

Random Signal Response of Linear Systems

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Discrete Time System

N Gaussian random variables

Definition

$$y(t) = \int_{-\infty}^{+\infty} x(\xi)h(t - \xi)d\xi$$

$$y(t) = \int_{-\infty}^{+\infty} h(\xi)x(t - \xi)d\xi$$

$$Y(t) = \int_{-\infty}^{+\infty} h(\xi)X(t - \xi)d\xi$$

Mean Value of a System Response

N Gaussian random variables

Definition

$$\begin{aligned} E[Y(t)] &= E \left[\int_{-\infty}^{+\infty} h(\xi) X(t - \xi) d\xi \right] \\ &= \int_{-\infty}^{+\infty} h(\xi) E[X(t - \xi)] d\xi \\ &= \bar{X} \int_{-\infty}^{+\infty} h(\xi) d\xi = \bar{Y} \end{aligned}$$

Interchanging expectation and integration operations

N Gaussian random variables

Definition

$$E \left[\int_{t_1}^{t_2} W(t)h(t)dt \right] = \int_{t_1}^{t_2} E[W(t)]h(t)dt$$

$$\int_{t_1}^{t_2} E[W(t)]|h(t)|dt < \infty$$

Mean Square Value of a System Response

N Gaussian random variables

Definition

$$E[Y^2(t)] = E \left[\int_{-\infty}^{+\infty} h(\xi_1) X(t - \xi_1) d\xi_1 \int_{-\infty}^{+\infty} h(\xi_2) X(t - \xi_2) d\xi_2 \right]$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E[X(t - \xi_1) X(t - \xi_2)] h(\xi_1) h(\xi_2) d\xi_1 d\xi_2$$

$$\overline{Y^2} = E[Y^2(t)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R_{XX}(\xi_1 - \xi_2) h(\xi_1) h(\xi_2) d\xi_1 d\xi_2$$

Autocorrelation Function of a System Response

N Gaussian random variables

Definition

$$\begin{aligned}R_{YY}(t, t + \tau) &= E[Y(t)Y(t + \tau)] \\&= E \left[\int_{-\infty}^{+\infty} h(\xi_1)X(t - \xi_1)d\xi_1 \int_{-\infty}^{+\infty} h(\xi_2)X(t + \tau - \xi_2)d\xi_2 \right] \\&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E[X(t - \xi_1)X(t + \tau - \xi_2)] h(\xi_1)h(\xi_2)d\xi_1 d\xi_2 \\R_{YY}(\tau) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R_{XX}(\tau + \xi_1 - \xi_2)h(\xi_1)h(\xi_2)d\xi_1 d\xi_2 \\R_{YY}(\tau) &= R_{XX}(\tau) * h(-\tau) * h(\tau)\end{aligned}$$

Crosscorrelation Function of Input and Output (1)

N Gaussian random variables

Definition

$$\begin{aligned} R_{XY}(t, t + \tau) &= E[X(t)Y(t + \tau)] \\ &= E \left[X(t) \int_{-\infty}^{+\infty} h(\xi_1) X(t + \tau - \xi_1) d\xi_1 \right] \\ &= \int_{-\infty}^{+\infty} E[X(t)X(t + \tau - \xi_1)] h(\xi_1) d\xi_1 \end{aligned}$$

Crosscorrelation Function of Input and Output (3)

N Gaussian random variables

Definition

$$R_{YY}(\tau) = \int_{-\infty}^{+\infty} R_{XY}(\tau - \xi_1) h(-\xi_1) d\xi_1$$

$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)$$

$$R_{YX}(\tau) = \int_{-\infty}^{+\infty} R_{YX}(\tau - \xi_2) h(\xi_2) d\xi_2$$

$$R_{YX}(\tau) = R_{YX}(\tau) * h(\tau)$$

Crosscorrelation Function of Input and Output (4)

N Gaussian random variables

Definition

$$R_{XY}(t, t + \tau) = \int_{-\infty}^{+\infty} E[X(t)X(t + \tau - \xi_1)] h(\xi_1) d\xi_1$$

$$R_{XY}(\tau) = \int_{-\infty}^{+\infty} R_{XX}(\tau - \xi_1) h(\xi_1) d\xi_1$$

$$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$$

$$R_{YX}(\tau) = \int_{-\infty}^{+\infty} R_{XX}(\tau - \xi_1) h(-\xi_1) d\xi_1$$

$$R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau)$$

Discrete Time LTI System

N Gaussian random variables

Definition

$$Y[n] = \sum_{m=-\infty}^{+\infty} X[m]h[n-m]$$

$$Y[n] = \sum_{m=-\infty}^{+\infty} h[m]X[n-m]$$

Mean of Discrete Time LTI System Response

N Gaussian random variables

Definition

$$\begin{aligned} E[Y[n]] &= E \left[\sum_{m=-\infty}^{+\infty} h[m]X[n-m] \right] \\ &= \sum_{m=-\infty}^{+\infty} h[m]E[X[n-m]] \\ &= \bar{X} \sum_{m=-\infty}^{+\infty} h[m] \\ &= \bar{Y} \end{aligned}$$

Cross Correlation of Discrete Time LTI System Response

N Gaussian random variables

Definition

$$\begin{aligned}R_{XY}[n, n+m] &= E[X[n]Y[n+m]] \\&= E\left[X[n] \sum_{k=-\infty}^{+\infty} h[k]X[n+m-k]\right] \\&= \sum_{k=-\infty}^{+\infty} h[k]E[X[n]X[n+m-k]] \\&= \sum_{k=-\infty}^{+\infty} h[k]R_{XX}[m-k] \\R_{XY}[m] &= R_{XX}[m] * h[m]\end{aligned}$$

Cross-Correlation of Discrete Time LTI System Response

N Gaussian random variables

Definition

$$R_{XY}[n, n+m] == \sum_{k=-\infty}^{+\infty} h[m] R_{XX}[m-k]$$

$$R_{YX}[n, n+m] == \sum_{k=-\infty}^{+\infty} h[-k] R_{XX}[m-k]$$

$$R_{XY}[n, n+m] == h[m] * R_{XX}[m]$$

$$R_{YX}[n, n+m] == h[-m] * R_{XX}[m]$$

Auto-Correlation of Discrete Time LTI System Response (1)

N Gaussian random variables

Definition

$$\begin{aligned}R_{YY}[n, n+m] &= E[Y[n]Y[n+m]] \\&= E\left[Y[n+m] \sum_{k=-\infty}^{+\infty} h[k]X[n-k]\right] \\&= E\left[\sum_{k=-\infty}^{+\infty} h[k]X[n-k]Y[n+m]\right] \\&= \sum_{k=-\infty}^{+\infty} h[k]E[X[n-k]Y[n+m]]\end{aligned}$$

substituting $t = n - k$

Auto-Correlation of Discrete Time LTI System Response (2)

N Gaussian random variables

Definition

$$R_{YY}[n, n+m] = \sum_{k=-\infty}^{+\infty} h[k] E[X[n-k]Y[n+m]]$$

substituting $t = n - k$

$$\begin{aligned} R_{YY}[m] &= \sum_{k=-\infty}^{+\infty} h[k] E[X[t]Y[t+(m+k)]] \\ &= \sum_{k=-\infty}^{+\infty} h[k] R_{XY}[m+k] \\ &= h[-m] * R_{XY}[m] \end{aligned}$$

Auto-Correlation of Discrete Time LTI System Response (3)

N Gaussian random variables

Definition

$$R_{YY}[m] == h[-m] * R_{XY}[m]$$

$$R_{YY}[m] == h[m] * R_{YX}[m]$$

$$R_{YY}[m] == h[-m] * h[m] * R_{XX}[m]$$

Mean Square Value of Discrete Time LTI System Response

(1)

N Gaussian random variables

Definition

$$\begin{aligned} E[Y^2[n]] &= E \left[\sum_{k=-\infty}^{+\infty} h[k]X[n-k] \sum_{m=-\infty}^{+\infty} h[m]X[n-m] \right] \\ &= E \left[\sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} h[k]h[m]X[n-k]X[n-m] \right] \\ &= \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} h[k]h[m]E[X[n-k]X[n-m]] \end{aligned}$$

assume WSS

Mean Square Value of Discrete Time LTI System Response

(2)

N Gaussian random variables

Definition

$$E[Y^2[n]] = \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} h[k]h[m]E[X[n-k]X[n-m]]$$

assum WSS

$$E[X[n-k]X[n-m]] = R_{XX}[k-m]$$

$$E[Y^2[n]] = \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} h[k]h[m]R_{XX}[k-m] = \overline{Y^2}$$

