponential Total Response



input applied at t = -∞

zero initial conditions

$(1, a_1, \cdots, a_{N-1}, a_N)$

$\delta(t)$

$h(t)$

0 0 ⋮ 0

$h(t)$

e^{st}

$H(s)e^{st}$

ZSR

$(b_0, b_1, \cdots, b_{N-1}, b_N)$

$for\ a\ given\ s = \zeta$

ZSR

$$y(t) = H(\zeta)e^{\zeta t} \qquad -\infty < t < +\infty$$

$$\boxed{y(t) = H(\zeta)x(t)} \qquad x(t) = e^{\zeta t}$$

$$\boxed{Y(s) = H(s)X(s)}$$

Laplace Transform of h(t)

$$\boxed{H(s) = \int_0^{+\infty} h(\tau)e^{-s\tau}d\tau}$$

Polynomials of Differential Equation

$$\boxed{H(s) = \frac{P(s)}{Q(s)}}$$

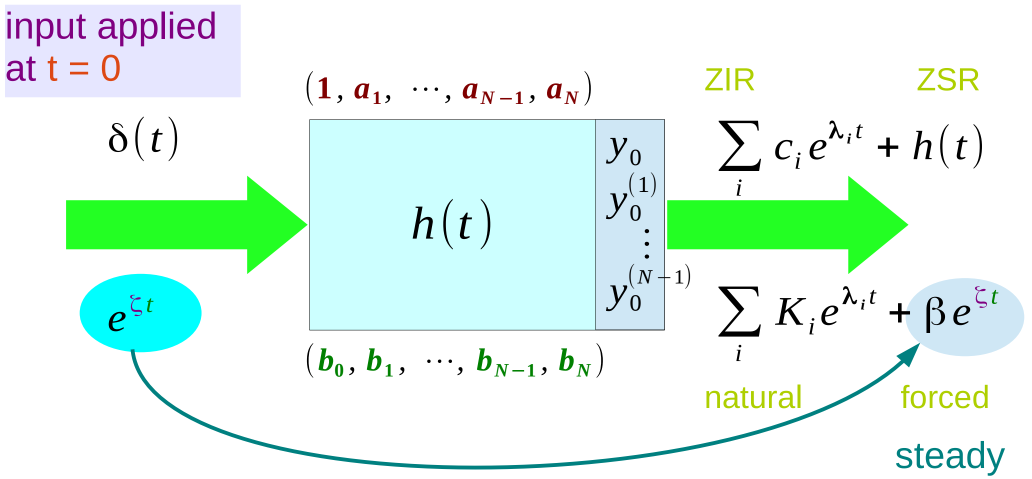
Transfer function (t-domain)

$$\boxed{H(s) = \left[\frac{y(t)}{x(t)}\right]_{x(t)=e^{st}}}$$

Transfer function (s-domain)

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

<u>Causal</u> Exponential Total Response



for a given $s = \zeta$

natural forced

$$y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta) e^{\zeta t} \qquad t \geq 0$$

$$\boxed{y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta) x(t)} \qquad x(t) = e^{\zeta t}$$

$$\boxed{Y(s) = \left[\sum_i \frac{K_i}{(s - \lambda_i)} + H(s) \right] X(s)}$$

Laplace Transform of h(t)

$$\boxed{H(s) = \int_0^{+\infty} h(\tau) e^{-s\tau} d\tau}$$

Polynomials of Differential Equation

$$\boxed{H(s) = \frac{P(s)}{Q(s)}}$$

Transfer function (t-domain)

$$\boxed{H(s) = \left[\frac{y(t)}{x(t)} \right]_{x(t) = e^{st}}}$$

Transfer function (s-domain)

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

Impulse Matching Example

$$\frac{d^2 y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + 1x(t)$$

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$

$$(\lambda^2 + 5\lambda + 6) = (\lambda+2)(\lambda+3) = 0$$

$$h(t) = (c_1 e^{-2t} + c_2 e^{-3t}) \cdot u(t)$$

$$\dot{h}(t) = (-2c_1 e^{-2t} - 3c_2 e^{-3t}) \cdot u(t) + (c_1 e^{-2t} + c_2 e^{-3t}) \cdot \delta(t)$$

$$\begin{cases} h(0^+) = (c_1 e^0 + c_2 e^0) \cdot u(t) \\ \dot{h}(0^+) = (-2c_1 e^0 - 3c_2 e^0) \cdot u(t) \end{cases}$$

$$\begin{cases} h(0^+) = c_1 + c_2 = +1 = K_1 \\ \dot{h}(0^+) = -2c_1 - 3c_2 = -4 = K_2 \end{cases}$$

$$c_1 = -1$$
$$c_2 = +2$$

$$\boxed{h(t) = (-e^{-2t} + 2e^{-3t}) \cdot u(t)}$$

$$\ddot{h}(t) + 5\dot{h}(t) + 6h(t) = \dot{\delta}(t) + 1\delta(t)$$

$$\begin{cases} h(0^-) = 0 \\ \dot{h}(0^-) = 0 \end{cases} \qquad \begin{cases} h(0^+) = K_1 \\ \dot{h}(0^+) = K_2 \end{cases}$$

$$\begin{cases} \dot{h}(0) = K_1 \delta(t) \\ \ddot{h}(0) = K_1 \dot{\delta}(t) + K_2 \delta(t) \end{cases}$$

$$\ddot{h}(0) + 5\dot{h}(0) + 6h(0) = \dot{\delta}(t) + 1\delta(t)$$

$$(K_1 \dot{\delta}(t) + K_2 \delta(t)) + 5K_1 \delta(t) + 6h(0)$$

$$K_1 \dot{\delta}(t) + (5K_1 + K_2)\delta(t) + \cdots = \dot{\delta}(t) + 1\delta(t)$$

$$K_1 = 1$$
$$5K_1 + K_2 = 1$$

$$\begin{cases} K_1 = 1 = h(0^+) \\ K_2 = -4 = \dot{h}(0^+) \end{cases}$$

Simplified Impulse Matching Example

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + 1x(t)$$

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$

$$(\lambda^2 + 5\lambda + 6) = (\lambda+2)(\lambda+3) = 0$$

$$\begin{cases} y_n(t) = (c_1e^{-2t}+c_2e^{-3t}) \\ \dot{y}_n(t) = (-2c_1e^{-2t}-3c_2e^{-3t}) \end{cases} \quad \begin{cases} y_n(0) = 0 \\ \dot{y}_n(0) = 1 \end{cases}$$

$$\begin{cases} y_n(0) = (c_1e^0+c_2e^0) & = 0 \\ \dot{y}_n(0) = (-2c_1e^0-3c_2e^0) & = 1 \end{cases}$$

$$\begin{cases} y_n(0) = c_1+c_2 & = 0 \\ \dot{y}_n(0) = -2c_1-3c_2 & = 1 \end{cases}$$

$$c_1 = +1$$
$$c_2 = -1$$

$$y_n(t) = (+e^{-2t}-e^{-3t})$$

$$h(t) = b_0\delta(t) + [P(D)y_n(t)]u(t)$$

$$(D+1)y_n(t) = (-2e^{-2t}+3e^{-3t})+(e^{-2t}-e^{-3t})$$
$$= (-e^{-2t}+2e^{-3t})$$

$$h(t) = (-e^{-2t}+2e^{-3t})\cdot u(t)$$

Step Response

$$\frac{d^2 y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + 1x(t)$$

$$\boxed{h(t) = (-e^{-2t} + 2e^{-3t})\cdot u(t)}$$

$$\begin{tikzcd}
h(t) \arrow[d] & \boxed{h(t)} \arrow[l] & \delta(t) \arrow[l] \arrow[d] \\
\boxed{\int \cdot\, dt} \arrow[d] & & \boxed{\int \cdot\, dt} \arrow[d] \\
s(t) & \boxed{h(t)} \arrow[l] & u(t) \arrow[l]
\end{tikzcd}$$

$$s(t) = h(t) * u(t)$$

$$s(t) = h(t) * u(t) = \int_{-\infty}^{+\infty} h(\tau)u(t-\tau)\,d\tau$$

$$h(\tau) = (-e^{-2\tau} + 2e^{-3\tau})\cdot u(\tau)$$

$$s(t) = \int_{-\infty}^{+\infty} (-e^{-2\tau} + 2e^{-3\tau})\cdot u(\tau)\cdot u(t-\tau)\,d\tau$$

$$= \int_{0}^{t} (-e^{-2\tau} + 2e^{-3\tau})\,d\tau$$

$$s(t) = \int_{0^-}^{t} h(\tau)\,d\tau$$

$$s(t) = \int_{0^-}^{t} (-e^{-2\tau} + 2e^{-3\tau})\cdot u(\tau)\,d\tau$$

$$= \left[\frac{1}{2}e^{-2\tau} - \frac{2}{3}e^{-3\tau}\right]_0^t$$

$$= \frac{1}{2}e^{-2t} - \frac{2}{3}e^{-3t} + \frac{1}{6}$$

Step Response : ZSR of u(t)

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + 1x(t)$$

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \delta(t) + u(t)$$

⬆ ⬆ ⬆

$\delta(t)+\cdots$ $u(t)+\cdots$ $tu(t)+\cdots$ ⬅ the highest order singularities

unit jump no jump

$$\begin{cases} y(0^+) = y(0^-) \\ \dot{y}(0^+) = \dot{y}(0^-) + 1 \end{cases} \qquad \begin{cases} y(0^+) = 0 \\ \dot{y}(0^+) = 1 \end{cases}$$

$$\begin{cases} y(0) = (c_1+c_2) + \frac{1}{6} & = 0 \\ \dot{y}(t) = (-2c_1e^{-2t} - 3c_2e^{-3t}) & = 1 \end{cases}$$

$$s(t) = \frac{1}{2}e^{-2t} - \frac{2}{3}e^{-3t} + \frac{1}{6} \qquad (t>0)$$

The Method of Direct Inspection

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$

$$(\lambda^2 + 5\lambda + 6) = (\lambda+2)(\lambda+3) = 0$$

$$y_h(t) = (c_1e^{-2t} + c_2e^{-3t}) \qquad (t>0)$$

$$x(t) = u(t) \Rightarrow x(t) = 1 \qquad (t>0)$$

$$\ddot{y}_p(t) + 5\dot{y}_p(t) + 6y_p(t) = 1 \qquad (t>0)$$

$$y_p(t) = A \Rightarrow 6A = 1 \Rightarrow y_p(t) = \frac{1}{6}$$

$$y(t) = y_h(t) + y_p(t)$$

$$y(t) = (c_1e^{-2t} + c_2e^{-3t}) + \frac{1}{6} \qquad (t>0)$$

$$\dot{y}(t) = (-2c_1e^{-2t} - 3c_2e^{-3t}) \qquad (t>0)$$

$$y(0) = (c_1+c_2) + \frac{1}{6} \qquad (t>0)$$

$$\dot{y}(0) = (-2c_1 - 3c_2) \qquad (t>0)$$

Step Response : ZSR of u(t)

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + 1x(t)$$

The Method of balancing singularities

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$

$$(\lambda^2 + 5\lambda + 6) = (\lambda+2)(\lambda+3) = 0$$

$$y_h(t) = (c_1e^{-2t}+c_2e^{-3t}) \quad (t>0)$$

$$x(t) = u(t) \Rightarrow x(t) = 1 \quad (t>0)$$

$$\ddot{y}_p(t) + 5\dot{y}_p(t) + 6y_p(t) = 1 \quad (t>0)$$

$$y_p(t) = A \Rightarrow 6A = 1 \Rightarrow y_p(t) = \frac{1}{6}$$

$$\begin{aligned} y(t) &= (y_h(t) + y_p(t)) \cdot u(t) \\ &= (c_1e^{-2t}+c_2e^{-3t} + \tfrac{1}{6}) \cdot u(t) \end{aligned}$$

$$\begin{aligned} \dot{y}(t) &= (c_1e^{-2t}+c_2e^{-3t} + \tfrac{1}{6}) \cdot \delta(t) \\ &\quad + (-2c_1e^{-2t}-3c_2e^{-3t}) \cdot u(t) \\ &= (c_1+c_2 + \tfrac{1}{6}) \cdot \delta(t) + (-2c_1e^{-2t}-3c_2e^{-3t}) \cdot u(t) \end{aligned}$$

$$\begin{aligned} \ddot{y}(t) &= (c_1+c_2 + \tfrac{1}{6}) \cdot \dot{\delta}(t) + (-2c_1-3c_2) \cdot \delta(t) \\ &\quad + (4c_1e^{-2t}+9c_2e^{-3t}) \cdot u(t) \end{aligned}$$

$$5\dot{y}(t) = (5c_1+5c_2 + \tfrac{5}{6}) \cdot \delta(t) + (-10c_1e^{-2t}-15c_2e^{-3t}) \cdot u(t)$$

$$6y(t) = (6c_1e^{-2t}+6c_2e^{-3t} + 1) \cdot u(t)$$

$$\begin{cases} (c_1+c_2 + \tfrac{1}{6}) = 0 \\ (3c_1+2c_2 + \tfrac{5}{6}) = 1 \end{cases} \quad \begin{cases} c_1+c_2 = -\tfrac{1}{6} \\ 3c_1+2c_2 = \tfrac{1}{6} \end{cases} \quad \begin{cases} c_1 = +\tfrac{1}{2} \\ c_2 = -\tfrac{2}{3} \end{cases}$$

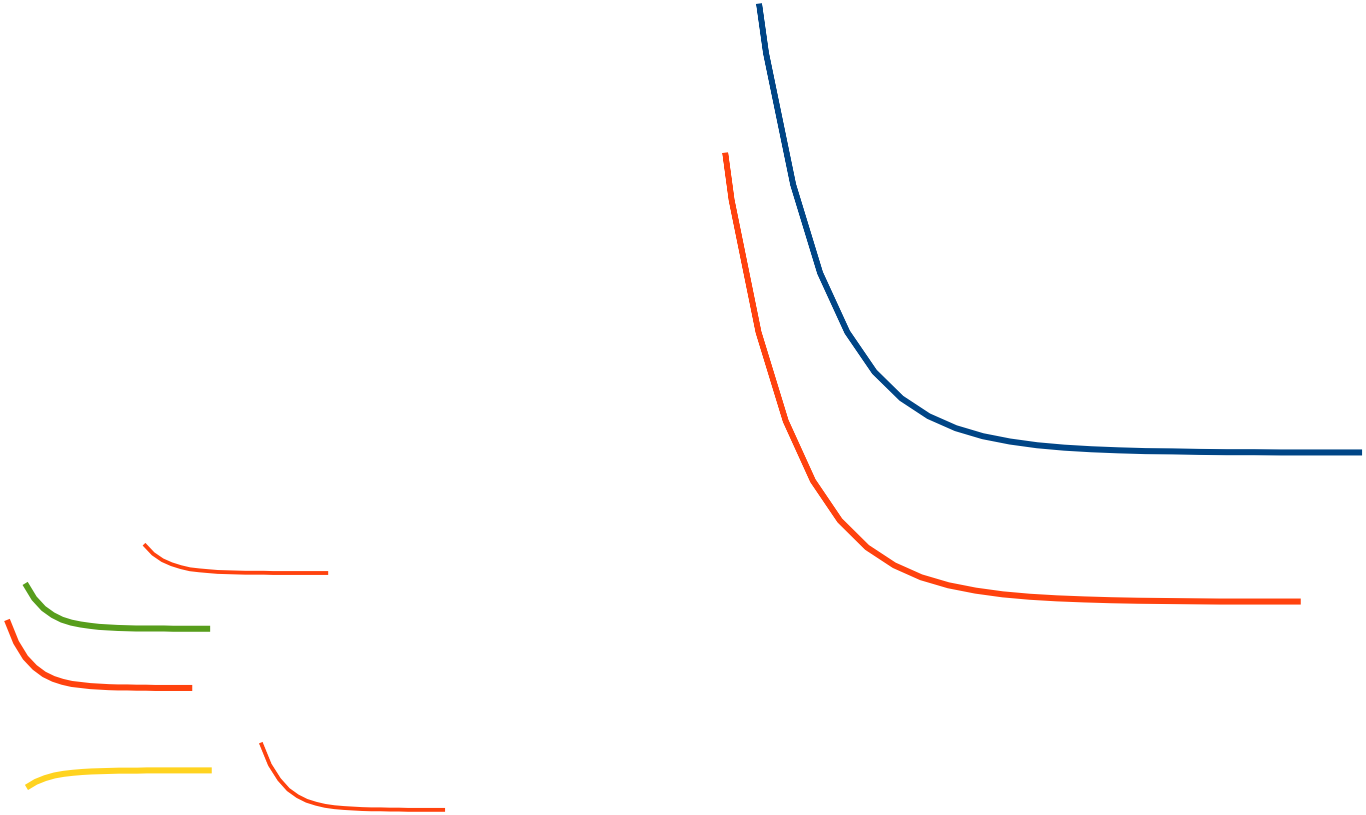
$$s(t) = \left(\frac{1}{2}e^{-2t} - \frac{2}{3}e^{-3t} + \frac{1}{6}\right) \cdot u(t)$$

Step Response : ZSR of u(t)

The Method of Balancing Singularity Function

$$\frac{d^2 y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + 1x(t)$$

Impulse Response h(t)



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