# Propositional Logic (2A)

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# Formal Language

Formal Language :

a set of words / expressions

using alphabet and rules

Well-formed strings of symbols

Alphabet :

The set of symbols

Rules :

The syntax of the language

# Propositional Logic

Consists of a formal language and semantics

That give meaning  to the well-formed strings

(propositions)

# Alphabet of Propositional Logic

1. the English alphabet letters
2. the logical value T and F
3. special symbols

| | |
|---|---|
| ¬ | NOT |
| ∧ | AND |
| ∨ | OR |
| → | if-then |
| ↔ | if and only if |
| ( ) | grouping |

# Rules of Propositional Logic

1. Atomic Propositions

    All letters, all indexed letters, T & F

2. Compound Propositions

    If *A* and *B* are propositions,

    $\neg A$, $A \wedge B$, $A \vee B$, $A \rightarrow B$, $A \leftrightarrow B$, $( A )$ are all propositions

Italic fonts for propositions :

Denote variable whose value may be

atomic or compound propositions.

# Semantics of Propositional Logic

The semantics gives meaning to the propositions

The semantics consists of rules

for assigning either T or F to every proposition

(truth value)

Rules:

1. The logical value true ← the value T

   The logical value false ← the value F

2. Atomic proposition ← either T or F
3. the truth tables of connectives
4. (a) the grouping ()

   (b) the precedence order (¬, ∧, ∨, →, ↔)

   (c) left to right

# Tautology and Logical Implication

Tautology:

a proposition that is true in all possible world

Contradiction:

A proposition that is false in all possible world

Logically Equivalent :

If A ↔ B is a tautology,

A and B are logically equivalent A

# Some Logical Equivalences

from en.wikipedia.org

| ***Equivalence*** | ***Name*** |
|---|---|
| $p \wedge \mathbf{T} \equiv p$<br>$p \vee \mathbf{F} \equiv p$ | Identity laws |
| $p \vee \mathbf{T} \equiv \mathbf{T}$<br>$p \wedge \mathbf{F} \equiv \mathbf{F}$ | Domination laws |
| $p \vee p \equiv p$<br>$p \wedge p \equiv p$ | Idempotent laws |
| $\neg(\neg p) \equiv p$ | Double negation law |
| $p \vee q \equiv q \vee p$<br>$p \wedge q \equiv q \wedge p$ | Commutative laws |
| $(p \vee q) \vee r \equiv p \vee (q \vee r)$<br>$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | Associative laws |
| $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$<br>$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | Distributive laws |
| $\neg(p \wedge q) \equiv \neg p \vee \neg q$<br>$\neg(p \vee q) \equiv \neg p \wedge \neg q$ | De Morgan's laws |
| $p \vee (p \wedge q) \equiv p$<br>$p \wedge (p \vee q) \equiv p$ | Absorption laws |
| $p \vee \neg p \equiv \mathbf{T}$<br>$p \wedge \neg p \equiv \mathbf{F}$ | Negation laws |

# Logical Arguments

An argument consists of

A set of propositions (premises) and

A proposition (conclusion)

The premises  entail the conclusion

If in every model in which all the premises are true,

the conclusion is also true

The argument is sound:

If the premises entail the conclusion

Otherwise, the argument is a fallacy

# Derivation Systems

To prove whether an argument is sound or a fallacy

Using truth tables is too difficult

- n premises
- $2^n$ rows in the truth table
- Exponential time complexity
- Not like human

Using inference rules

| | | |
|---|---|---|
| $\dfrac{p \to q}{\therefore \neg p \lor q}$ | $(p \to q) \to (\neg p \lor q)$ | Material implication |
| $\dfrac{(p \lor q) \land r}{\therefore (p \land r) \lor (q \land r)}$ | $((p \lor q) \land r) \to ((p \land r) \lor (q \land r))$ | Distributive |
| $\dfrac{p \to q}{\therefore p \to (p \land q)}$ | $(p \to q) \to (p \to (p \land q))$ | Absorption |
| $\dfrac{p \lor q \quad \neg p}{\therefore q}$ | $((p \lor q) \land \neg p) \to q$ | Disjunctive syllogism |
| $\dfrac{p}{\therefore p \lor q}$ | $p \to (p \lor q)$ | Addition |
| $\dfrac{p \land q}{\therefore p}$ | $(p \land q) \to p$ | Simplification |
| $\dfrac{p \quad q}{\therefore p \land q}$ | $((p) \land (q)) \to (p \land q)$ | Conjunction |
| $\dfrac{p}{\therefore \neg\neg p}$ | $p \to (\neg\neg p)$ | Double negation |
| $\dfrac{p \lor p}{\therefore p}$ | $(p \lor p) \to p$ | Disjunctive simplification |
| $\dfrac{p \lor q \quad \neg p \lor r}{\therefore q \lor r}$ | $((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$ | Resolution |

# Inference Rules

| Rules of inference | Tautology | Name |
|---|---|---|
| $p$ <br> $p \to q$ <br> $\therefore \overline{q}$ | $((p \land (p \to q)) \to q$ | Modus ponens |
| $\neg q$ <br> $p \to q$ <br> $\therefore \overline{\neg p}$ | $((\neg q \land (p \to q)) \to \neg p$ | Modus tollens |
| $(p \lor q) \lor r$ <br> $\therefore \overline{p \lor (q \lor r)}$ | $((p \lor q) \lor r) \to (p \lor (q \lor r))$ | Associative |
| $p \land q$ <br> $\therefore \overline{q \land p}$ | $(p \land q) \to (q \land p)$ | Commutative |
| $p \to q$ <br> $q \to p$ <br> $\therefore \overline{p \leftrightarrow q}$ | $((p \to q) \land (q \to p)) \to (p \leftrightarrow q)$ | Law of biconditional propositions |
| $(p \land q) \to r$ <br> $\therefore \overline{p \to (q \to r)}$ | $((p \land q) \to r) \to (p \to (q \to r))$ | Exportation |
| $p \to q$ <br> $\therefore \overline{\neg q \to \neg p}$ | $(p \to q) \to (\neg q \to \neg p)$ | Transposition or contraposition law |
| $p \to q$ <br> $q \to r$ <br> $\therefore \overline{p \to r}$ | $((p \to q) \land (q \to r)) \to (p \to r)$ | Hypothetical syllogism |

# Sound and Complete

Deduction system

A set of inference rules

A deduction system is sound

If it only derives sound arguments

A deduction system is complete

If it can derive every sound argument

# Implication

$P \longrightarrow Q$

If P, then Q.

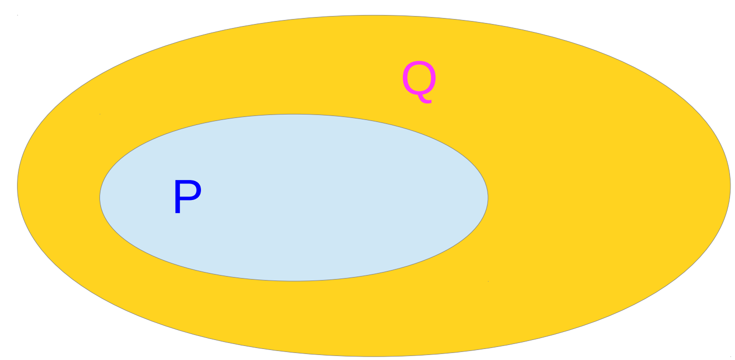
P implies Q.

P only if Q.

Q whenever P.

P is sufficient for Q.

Q is necessary P.




http://en.wikipedia.org/wiki/

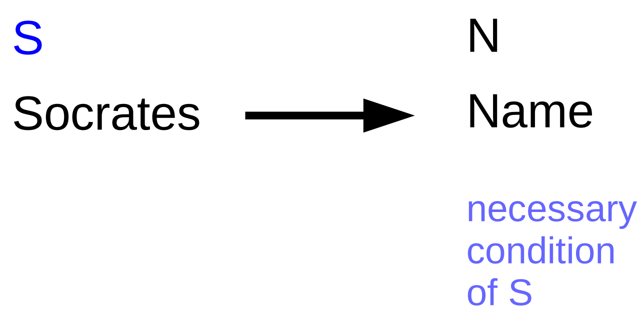
P only if Q        not P if not Q

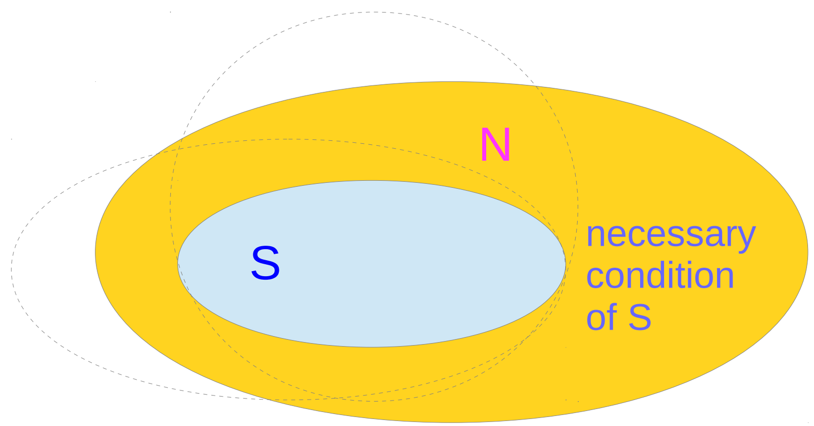
# Necessity

A true necessary condition in a conditional statement makes the statement true.

a consequent N is a necessary condition for an antecedent S, in the conditional statement,
"N if S ",
"N is implied by S ", or
“N ← S”.

"N is weaker than S " or "S cannot occur without N ". For example, it is necessary to be Named, to be called "Socrates".




http://en.wikipedia.org/wiki/

S only if N    not S if not N

# Sufficiency

A true sufficient condition in a conditional statement ties the statement's truth to its consequent.

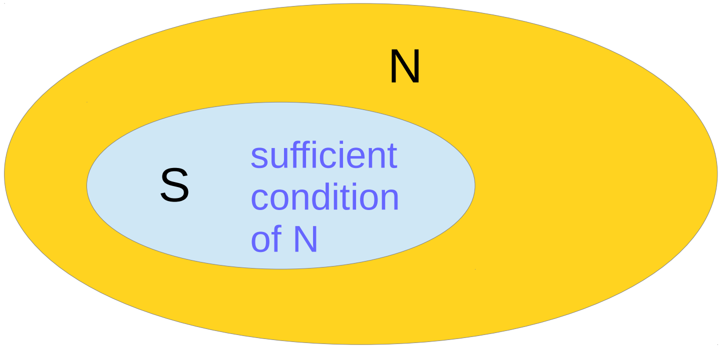
an antecedent S is a sufficient condition for a consequent N, in the conditional statement,
"if S, then N ",
"S implies N ", or
“S → N”

"S is stronger than N " or "S guarantees N ". For example, "Socrates" suffices for a Name.

S N

Socrates → Name

sufficient
condition
of N




http://en.wikipedia.org/wiki/Derivative

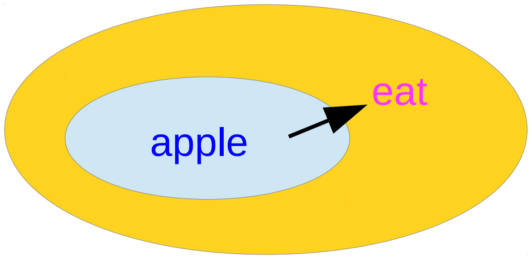
# IF / Only IF

"Madison will eat the fruit if it is an apple."

"Only if Madison will eat the fruit, is it an apple;"

"Madison will eat the fruit ← fruit is an apple"

- This states simply that Madison will eat fruits that are apples.
- It does not, however, exclude the possibility that Madison might also eat bananas or other types of fruit.
- All that is known for certain is that she will eat any and all apples that she happens upon.
- That the fruit is an apple is **a sufficient condition** for Madison to eat the fruit.

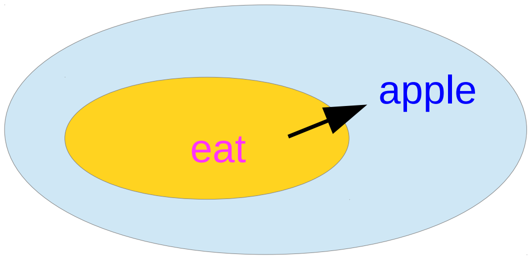



"Madison will eat the fruit only if it is an apple."

"If Madison will eat the fruit, then it is an apple"

"Madison will eat the fruit → fruit is an apple"

- This states that the only fruit Madison will eat is an apple.
- It does not, however, exclude the possibility that Madison will refuse an apple if it is made available
- in contrast with (1), which requires Madison to eat any available apple.
- In this case, that a given fruit is an apple is **a necessary condition** for Madison eating it.
- It is not a sufficient condition since Madison might not eat all the apples she is given.




http://en.wikipedia.org/wiki/Derivative

# IFF

"Madison will eat the fruit if and only if it is an apple"

"Madison will eat the fruit ↔ fruit is an apple"

- This statement makes it clear that Madison will eat all and only those fruits that are apples.
- She will not leave any apple uneaten, and
- she will not eat any other type of fruit.
- That a given fruit is an apple is both a necessary and a sufficient condition for Madison to eat the fruit.

http://en.wikipedia.org/wiki/Derivative

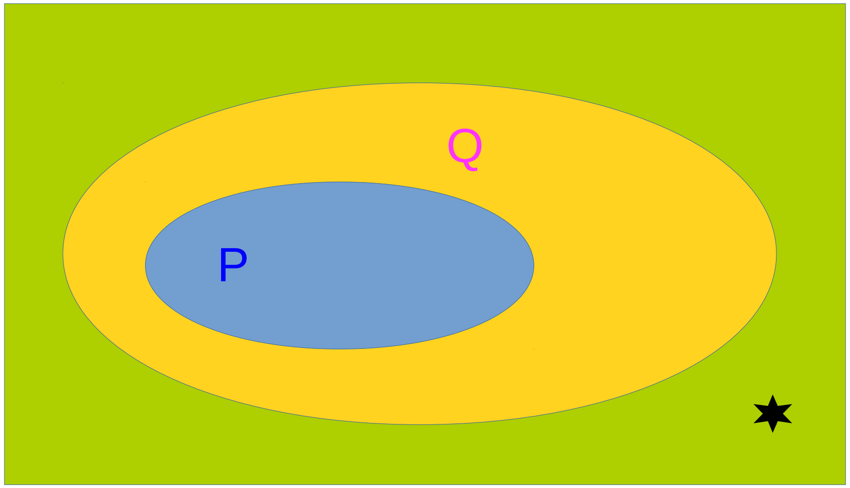
# To prove implications by contradiction

P ⟶ Q

Assume this is false

$$\neg(\neg p \vee q)$$

$$p \wedge \neg q$$




Assume P is true and Q is false

Derive contradiction

http://en.wikipedia.org/wiki/Derivative

# Indirect Proof

contradiction

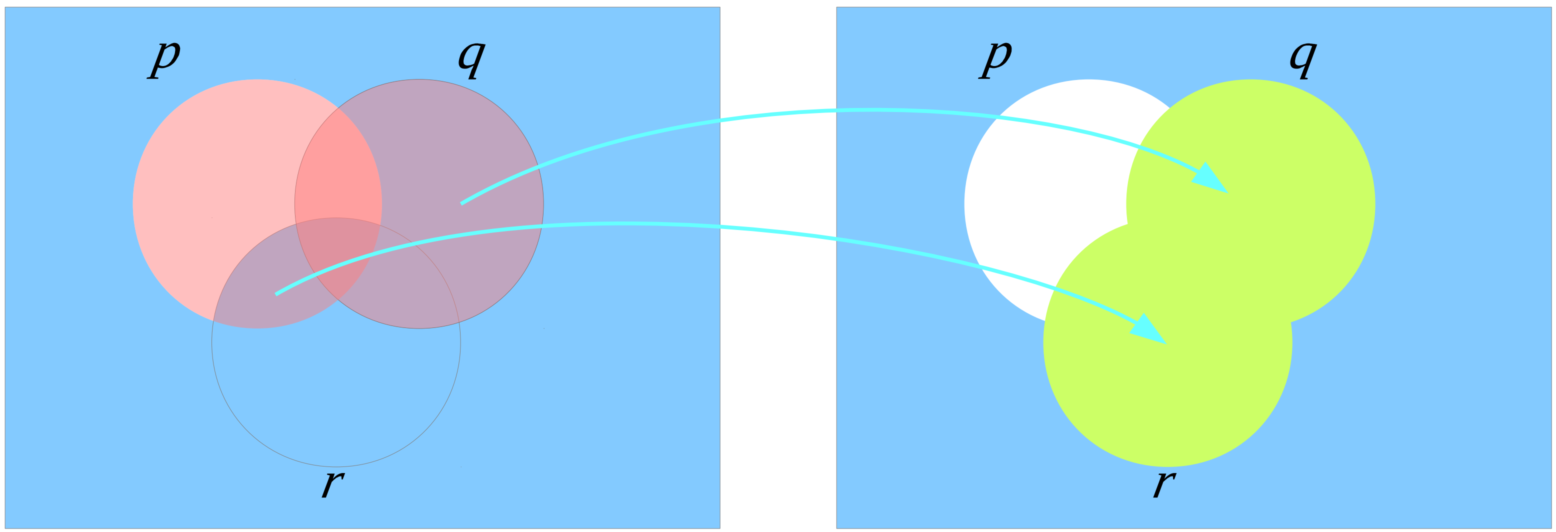
| $\neg p$ | $p$ | $q$ | $r$ | $p \rightarrow q$ | $p \wedge \neg q$ | $r \wedge \neg r$ | $(p \wedge \neg q) \rightarrow (r \wedge \neg r)$ |
|---|---|---|---|---|---|---|---|
| $F$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |

# Resolution Example

$p \lor q$

$\neg p \lor r$

$q \lor r$



# Resolution Example

| $\neg p$ | $p$ | $q$ | $r$ | $p \vee q$ | $\neg p \vee r$ | $(p \vee q) \wedge (\neg p \vee r)$ | | $q \vee r$ |
|---|---|---|---|---|---|---|---|---|
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | → | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | → | $T$ |
| $F$ | $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | | $F$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | → | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | → | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $T$ | $F$ | | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $T$ | $F$ | | $F$ |

# A simple example

Case 1: *p* is false

Case 2: *p* is true

$$\frac{p \lor q \qquad \neg p \lor r}{q \lor r}$$

$$\frac{F \lor q \qquad T \lor r}{q}$$

$$\frac{T \lor q \qquad F \lor r}{r}$$

$$\frac{a \lor b, \quad \neg a \lor c}{b \lor c}$$

In plain language: Suppose $a$ is false. In order for the premise $a \lor b$ to be true, $b$ must be true. Alternatively, suppose $a$ is true. In order for the premise $\neg a \lor c$ to be true, $c$ must be true. Therefore regardless of falsehood or veracity of $a$, if both premises hold, then the conclusion $b \lor c$ is true.

http://en.wikipedia.org/wiki/Derivative

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