Angle Recoding CORDIC
2. Wu

20180514

Extended EAS (EEAS) - Wu

more flexible way of decomposing the rotation angle

better
the number of iterations
the error performance

$$S_{EAS} = \{ (\sigma \cdot \tan^{-1}(2^{-r})) : \sigma \in \{+1, 0, -1\}, \ r \in \{1, 2, \cdots, n-1\} \}$$

$$S_{EEAS} = \{ (\sigma_1 \cdot \tan^{-1}(2^{-r_1}) + \sigma_2 \cdot \tan^{-1}(2^{-r_2})) : \sigma_1, \sigma_2 \in \{+1, 0, -1\}, \ r_1, r_2 \in \{1, 2, \cdots, n-1\} \}$$

The pse do-rotation
for i-th micro rotations

$$x_{i+1} = x_i - [\sigma_1(i) \cdot 2^{-r_1(i)} + \sigma_2(i) \cdot 2^{-r_2(i)}] \, y_i$$
$$y_{i+1} = y_i + [\sigma_1(i) \cdot 2^{-r_1(i)} + \sigma_2(i) \cdot 2^{-r_2(i)}] \, x_i$$

The pseudo-rotated vector $[x_{R_m}, y_{R_m}]$
after R_m (the required number of micro-rotations)

Needs to be scaled by a factor $K = \prod K_i$

$$K_i = \left[1 + \left(\sigma_1(i) \cdot 2^{-r_1(i)} + \sigma_2(i) \cdot 2^{-r_2(i)} \right)^2 \right]^{-\frac{1}{2}}$$

$$\tilde{x}_{i+1} = \tilde{x}_i - [k_1(i) \cdot 2^{-s_1(i)} + k_2(i) \cdot 2^{-s_2(i)}] \, \tilde{y}_i$$
$$\tilde{y}_{i+1} = \tilde{y}_i + [k_1(i) \cdot 2^{-s_1(i)} + k_2(i) \cdot 2^{-s_2(i)}] \, \tilde{x}_i$$

$\tilde{x}_0 = x_{R_m}$ $\quad\quad k_1, k_2 \in \{-1, 0, 1\}$

$\tilde{y}_0 = y_{R_m}$ $\quad\quad s_1, s_2 \in \{1, 2, \dots, n-1\}$

[21] C.-S. Wu, A.-Y. Wu, and C.-H. Lin, “A high-performance/low-latency vector rotational CORDIC architecture based on extended elementary angle set and trellis-based searching schemes,” *IEEE Trans. Circuits Syst. II: Anal. Digital Signal Process.*, vol. 50, no. 9, pp. 589–601, Sep. 2003.

IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS—I: FUNDAMENTAL THEORY AND APPLICATIONS, VOL. 49, NO. 10, OCTOBER 2002

A Unified View for Vector Rotational CORDIC Algorithms and Architectures Based on Angle Quantization Approach

An-Yeu Wu and Cheng-Shing Wu

AR : to approximate θ
with the combination
of selected angle elements
from a pre-defined **EAS**
(Elementary Angle Set)

EAS : all possible values of $\theta(j)$

EAS $\hat{S}_1 = \{ \tan^{-1} (\alpha^* \cdot 2^{-s^*}) \ : \ \alpha^* \in \{-1, 0, +1\},$
$s^* \in \{0, 1, \cdots, N-1\} \}$

EAS $\hat{S}_1$ consists of $\tan^{-1}$(Single **S**igned **p**ower of **t**wo)
$\tan^{-1}$(single **SPT**)
$\tan^{-1} (\alpha^* \cdot 2^{-s^*})$

SPT-based digital filter design

to increase the coefficient resolution

→ imploy more SPT terms to represent filter coefficients

[12] H. Samueli, "An improved search algorithm for the design of multiplierless FIR filters with power-of-two coefficients," *IEEE Trans. Circuits Syst.*, vol. 36, pp. 1044–1047, July 1989.

[13] Y. C. Lim, R. Yang, D. Li, and J. Song, "Signed power-of-two term allocation scheme for the design of digital filters," *IEEE Trans. Circuits Syst. II*, vol. 46, pp. 577–584, May 1999.

EAS $\hat{S}_1$ consists of $\tan^{-1}$(Single signed power of two)

$\tan^{-1}$(Single SPT)

$\tan^{-1}(\alpha^* \cdot 2^{-s^*})$

EAS $\hat{S}_2$ consists of $\tan^{-1}$(two signed power of two)

$\tan^{-1}$(two SPT)

$\tan^{-1}(\alpha_0^* \cdot 2^{-s_0^*} + \alpha_1^* \cdot 2^{-s_1^*})$

Two Signed-Power-of-Two terms

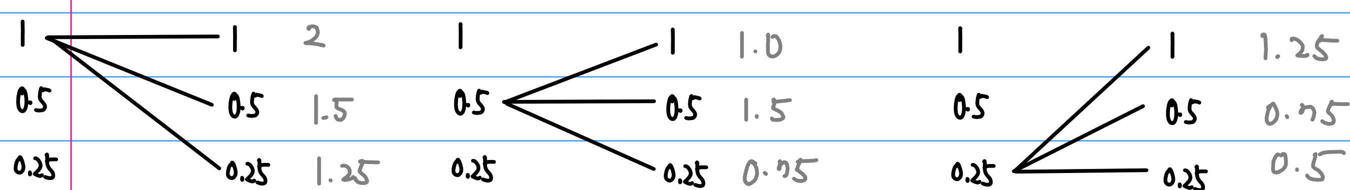
$$S_2 = \{ \tan^{-1}(\alpha_0^* \cdot 2^{-s_0^*} + \alpha_1^* \cdot 2^{-s_1^*}) : \alpha_0^*, \alpha_1^* \in \{-1, 0, +1\} \quad s_0^*, s_1^* \in \{0, 1, \cdots, w-1\} \}$$

S_1

1	$1 = 2^{-0}$	$\tan^{-1}(2^{-0})$
0.5	$\frac{1}{2} = 2^{-1}$	$\tan^{-1}(2^{-1})$
0.25	$\frac{1}{4} = 2^{-2}$	$\tan^{-1}(2^{-2})$

S_2

2	$1+1 = 2^{-0} + 2^{-0}$	$\pm\tan^{-1}(2^{-0} + 2^{-0})$
1.5	$1+\frac{1}{2} = 2^{-0} + 2^{-1}$	$\pm\tan^{-1}(2^{-0} + 2^{-1})$
1.25	$1+\frac{1}{4} = 2^{-0} + 2^{-2}$	$\pm\tan^{-1}(2^{-0} + 2^{-2})$
1.0	$1 = 2^{-0}$	$\pm\tan^{-1}(2^{-0})$
0.75	$\frac{1}{2}+\frac{1}{4} = 2^{-1} + 2^{-2}$	$\pm\tan^{-1}(2^{-1} + 2^{-2})$
0.5	$\frac{1}{2} = 2^{-1}$	$\pm\tan^{-1}(2^{-1})$
0.25	$\frac{1}{4} = 2^{-2}$	$\pm\tan^{-1}(2^{-2})$



$2^{-0}, \quad 2^{-1}, \quad 2^{-2}$ $\qquad \{0, 1, 2\} = \{0, 1, W-1\}$

$W=3$

$s_0^*, s_1^* \in \{0, 1, 2\}$

$2^{s_0^*}, 2^{s_1^*} \in \{2^{-0}, 2^{-1}, 2^{-2}\}$

as the word size W increases,

the size of the set S_2 increases exponentially

$$\theta_i = \tan^{-1}\left(\alpha_0(j)\cdot 2^{-S_0(j)} + \alpha_1(j)\cdot 2^{-S_1(j)}\right)$$

R_m: the number of the subangle N_A

$$S_2 = \{\theta_i \mid i = 0, 1, \cdots R_m\}$$

$$S_2 = \{\tan^{-1}\left(\alpha_0^* \cdot 2^{-S_0^*} + \alpha_1^* \cdot 2^{-S_1^*}\right) : \alpha_0^*, \alpha_1^* \in \{-1, 0, +1\}$$
$$S_0^*, S_1^* \in \{0, 1, \cdots, W-1\}\}$$

the optimization problem of the EEAS-based CORDIC algorithm

given θ and R_m

find $\alpha_0(j)$, $\alpha_1(j)$, $S_0(j)$, and $S_1(j)$

the combination of elementary angles

from EEAS $\hat{S}_2$

minimize the angle quantization error

$$|\xi_{m,\,EEAS}| \triangleq \theta - \sum_{j=0}^{R_m-1} \tan^{-1}\left(\alpha_0(j)\, 2^{-S_0(j)} + \alpha_1(j)\, 2^{-S_1(j)}\right)$$

given $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$

$$\begin{bmatrix} x(j+1) \\ y(j+1) \end{bmatrix} = \begin{bmatrix} 1 & \alpha_0(j)\, 2^{-S_0(j)} + \alpha_1(j)\, 2^{-S_1(j)} \\ \alpha_0(j)\, 2^{-S_0(j)} + \alpha_1(j)\, 2^{-S_1(j)} & 1 \end{bmatrix} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix}$$

$$\begin{bmatrix} x_f \\ y_f \end{bmatrix} = P \begin{bmatrix} x(R_m) \\ y(R_m) \end{bmatrix} = \frac{1}{\prod_{j=0}^{R_s-1} \sqrt{1 + [\alpha_0(j) \cdot 2^{-S_0(j)} + \alpha_1(j) \cdot 2^{-S_1(j)}]^2}} \begin{bmatrix} x(R_m) \\ y(R_m) \end{bmatrix}$$

micro rotation procedure

the scaling operation

4 additions

increased hardware

reduced iteration steps

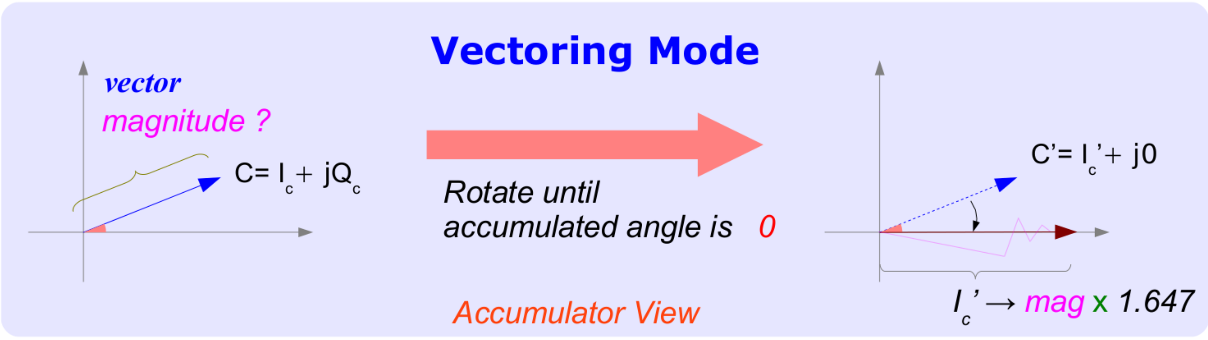
MVR (Modified Vector Rotation)

1) Repeat of Elementary Angles θ_i, σ_i

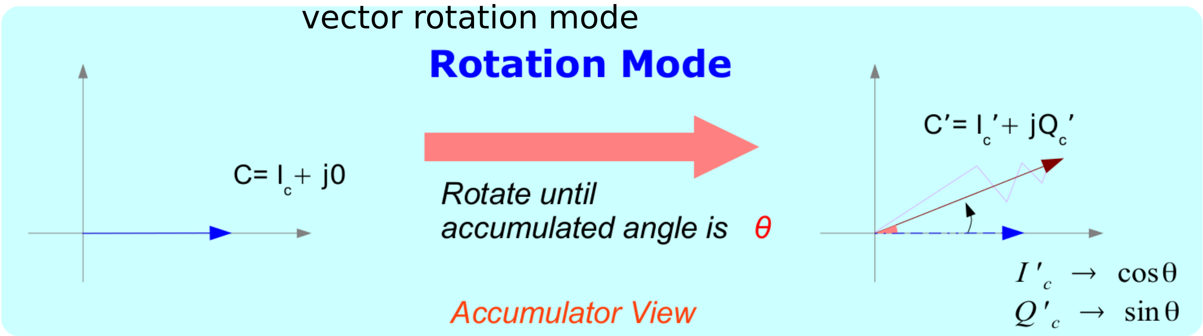
2) fixed total micro-rotation Number R_m

* Vector Rotation Mode

* and the rotation angles are known in advance



vector rotation mode



Modified Vector Rotational MVR CORDIC

- reduce the iteration number
- maintaining the SQNR performance
- modifying the basic microrotation procedure

Three Searching Algorithm

① the selective prerotation
② the selective scaling
③ iteration-tradeoff scheme

Angle Quantization

Quantization process on the rotational angle θ

decompose θ into several subangles θ_i's

the angle quantization error

$$\xi_m \triangleq \theta - \sum_{i=0}^{N_A-1} \theta_i$$

N_A the number of subangles

$\theta_0, \theta_1, \cdots, \theta_{N_A-1}$

$$\theta = \theta_0 + \theta_1 + \cdots + \theta_{N_A-1} + \xi_m$$

data : W-bit word length

the iteration number : N $\qquad N \leqslant W$

the restricted iteration number : R_m $\qquad R_m \ll W$

AQ Process : 2 Design Issues

① need to determine the sub-angles θ_i

② select / combine subangles
to <u>minimize</u> the angle quantization angle ξ_m

CSD (Canonical Signed Digit) Quantization

digital filter designs

coefficients are recoded

in terms of SPT (Signed Power of Two) terms

multiplication can be easily realized

with shift-and-add operations

$h_2 = (-0.156249)_{10} \Rightarrow (0.0\bar{1}011)_2$

$W=8$, 3 non-zero digits

① CSD quantization decomposes

coefficients into several SPT terms

(sub-coefficients)

② the multiplication of a coefficient

can be reformed

through the combination of

the non-zero SPT sub-coefficients

quantize the rotation angle θ

decompose the rotation angle θ
into several sub-angles θ_i's

the rotational operation of each θ_i
should be easily realized

If each θ_i can be realized
using only shift-and-add operations

the rotation of θ can be performed
through successive applications of
sub-angle rotations
in a cost-effective way

Approximation target	Coefficient h_i	Rotation angle θ
Basic Element	Non-zero digit 2^{i}	Sub-angle $a(i) = \tan^{-1}(2^{-i})$
Basic Operation	shift-and-add operation	2 shift-and-add Operations
Approximation Equation	$h_i \approx \sum_{j=0}^{N_D-1} g_j \cdot 2^{-d_j}$	$\theta \approx \sum_{j=0}^{R_m-1} \alpha(j) \cdot a(s(j))$
	$g_j \in \{-1, 0, +1\}$ $d_j \in \{0, 1, \cdots, W-1\}$ N_D = the number of non-zero digits	N_A = the number of sub-angles

Vector Rotation CORDIC Family

0. Conventional CORDIC
1. AR
2. MVR
3. EEAS

⓪ Conventional CORDIC

elementary angle $a(i) = \tan^{-1}(2^{-i})$

the number of elementary angles N

the rotation sequence $\mu(i) = \{-1, +1\}$

+1, −1, −1, +1, +1, ……

the i-the rotation angle $a(i)$

the W-bit word length

the iteration number $N \leq W$

the angle quantization error

$$\xi_{m,\,CORDIC} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\, a(i)$$

① AR [Hu]

skip certain micro rotations

the rotation sequence $\mu(i) = \{-1, 0, +1\}$

$\mu(i) = 0 \rightarrow$ skip

desire to minimize

$$\boxed{\sum_{i=0}^{N} |\mu(i)|}$$

so that the total number of CORDIC iterations can be minimized

Angle Recoding ← Multiplier Recoding

angle recoding method for efficient
implementation of the CORDIC algorithm
Hu & Naganathan, ISCAS 89

Greedy algorithm

$\theta(0) = \theta$, $\{\mu(i) = 0,\ 0 \leq i \leq N-1\}$, $k=0$.

repeat until $|\theta(k)| < a(N-1)$ Do

Choose i_k, $0 \leq i_k \leq N-1$

$$\big|\,|\theta(k)| - a(i_k)\big| = \min_{0 \leq i \leq N-1} \big|\,|\theta(k)| - a(i_k)\big|$$

$$\theta(k+1) = \theta(k) - \mu(i_k)\, a(i_k)$$

$$\mu(i_k) = \mathrm{Sign}(\theta(k))$$

try to approach the target rotation angle θ
step by step

decisions are made in each step
by choosing the best combination of $\alpha(i)\, a(s(i))$

so as to minimize $|\xi_m|$

$\alpha(i)$, $a(i)$ are determined such that
the error function is minimized

$$J(i) = |\theta(i) - \alpha(i)\, a(s(i))|$$

$$\theta(i) = \theta - \sum_{m=0}^{i-1} \alpha(m)\, a(s(m))$$

terminated if no further improvement can be found

$$J(i) \geqslant J(i-1)$$

or $\alpha(R_m - 1)$ and $s(R_m - 1)$
are determined at the end

$$\xi_{m,\,\mathrm{CORDIC}} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\,\alpha(i) \qquad \mu(i) = \{-1,\ 0,\ +1\}$$

$$= \theta - \sum_{j=0}^{N'} \tilde{\theta}(j)$$

$$N' \equiv \sum_{i=0}^{N-1} |\mu(i)| \qquad \{|-1|,\ |0|,\ |+1|\}$$

the effective iteration number N'

$S(j)$ the rotational sequence

determines the micro-rotation angle in the j-th iteration

$$i \leftarrow s(j)$$

index $0, 1, \cdots, N'$ (reduced number)

index $0, 1, \cdots, N$ (original number)

$$\mu(s(j)) \leftarrow \alpha(j)$$

$\Downarrow$ $\Uparrow$

$\{-1, +1\}$

$$\mu(i) = \begin{cases} \mu(s(j)) & i = s(j) \\ 0 & i \neq s(j) \end{cases}$$

--- reduced index

er

$$
\begin{array}{rl}
i = & 0,\ 1,\ 2,\ 3,\ \cdots,\ N-1 \\
s(j) = & 0,\ 1,\ 2,\ 3,\ \cdots,\ N-1 \\
\alpha(j) = & -1,\ 0,\ 0,\ +1,\ \cdots,\ -1 \\
j = & 0,\ -,\ -,\ 1,\ \cdots,\ N'-1
\end{array}
$$

rotational sequence

directional sequence

effective iteration number

$N' = N-2$

the j-th micro-rotation of $a(s(j))$

elementary angle

$$a(i) = \tan^{-1}(2^{-i})$$
$$a(s(j)) = \tan^{-1}(2^{-s(j)})$$

$$\alpha(j)\, a(s(j)) = \alpha(j) \tan^{-1}(2^{-s(j)}) \qquad \alpha(j) \in \{-1, +1\}$$

$$\Leftrightarrow \mu(i)\, a(i) \qquad \mu(i) \in \{-1, 0, +1\}$$

$$\xi_{m,\,CORDIC} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\, a(i) \qquad \mu(i) \in \{-1, 0, +1\}$$

$$= \theta - \left[\sum_{j=0}^{N'} \tilde{\theta}(j) \right]$$

$$= \theta - \left[\sum_{j=0}^{N'} \tan^{-1}\left(\alpha(j) \cdot 2^{-s(j)} \right) \right] \qquad \alpha(j) \in \{-1, +1\}$$

$$\tilde{\theta}(j) = \alpha(j) \tan^{-1}\left(2^{-s(j)} \right)$$

$$= \tan^{-1}\left(\alpha(j) \cdot 2^{-s(j)} \right)$$

$$S_1 = \left\{ \tan^{-1}\left(\boxed{\alpha} \cdot 2^{-\boxed{s}} \right) \qquad \boxed{\alpha} \in \{-1, 0, +1\}, \quad \boxed{s} \in \{0, 1, 2, \cdots N-1\} \right\}$$

② MVR (Modified Vector Rotational)

two modifications

① repeat of elementary angles

each micro-rotation of elementary angle can be performed repeatedly

- more possible combinations
- smaller ξ_m

② confines of total micro-rotation number

confine the iteration number in the micro-rotation phase to R_m $(R_m \ll W)$

The role of R_m is quite similar to the number of non-zero digit N_D in CSD recoding scheme

$$\xi_{m,MVR} \triangleq \theta - \sum_{i=0}^{R_m-1} \alpha(i)\, a(s(i))$$

the rotational sequence

$$s(i) \in \{0, 1, \cdots, W-1\}$$

determines the micro-rotation angle in the i-th iteration

the directional sequence

$$\alpha(i) \in \{-1, 0, +1\}$$

controls the direction of the i-th micro-rotation of $a(s(i))$

$$\alpha(i)\, a(s(i)) = \tilde{\theta}(j)$$

$$\xi_{m,MVR} \triangleq \theta - \sum_{j=0}^{R_m-1} \alpha(j)\, a(s(j))$$

the rotational sequence $s(j)$

$$j = 0, 1, 2, \cdots, R_m-1$$

$$\downarrow$$

$$s(j) \in \{0, 1, \cdots, W-1\}$$

determines the micro-rotation angle $a(s(j))$
in the j-th iteration

the directional sequence $\alpha(j)$

$$\alpha(j) \in \{-1, 0, +1\}$$

controls the direction of the j-th
micro-rotation of $a(s(j))$

$$\alpha(j)\, a(s(j)) = \tilde{\theta}(j)$$

$i =$	0,	1,	2,	3,	$\cdots$,	$W-1$		
$s(j) =$	0,	1,	2,	3,	$\cdots$,	$W-1$	rotational sequence	
$\alpha(j) =$	-1,	0,	0,	$+1$,	$\cdots$,	-1	directional sequence	
$j =$	0,	$-$,	$-$,	1,	$\cdots$,	R_m-1	effective iteration number	

$R_m \ll W$

sub-angle $(\alpha(j)\, a(s(j))) \sim \tilde{\theta}(j)$

$$\xi_{m,AR} = \theta - \left[\sum_{j=0}^{N'-1} \tan^{-1}\left(\alpha(j) \cdot 2^{-s(j)}\right)\right]$$

$$= \theta - \left[\sum_{j=0}^{N'-1} \tilde{\theta}(j)\right], \qquad \tilde{\theta}(j) = \tan^{-1}\left(\alpha(j) \cdot 2^{-s(j)}\right)$$

$$N' \triangleq \sum_{j=0}^{N-1} |\mu(j)| \quad \text{the effective iteration number}$$

EAS formed by MVR-CORDIC
is the same as AR
also performs AQ

The major difference

1) the total number of sub-angles N_A
the total iteration number
in the micro-rotation phase
is kept fixed to a pre-defined value of R_m
$N_A = R_m$

2) the sub-angle θ_i corresponds to $\alpha(j)\, a(s(j))$

$$\theta_j = \alpha(j)\, a(s(j)) = \tilde{\theta}_j$$

Optimization Problem

EAS point of view

Given θ, find the combination of R_m elementary angles from EAS S_1, such that the angle quantization error $|\xi_{m,MUR}|$ is minimized.

semi-greedy algorithm

trade offs between computational complexities and performance

key issue in the MVR-CORDIC
is to find the best sequences of
$s(i)$ and $\alpha(i)$ to minimize $|\xi_m|$
subject to the constraint that
the total iteration number is confined to R_m

1) Greedy Algorithm
2) Exhaustive Algorithm
3) Semigreedy Algorithm

1) Greedy Algorithm

try to approach the target rotation angle, θ,
step by step
in each step, decisions are made on $\alpha(i)$ and $s(i)$
by choosing the best combination of $\alpha(i)\, a(s(i))$
so as to minimize $|\xi_m|$

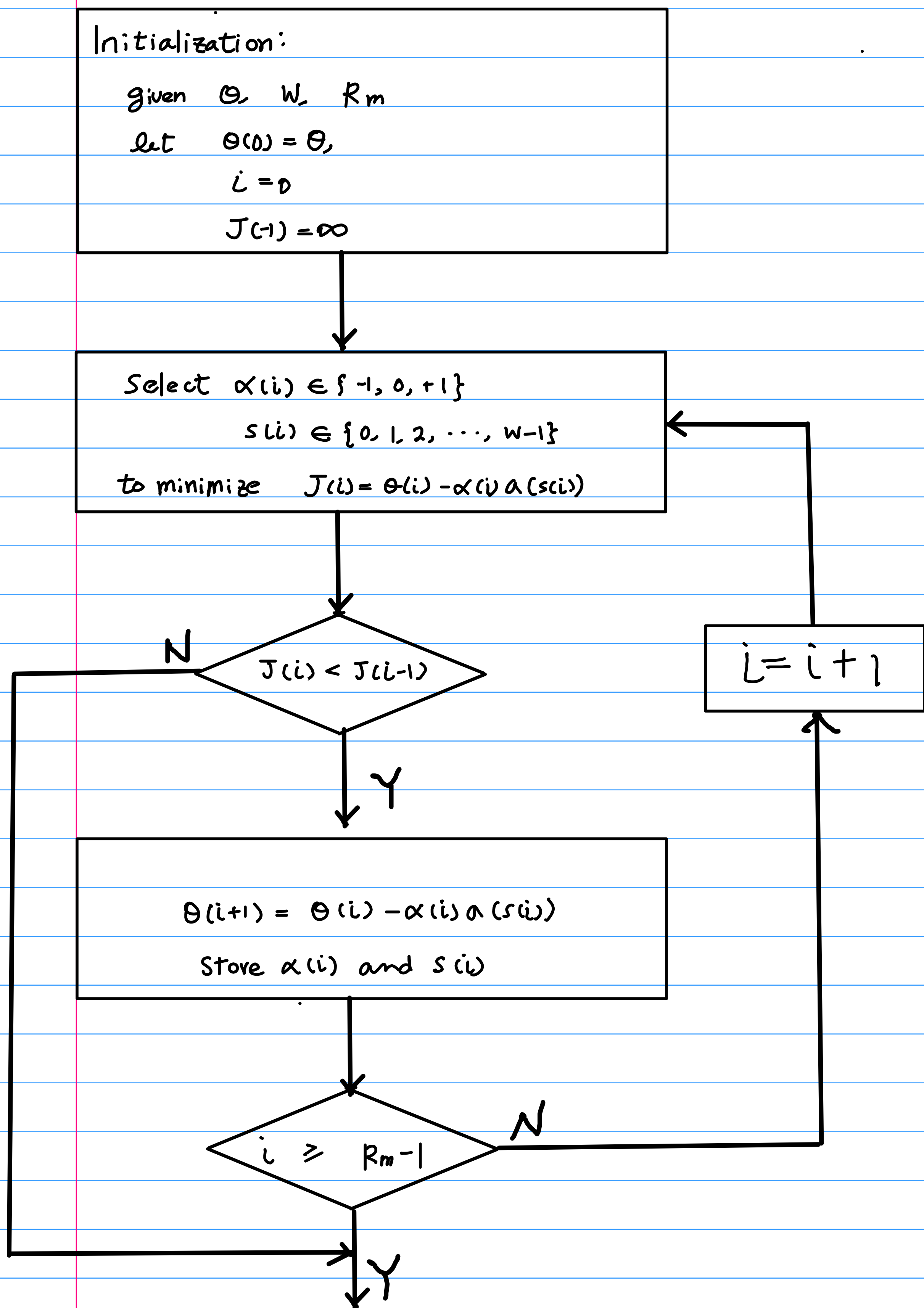
$\alpha(i)$ and $s(i)$ are determined such that
the error function $J(i) = |\theta(i) - \alpha(i)\, a(s(i))|$ is minimited

$\theta(i)$: the residue angle in the i-th step

$$\theta(i) = \theta - \sum_{m=0}^{i-1} \alpha(m)\, a(s(m))$$

the searching is terminated
if no further improvements can be found
$J(i) \geq J(i-1)$
$\alpha(R_m - 1)$ and $s(R_m - 1)$ are determined
at the end of the searching

the greedy algorithm terminates
only when the residue angle error
cannot be further reduced.


Initialization:
given Θ, W, Rm
let Θ(0) = Θ,
i = 0
J(-1) = ∞
Select α(i) ∈ {-1, 0, +1}
s(i) ∈ {0, 1, 2, ···, W-1}
to minimize J(i) = Θ(i) - α(i) a(s(i))
J(i) < J(i-1)
N
Y
Θ(i+1) = Θ(i) - α(i) a(s(i))
Store α(i) and s(i)
i ≥ Rm - 1
N
Y
i = i + 1

2) Exhaustive Algorithm

search for the entire solution space

all possible combinations of

$$\sum_{i=0}^{Rm-1} \alpha(i)\, a(s(i))$$

in a single step

decisions for $\alpha(i)$ and $s(i)$, $0 \leq i \leq Rm-1$

by minimizing the error function

$$J = \left| \theta - \sum_{i=0}^{Rm-1} \alpha(i)\, a(s(i)) \right|$$

global optimal solution

Initialization:

given Θ, W, R_m

let $\Theta(0) = \Theta$,

$i = 0$

$J(-1) = \infty$

$\downarrow$

Select $\alpha(i) \in \{-1, 0, +1\}$

$s(i) \in \{0, 1, 2, \cdots, W-1\}$

for $0 \leq i \leq R_m - 1$

to minimize $J(i) = \Theta - \sum_{i=0}^{R_m-1} \alpha(i)\, a(s(i))$

$\downarrow$

Store $\alpha(i)$ and $s(i)$

for $0 \leq i \leq R_m - 1$

3) Semi-greedy Algorithm

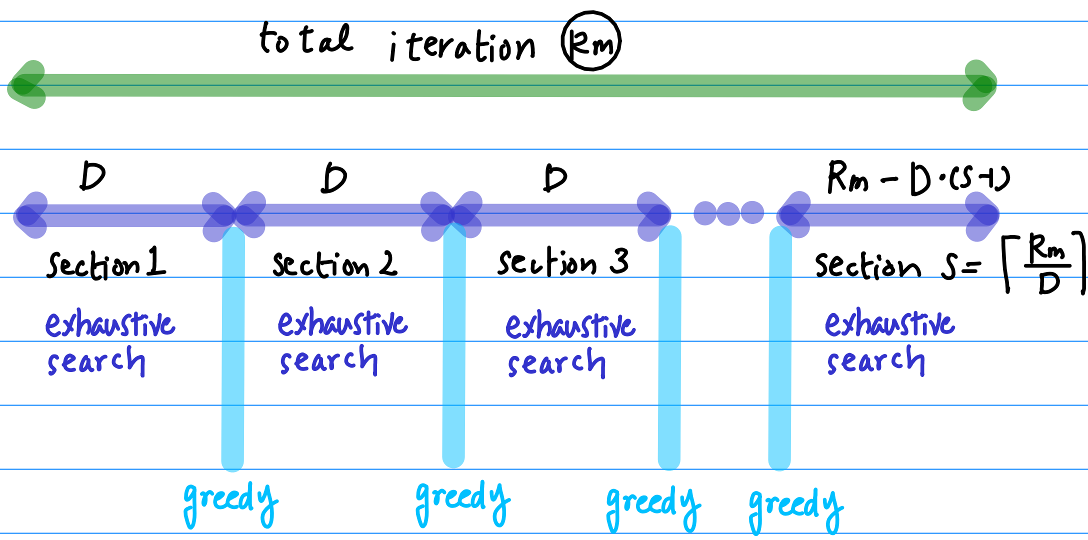
a combination of greedy and exhaustive algorithm

the search space of $\alpha(i)$ and $s(i)$ for $0 \le i \le R_m - 1$
are divided into several sections
with D iterations as a segment

D ↓ block length

segment ↓ block

the segmentation scheme



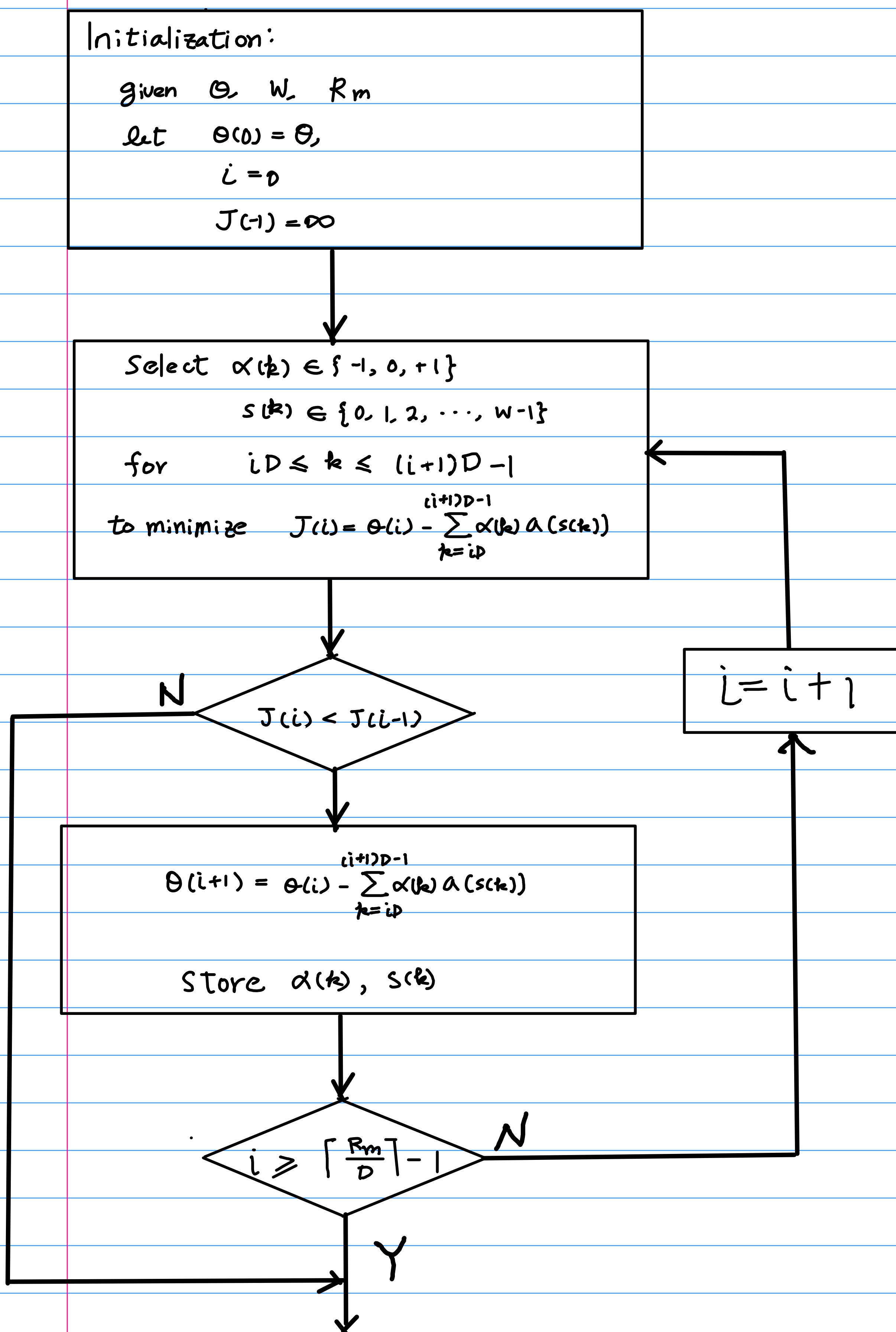
in the i-th block

decision of $\alpha(k)$ and $s(k)$ for $iD \leq k \leq (i+1)D-1$

minimizes $$J = \left| \theta(i) - \sum_{k=iD}^{(i+1)D-1} \alpha(k)\, a(s(k)) \right|$$

where $$\theta(i) = \theta - \sum_{m=0}^{i-1} \left[\sum_{k=mD}^{(m+1)D-1} \alpha(k)\, a(s(k)) \right]$$

the residue angle in the i-th step


Initialization:
given Θ, W, R_m
let $\Theta(0) = \Theta$,
$i = 0$
$J(-1) = \infty$
Select $\alpha(k) \in \{-1, 0, +1\}$
$s(k) \in \{0, 1, 2, \cdots, W-1\}$
for $iD \leq k \leq (i+1)D-1$
to minimize $J(i) = \Theta(i) - \sum_{k=iD}^{(i+1)D-1} \alpha(k)\, a(s(k))$
N
$J(i) < J(i-1)$
$i = i+1$
$\Theta(i+1) = \Theta(i) - \sum_{k=iD}^{(i+1)D-1} \alpha(k)\, a(s(k))$
Store $\alpha(k)$, $s(k)$
$i \geq \lceil \frac{R_m}{D} \rceil - 1$
N
Y