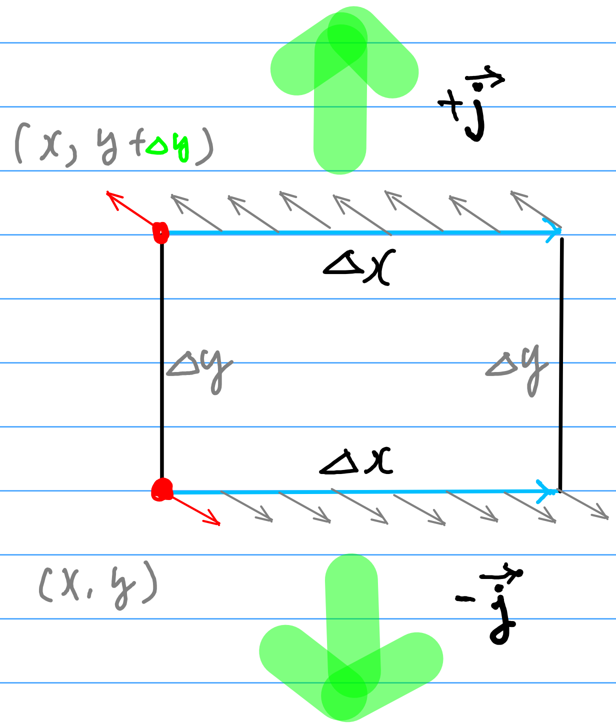
Vector Calculus (H.1) Divergence

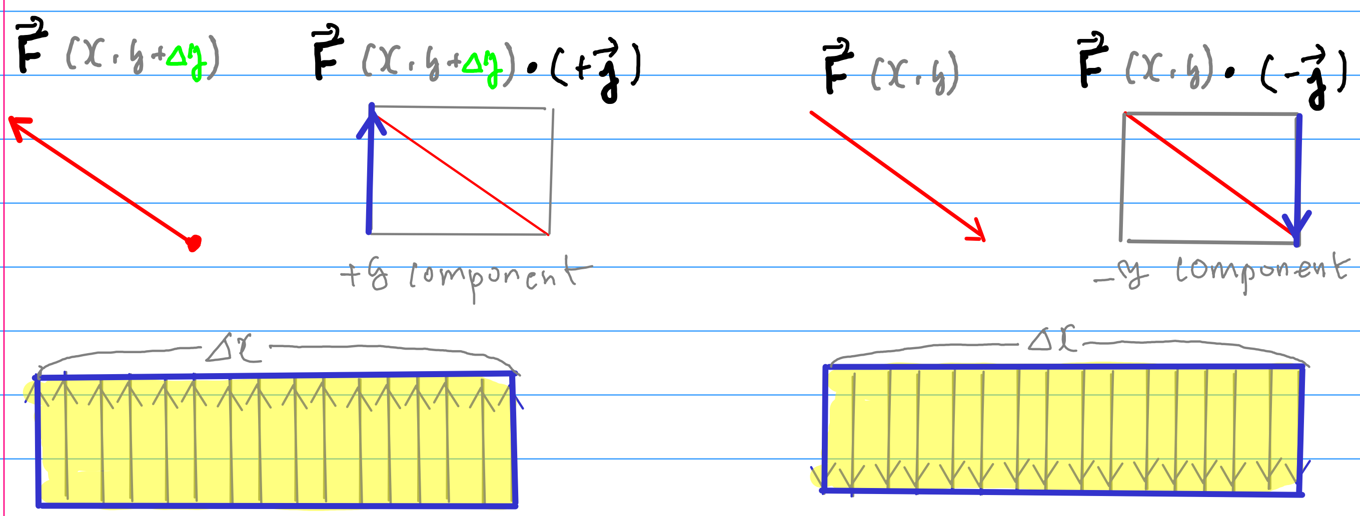
20160111

Divergence (Flux Density)

$$\boxed{\begin{array}{l}\text{Divergence (Flux Density)} \\ \quad \text{of a vector field } \vec{F} = M\vec{i} + N\vec{j} \\ \quad \text{div } \vec{F} = \left(\dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}\right)\end{array}}$$

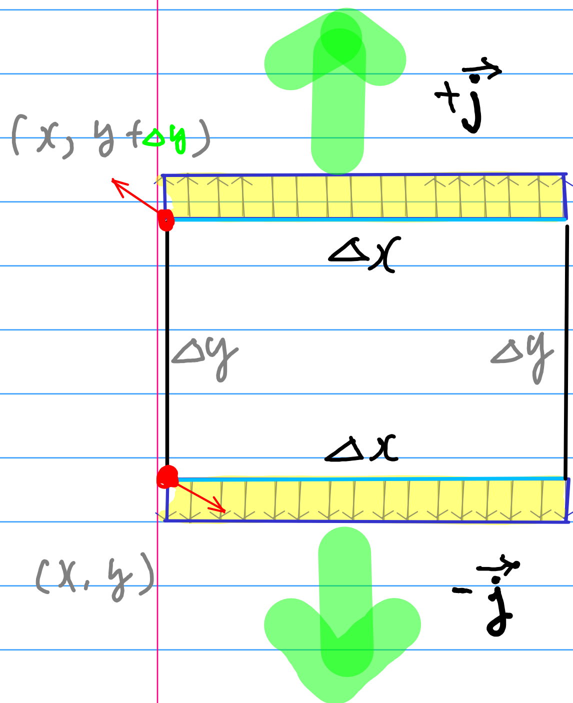
$\boxed{\vec{F}(x, y+\Delta y)\cdot(+\vec{j})\,\Delta x}$ $\boxed{\vec{F}(x, y)\cdot(-\vec{j})\,\Delta x}$





$\vec{F}(x, y+\Delta y)\cdot(+\vec{j})\,\Delta x$ $\vec{F}(x, y)\cdot(-\vec{j})\,\Delta x$

Consider the $+\vec{j}$ component of $\vec{F}$ only $\Rightarrow$ $N(x,y)$



$$\vec{F}(x, y+\Delta y)\cdot(+\vec{j})\,\Delta x$$

$$\Rightarrow N(x, y+\Delta y)\,\Delta x$$

$$\Rightarrow -N(x, y)\,\Delta x$$

$$\vec{F}(x, y)\cdot(-\vec{j})\,\Delta x$$

$$\vec{F}(x, y+\Delta y)\cdot(+\vec{j})\,\Delta x + \vec{F}(x, y)\cdot(-\vec{j})\,\Delta x$$

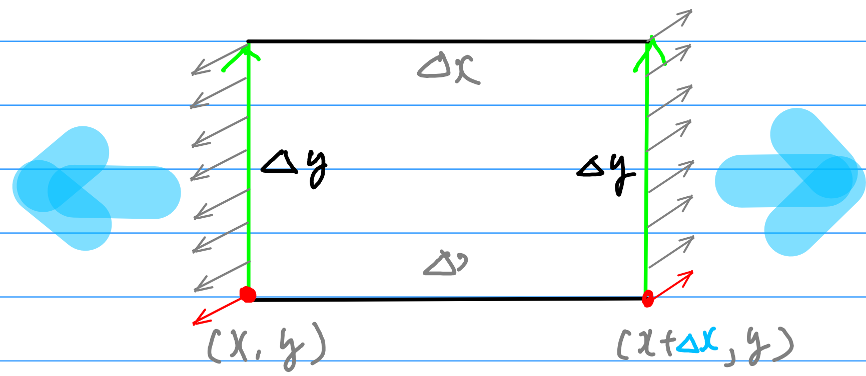
$$\Rightarrow N(x, y+\Delta y)\,\Delta x - N(x, y)\,\Delta x$$

$$\Rightarrow \left(N(x, y+\Delta y) - N(x, y)\right)\Delta x$$

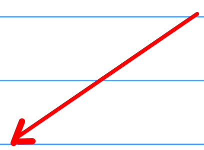
$$\Rightarrow \boxed{\left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x}$$

$$\boxed{\frac{\partial N}{\partial y} = \lim_{\Delta y \to 0}\frac{N(x, y+\Delta y) - N(x, y)}{\Delta y}}$$

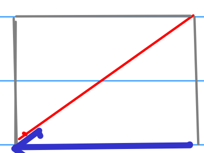
$\boxed{\vec{F}(x,y)\cdot(-\vec{i})\,\Delta y}$ $\boxed{\vec{F}(x+\Delta x,y)\cdot(+\vec{i})\,\Delta y}$



$\vec{F}(x,y)$

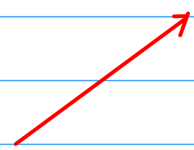


$\vec{F}(x,y)\cdot(-\vec{i})$

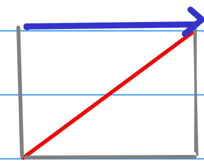


$-x$ component

$\vec{F}(x+\Delta x,y)$



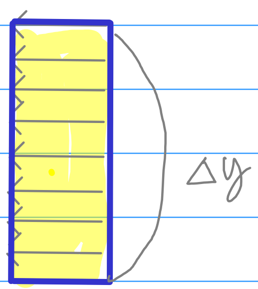
$\vec{F}(x+\Delta x,y)\cdot(+\vec{i})$

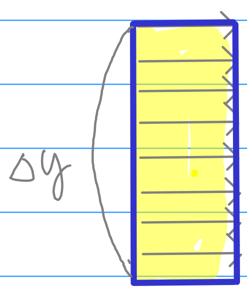


$+x$ component

$\vec{F}(x,y)\cdot(-\vec{i})\,\Delta y$

$\vec{F}(x+\Delta x,y)\cdot(+\vec{i})\,\Delta y$

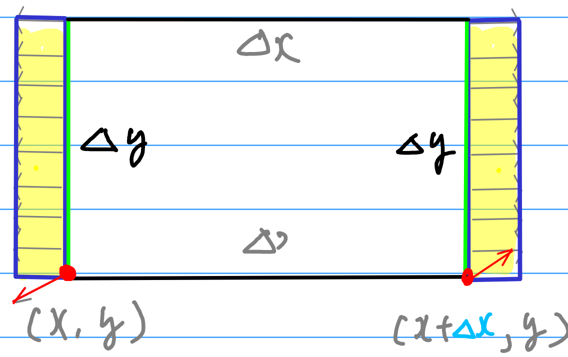




$$\boxed{\text{Consider the } +\vec{i} \text{ component of } \vec{F} \text{ only} \Rightarrow M(x,y)}$$

$\vec{F}(x,y)\cdot(-\vec{i})\,\Delta y$

$-M(x,y)\,\Delta y$



$\vec{F}(x+\Delta x,y)\cdot(+\vec{i})\,\Delta y$

$M(x+\Delta x,y)\,\Delta y$

$$\vec{F}(x+\Delta x,y)\cdot(+\vec{i})\,\Delta y + \vec{F}(x,y)\cdot(-\vec{i})\,\Delta y$$

$$\Rightarrow M(x+\Delta x,y)\,\Delta y - M(x,y)\,\Delta y$$

$$\Rightarrow \big(M(x+\Delta x,y) - M(x,y)\big)\,\Delta y$$

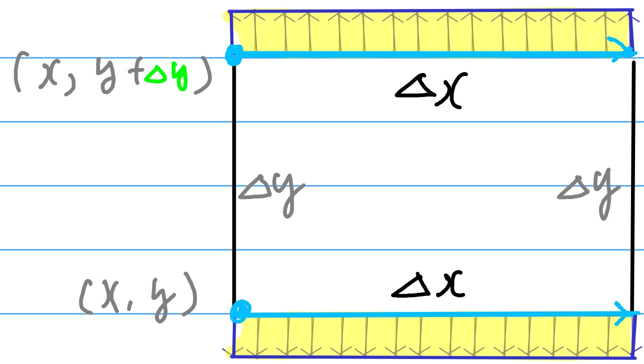
$$\Rightarrow \boxed{\left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y}$$

$$\boxed{\frac{\partial M}{\partial x} = \lim_{\Delta x \to 0} \frac{M(x+\Delta x,y) - M(x,y)}{\Delta x}}$$

Flux Density along $\vec{j}$ axis

$\vec{F}(x, y+\Delta y) \cdot (+\vec{j})\, \Delta x$

the rate at which the fluid leaves from the **top** edge



$\vec{F}(x, y) \cdot (-\vec{j})\, \Delta x$

the rate at which the fluid leaves from the **bottom** edge

$$\vec{F}(x, y+\Delta y) \cdot (+\vec{j})\, \Delta x + \vec{F}(x, y) \cdot (-\vec{j})\, \Delta x$$

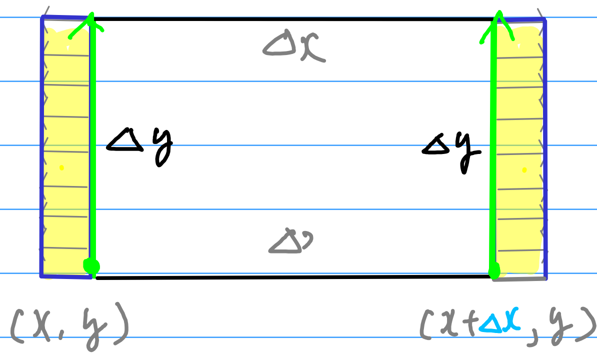
$$\Rightarrow \boxed{\left(\frac{\partial N}{\partial y} \Delta y\right) \Delta x}$$

$$\vec{F}(x, y) = M(x, y)\,\vec{i} + N(x, y)\,\vec{j}$$

Velocity field of a fluid flow in a plane

Flux density along $\vec{i}$ axis

$\vec{F}(x,y)\cdot(-\vec{i})\,\Delta y$



$\vec{F}(x+\Delta x,y)\cdot(+\vec{i})\,\Delta y$

the rate at which the fluid leaves from the left edge

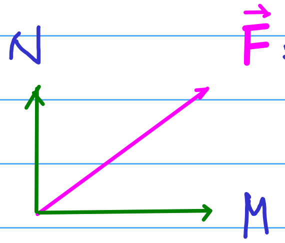
the rate at which the fluid leaves from the right edge

$$\vec{F}(x+\Delta x,y)\cdot(+\vec{i})\,\Delta y + \vec{F}(x,y)\cdot(-\vec{i})\,\Delta y$$

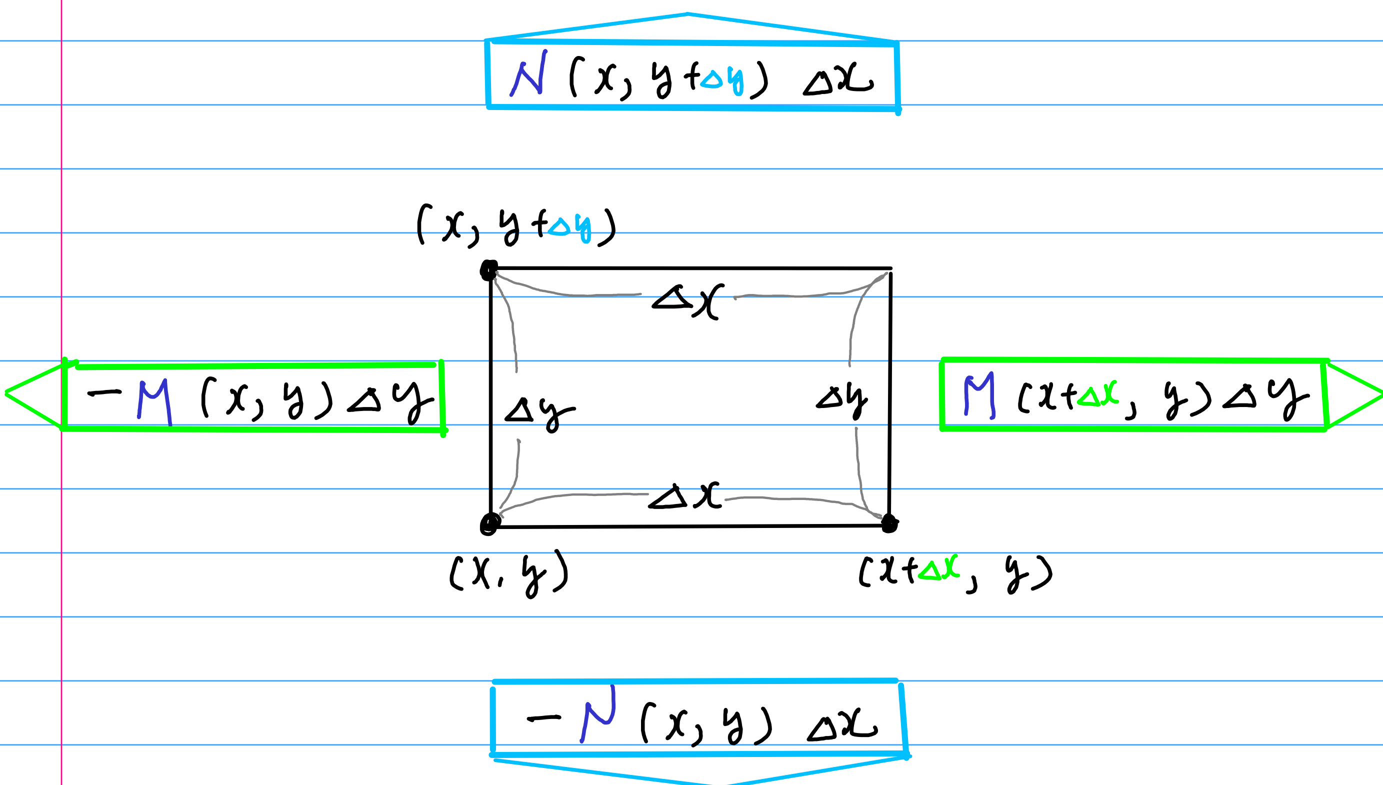
$$\Rightarrow \boxed{\left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y}$$

$$\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$$

Velocity field of a fluid flow in a plane



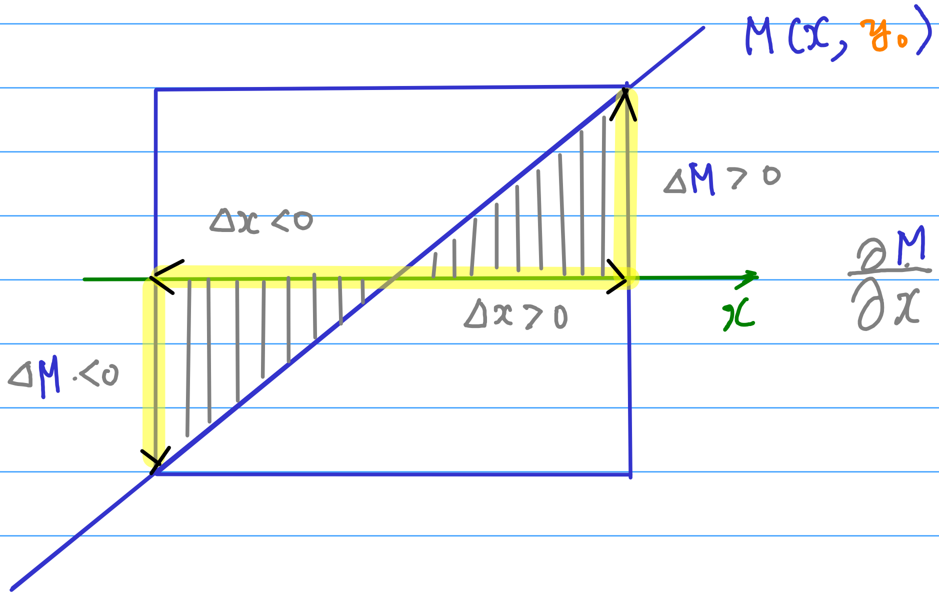
$$\vec{F} = M\vec{i} + N\vec{j} \implies \operatorname{div}\vec{F} = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$



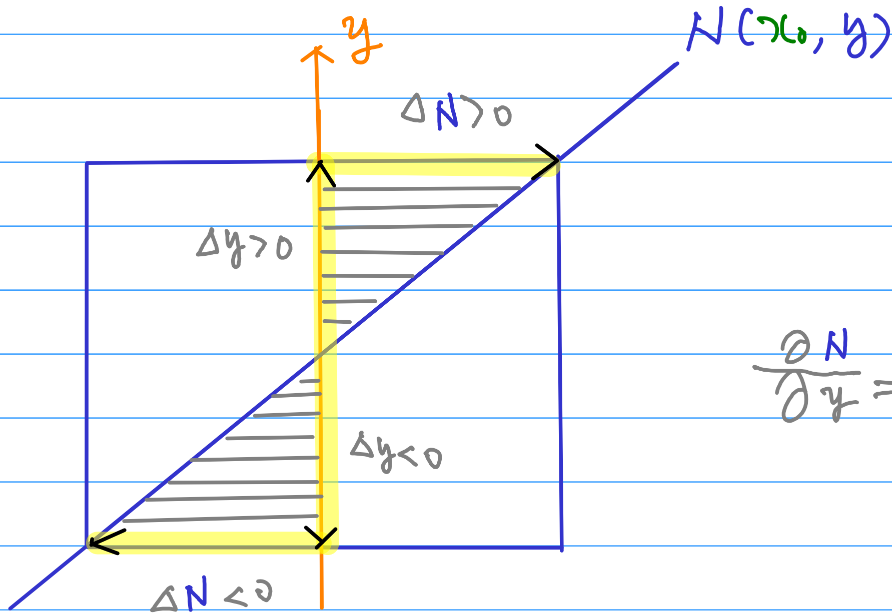
$$\left[N(x, y+\Delta y) - N(x, y)\right]\Delta x \approx \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

$$\left[M(x+\Delta x, y) - M(x, y)\right]\Delta y \approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$$

$$\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$

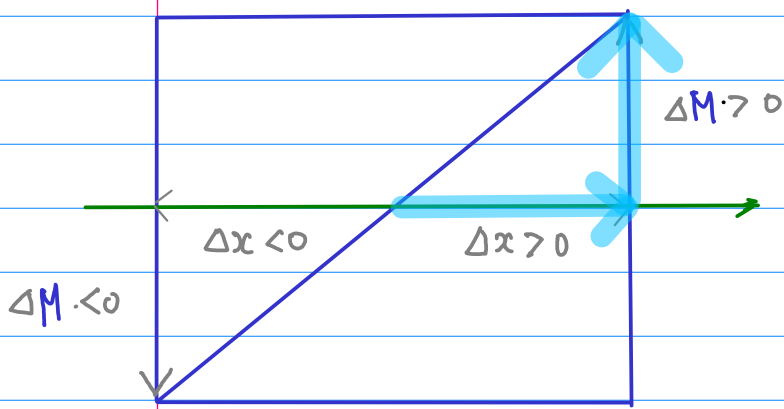


$$\frac{\partial M}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x}$$

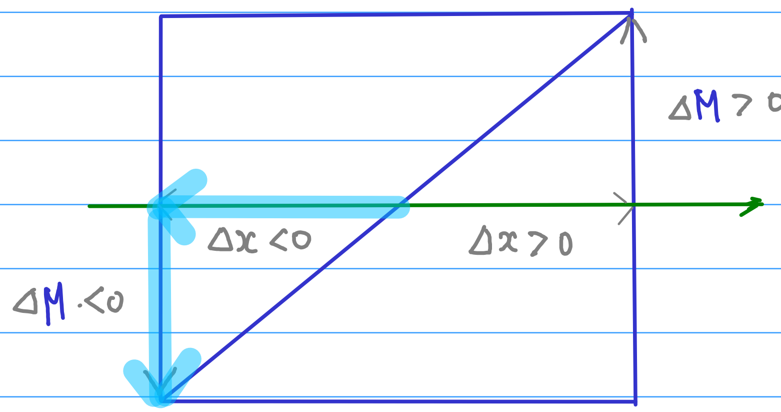


$$\frac{\partial N}{\partial y} = \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta y}$$

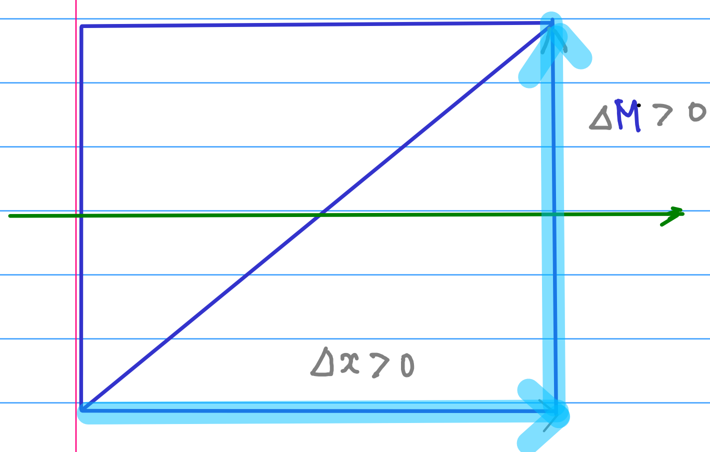
All the same $\frac{\partial M}{\partial x}$



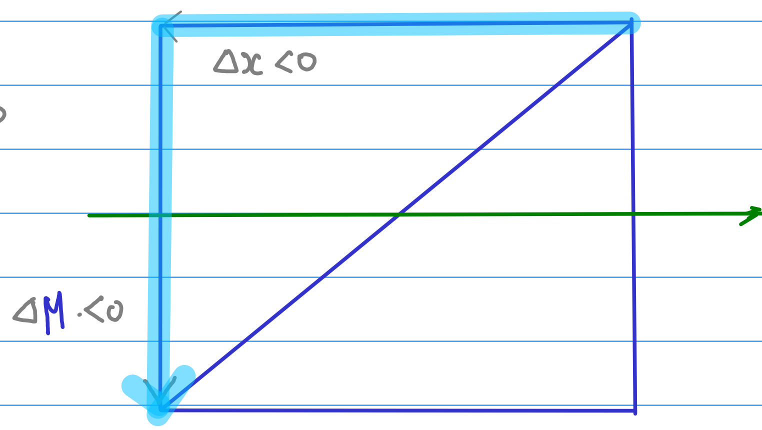
$$\frac{\partial M}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x}$$



$$\frac{\partial M}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x}$$

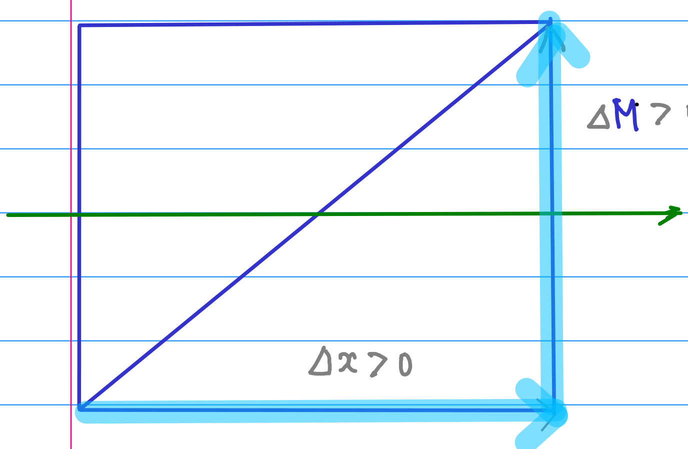


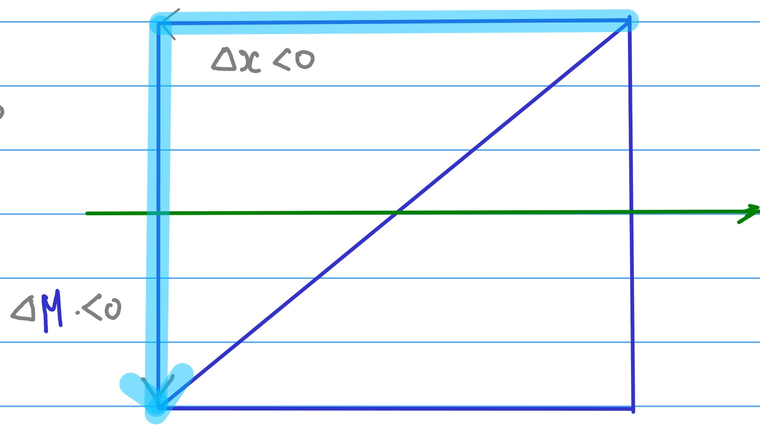
$$\frac{\partial M}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x}$$



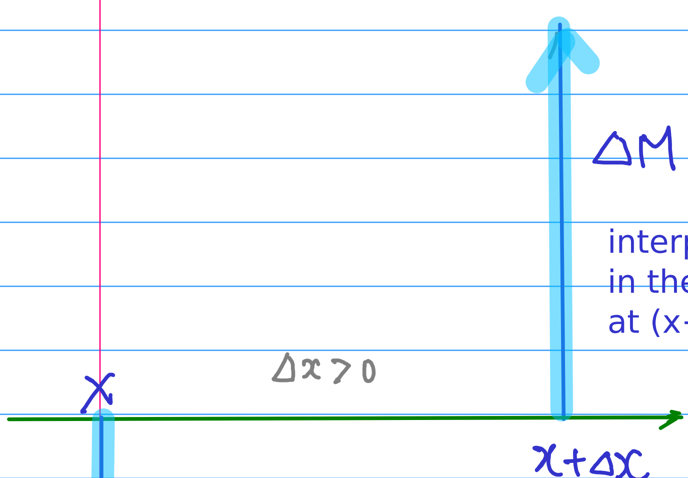
$$\frac{\partial M}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x}$$

Flux Interpretation





$$\frac{\partial M}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x} \qquad\qquad \frac{\partial M}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x}$$



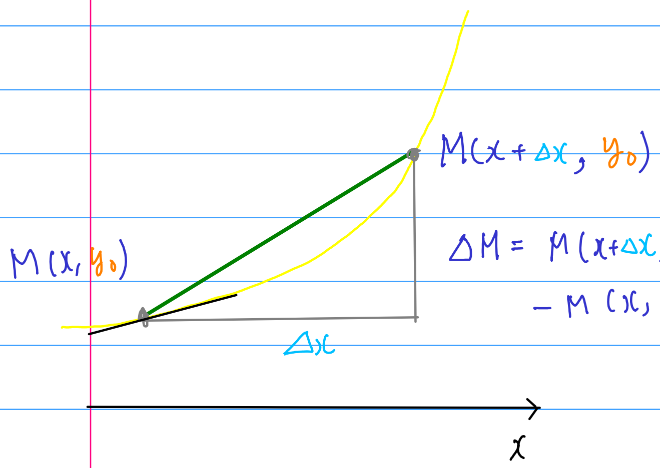
$$\Delta M \approx \frac{\partial M}{\partial x} \Delta x$$

interpret as a net outward flux
in the positive +i direction
at (x+dx, y0)

$$(-\Delta M) \approx \frac{\partial M}{\partial x}(-\Delta x)$$

interpret as a net outward flux
in the positive +i direction
at (x, y0)

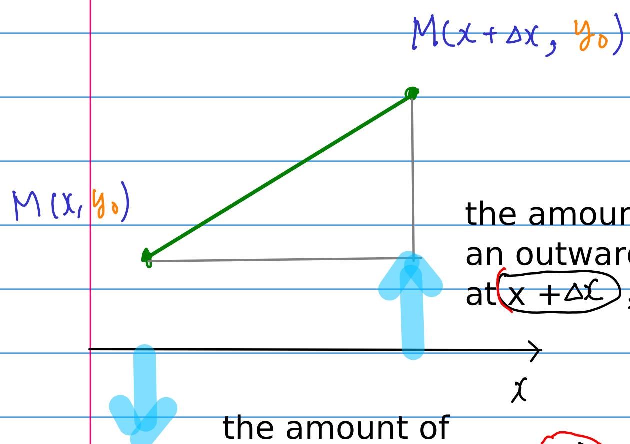
ΔM(x, y) Interpretation



$\Delta M = M(x+\Delta x, y_0) - M(x, y_0)$

the slope of a tangent :

$$\lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x} = \frac{\partial M}{\partial x}$$



the amount of an outward flux ⇒ $+\vec{i}$ direction at $(x+\Delta x, y_0)$

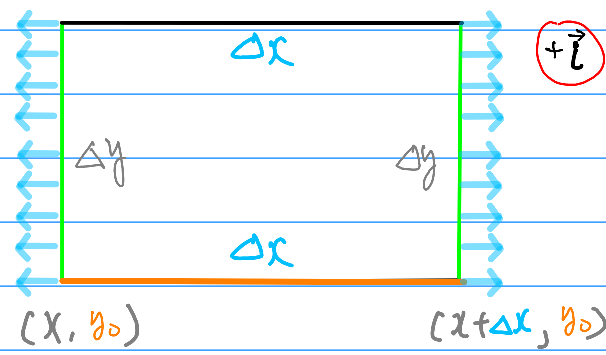
the amount of an outward flux ⇒ $-\vec{i}$ direction at (x, y_0)

the net outward flux :

$\Delta M = M(x+\Delta x, y_0) - M(x, y_0)$

$-\vec{i}$ diretion

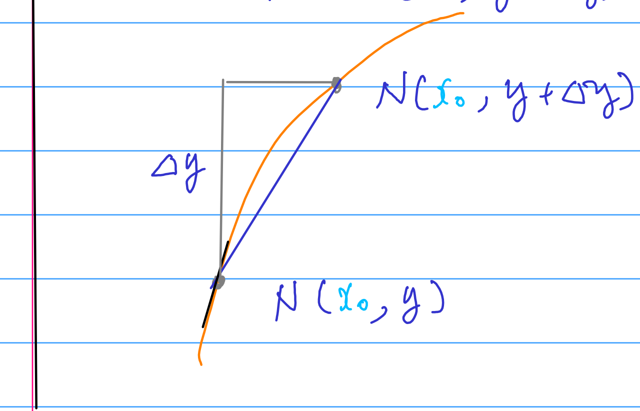
$\frac{\partial M}{\partial x}$



$+\vec{i}$ diretion

Δ N(x, y) Interpretation

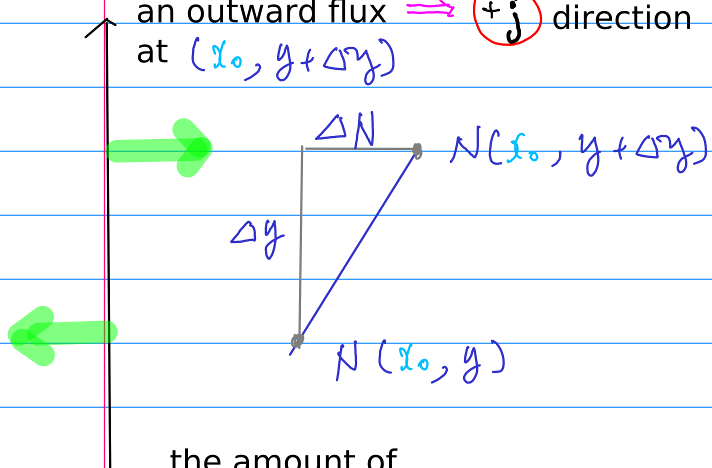
$$\Delta N = N(x_0, y+\Delta y) - N(x_0, y)$$



the slope of a tangent :

$$\lim_{\Delta y \to 0} \frac{\Delta N}{\Delta y} = \frac{\partial N}{\partial y}$$

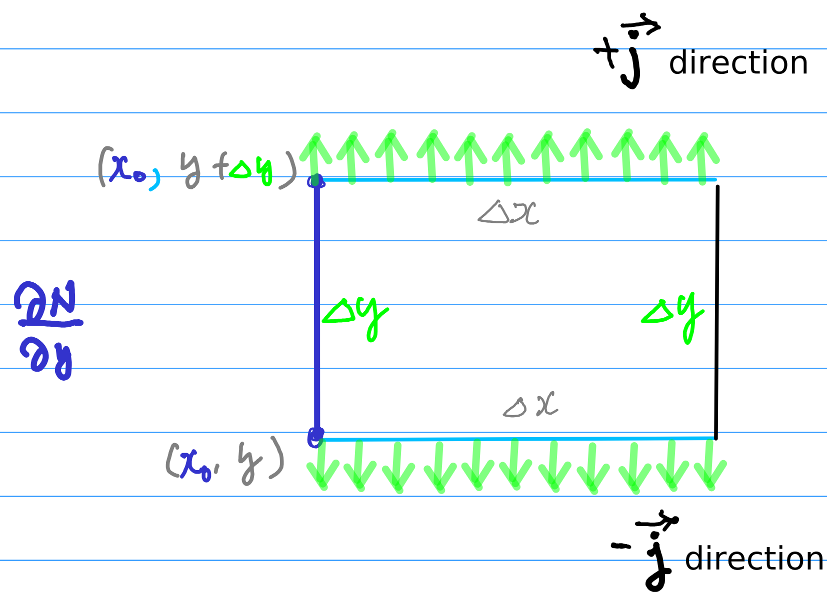
the amount of an outward flux at $(x_0, y+\Delta y)$ $\Rightarrow$ $+\vec{j}$ direction



the net outward flux :

$$\Delta N = N(x_0, y+\Delta y) - N(x_0, y)$$

the amount of an outward flux at (x_0, y) $\Rightarrow$ $-\vec{j}$ direction



Exit Rates

$\uparrow$ $\vec{F}(x, y+\Delta y) \cdot (+\vec{j})\,\Delta x = N(x, y+\Delta y)\,\Delta x$ scalar

$\downarrow$ $\vec{F}(x, y) \cdot (-\vec{j})\,\Delta x = -N(x, y)\,\Delta x$

$\rightarrow$ $\vec{F}(x+\Delta x, y) \cdot (+\vec{i})\,\Delta y = M(x+\Delta x, y)\,\Delta y$

$\leftarrow$ $\vec{F}(x, y) \cdot (-\vec{i})\,\Delta y = -M(x, y)\,\Delta y$

$$\big(N(x, y+\Delta y) - N(x, y)\big)\,\Delta x \approx \left(\frac{\partial N}{\partial y}\,\Delta y\right)\Delta x$$

$$\big(M(x+\Delta x, y) - M(x, y)\big)\,\Delta y \approx \left(\frac{\partial M}{\partial x}\,\Delta x\right)\Delta y$$

$$\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\underbrace{\Delta x\,\Delta y}_{R}$$

$$\frac{\text{Flux across rectangle boundary}}{\text{rectangle area}} = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

Divergence (Flux Density)

of a vector field $\vec{F} = M\vec{i} + N\vec{j}$

$$\operatorname{div}\vec{F} = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

$$f(x, y) = x^2 + xy + y^2$$

$$\frac{\partial f}{\partial x} = 2x + y = M(x, y)$$

$$\frac{\partial f}{\partial y} = x + 2y = N(x, y)$$

$$df = (2x+y)\,dx + (x+2y)\,dy$$ total differential

$$= M(x,y)\,dx + N(x,y)\,dy$$ $\left(\frac{\partial M}{\partial y} = 2 = \frac{\partial N}{\partial x}\right)$

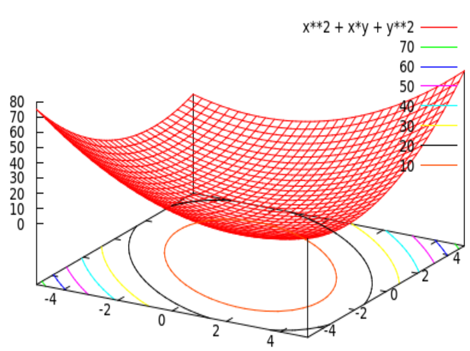
$$\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$$ gradient field

conservative field

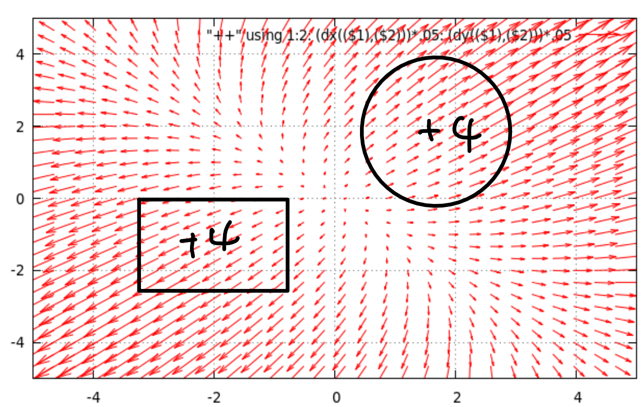
$\left(\frac{\partial M}{\partial y} = 2 = \frac{\partial N}{\partial x}\right)$

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 4$$ outward

$f(x,y) = x^2 + xy + y^2$



$\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$



$f(x, y)$ X

$$\frac{\partial f}{\partial x} = -y = M(x, y)$$

$$\frac{\partial f}{\partial y} = x = N(x, y)$$

no such f

$$\cancel{df} = (2x+y)\,dx + (x+2y)\,dy$$

$$= M(x,y)\,dx + N(x,y)\,dy$$

total differential X

$$\left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$$

$$\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$$

gradient field

conservative field X

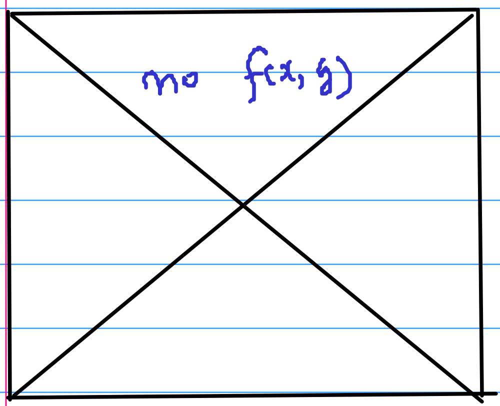
$$\left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$$

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$$

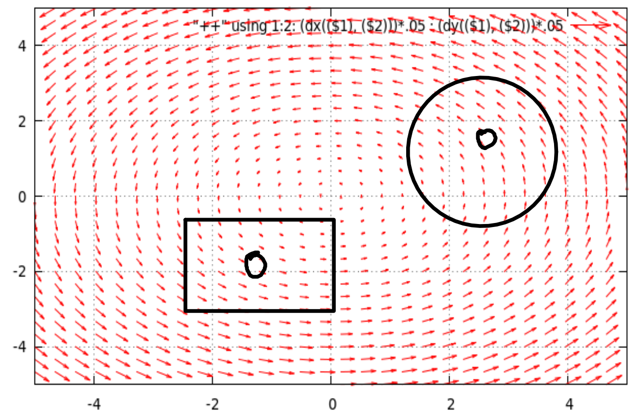
outward X

inward X

$f(x,y) = x^2 + xy + y^2$

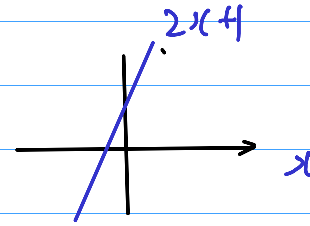


$$\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$$

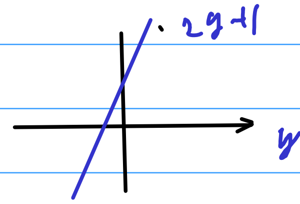


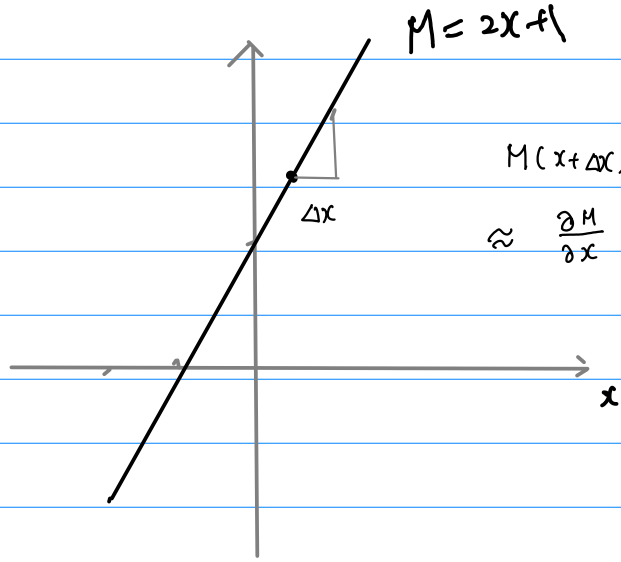
$f(1,1) = 1+1+1 = 3$

$y = 1 \Rightarrow \frac{\partial f}{\partial x} = 2x+1$



$x = 1 \Rightarrow \frac{\partial f}{\partial y} = 1+2y$





$M(x+\Delta x, y_0) - M(x, y_0)$

$\approx \frac{\partial M}{\partial x} \Delta x$