Random Signal Response of Linear Systems

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Definition

$$y(t) = \int_{-\infty}^{+\infty} x(\xi)h(t-\xi)d\xi$$

$$y(t) = \int_{-\infty}^{+\infty} h(\xi)x(t-\xi)d\xi$$

$$Y(t) = \int_{-\infty}^{+\infty} h(\xi)X(t-\xi)d\xi$$

Definition

$$E[Y(t)] = E\left[\int_{-\infty}^{+\infty} h(\xi)X(t-\xi)d\xi\right]$$

$$= \int_{-\infty}^{+\infty} h(\xi)E[X(t-\xi)]\,d\xi$$

$$= \overline{X}\int_{-\infty}^{+\infty} h(\xi)d\xi = \overline{Y}$$

Definition

$$E\left[\int_{t_1}^{t_2} W(t)h(t)dt\right] = \int_{t_1}^{t_2} E\left[W(t)\right]h(t)dt$$

$$\int_{t_1}^{t_2} E\left[W(t)\right]|h(t)|dt < \infty$$

Definition

$$E\left[Y^2(t)\right] = E\left[\int_{-\infty}^{+\infty} h(\xi_1)X(t-\xi_1)d\xi_1 \int_{-\infty}^{+\infty} h(\xi_2)X(t-\xi_2)d\xi_2\right]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E\left[X(t-\xi_1)X(t-\xi_2)\right] h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$$

$$\overline{Y^2} = E\left[Y^2(t)\right] == \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{XX}(\xi_1-\xi_2)h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$$

Definition

$$R_{YY}(t, t+\tau) = E\left[Y(t)Y(t+\tau)\right]$$

$$= E\left[\int_{-\infty}^{+\infty} h(\xi_1)X(t-\xi_1)d\xi_1 \int_{-\infty}^{+\infty} h(\xi_2)X(t+\tau-\xi_2)d\xi_2\right]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E\left[X(t-\xi_1)X(t+\tau-\xi_2)\right] h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$$

$$R_{YY}(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{XX}(\tau+\xi_1-\xi_2)h(\xi_1)h(\xi_2)d\xi_1 d\xi_2$$

$$R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau)$$

Definition

$$R_{XY}(t, t+\tau) = E\left[X(t)Y(t+\tau)\right]$$

$$= E\left[X(t)\int_{-\infty}^{+\infty} h(\xi_1)X(t+\tau-\xi_1)d\xi_1\right]$$

$$= \int_{-\infty}^{+\infty} E\left[X(t)X(t+\tau-\xi_1)\right]h(\xi_1)d\xi_1$$

Definition

$$R_{YY}(\tau) = \int_{-\infty}^{+\infty} R_{XY}(\tau - \xi_1) h(-\xi_1) d\xi_1$$

$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)$$

$$R_{YX}(\tau) = \int_{-\infty}^{+\infty} R_{YX}(\tau - \xi_2) h(\xi_2) d\xi_2$$

$$R_{YY}(\tau) = R_{YX}(\tau) * h(\tau)$$

Definition

$$R_{XY}(t,t+\tau) == \int_{-\infty}^{+\infty} E\left[X(t)X(t+\tau-\xi_1)\right] h(\xi_1) d\xi_1$$

$$R_{XY}(\tau) = \int_{-\infty}^{+\infty} R_{XX}(\tau-\xi_1) h(\xi_1) d\xi_1$$

$$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$$

$$R_{YX}(\tau) = \int_{-\infty}^{+\infty} R_{XX}(\tau-\xi_1) h(-\xi_1) d\xi_1$$

$$R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau)$$

Definition

$$Y[n] = \sum_{m=-\infty}^{+\infty} X[m]h[n-m]$$

$$Y[n] = \sum_{m=-\infty}^{+\infty} h[m]X[n-m]$$

Definition

$$E\left[Y[n]\right] = E\left[\sum_{m=-\infty}^{+\infty} h[m]X[n-m]\right]$$

$$= \sum_{m=-\infty}^{+\infty} h[m]E\left[X[n-m]\right]$$

$$= \overline{X} \sum_{m=-\infty}^{+\infty} h[m]$$

$$= \overline{Y}$$

Definition

$$R_{XY}[n, n+m] = E[X[n]Y[n+m]]$$

$$= E\left[X[n] \sum_{k=-\infty}^{+\infty} h[k]X[n+m-k]\right]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]E[X[n]X[n+m-k]]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]R_{XX}[m-k]$$

$$R_{XY}[m] = R_{XX}[m] * h[m]$$

Definition

$$R_{XY}[n, n+m] == \sum_{k=-\infty}^{+\infty} h[m] R_{XX}[m-k]$$

$$R_{YX}[n, n+m] == \sum_{k=-\infty}^{+\infty} h[-k] R_{XX}[m-k]$$

$$R_{XY}[n, n+m] == h[m] * R_{XX}[m]$$

$$R_{YX}[n, n+m] == h[-m] * R_{XX}[m]$$

Definition

$$R_{YY}[n,n+m] = E\left[Y[n]Y[n+m]\right]$$

$$= E\left[Y[n+m]\sum_{k=-\infty}^{+\infty} h[k]X[n-k]\right]$$

$$= E\left[\sum_{k=-\infty}^{+\infty} h[k]X[n-k]Y[n+m]\right]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]E\left[X[n-k]Y[n+m]\right]$$

substituting $t = n - k$

Definition

$$R_{YY}[n, n+m] = \sum_{k=-\infty}^{+\infty} h[k] E\left[X[n-k] Y[n+m]\right]$$

substituting $t = n - k$

$$R_{YY}[m] = \sum_{k=-\infty}^{+\infty} h[k] E\left[X[t] Y[t+(m+k)]\right]$$

$$= \sum_{k=-\infty}^{+\infty} h[k] R_{XY}[m+k]$$

$$= h[-m] * R_{XY}[m]$$

Definition

$$R_{YY}[m] == h[-m] * R_{XY}[m]$$

$$R_{YY}[m] == h[m] * R_{YX}[m]$$

$$R_{YY}[m] == h[-m] * h[m] * R_{XX}[m]$$

Definition

$$E\left[Y^2[n]\right] = E\left[\sum_{k=-\infty}^{+\infty} h[k]X[n-k] \sum_{m=-\infty}^{+\infty} h[m]X[n-m]\right]$$

$$= E\left[\sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} h[k]h[m]X[n-k]X[n-m]\right]$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} h[k]h[m]E\left[X[n-k]X[n-m]\right]$$

assume WSS

Definition

$$E\left[Y^2[n]\right] = \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} h[k]h[m]E\left[X[n-k]X[n-m]\right]$$

assum WSS

$$E\left[X[n-k]X[n-m]\right] = R_{XX}[k-m]$$

$$E\left[Y^2[n]\right] = \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} h[k]h[m]R_{XX}[k-m] = \overline{Y^2}$$