First Order Logic – Semantics (3A)

Young W. Lim
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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Based on

Contemporary Artificial Intelligence,
R.E. Neapolitan & X. Jiang

Logic and Its Applications,
Burkey & Foxley

Terms and Formulas

Terms

1. **Variables**
2. **Functions**

$x \quad y \quad f(x) \quad g(x, y)$

Formulas

Predicate symbols.

Equality.

Negation.

Binary connectives.

Quantifiers.

$P(x) \qquad Q(x, y)$

$x = f(y)$

$\neg\, Q(x, y)$

$P(x) \wedge \neg Q(x, y)$

$\forall x, y \;\; (P(x) \wedge \neg Q(x, y))$

https://en.wikipedia.org/wiki/First-order_logic#Formation_rules

Examples of Terms

no expression involving a predicate symbol is a **term**.

$x \quad y \quad f(x) \quad g(x, y)$

$father(x)$	A function returns neither True nor False The father of x	**term**
$Father(x)$	A predicate returns always True or False Is x a father?	~~**term**~~

∀x love(x,y) : free variable y
∀x tall(x) : no free variable

Bound variable x
Free variable y

https://en.wikipedia.org/wiki/First-order_logic#Formation_rules

Terms

Terms

1. **Variables**. Any variable is a term.
2. **Functions**. Any expression $\mathbf{f}(t_1,...,t_n)$ of n arguments is a term
   where each argument t_i is a term and **f** is a function symbol of valence n
   In particular, symbols denoting individual constants are
   0-ary function symbols, and are thus terms.

Only expressions which can be obtained
by finitely many applications of rules 1 and 2 are terms.

no expression involving a predicate symbol is a term.

https://en.wikipedia.org/wiki/First-order_logic#Formation_rules

Formulas (1)

Formulas (wffs)

Predicate symbols.	$P(x) \qquad Q(x, y)$
Equality.	$x = f(y)$
Negation.	$\neg Q(x, y)$
Binary connectives.	$P(x) \wedge \neg Q(x, y)$
Quantifiers.	$\forall x, y \; (P(x) \wedge \neg Q(x, y))$

Only expressions which can be obtained
by finitely many applications of rules 1–5 are formulas.

The formulas obtained from the first two rules
are said to be **atomic formulas**.

https://en.wikipedia.org/wiki/First-order_logic#Formation_rules

Formulas (2)

Formulas (wffs)

Predicate symbols.

If **P** is an n-ary predicate symbol

and t_1, ..., t_n are terms

then $P(t_1,...,t_n)$ is a formula.

Equality.

If the equality symbol is considered part of logic,

and t_1 and t_2 are terms,

then $t_1 = t_2$ is a formula.

$P(x)$ $Q(x, y)$

$x = f(y)$

https://en.wikipedia.org/wiki/First-order_logic#Formation_rules

Formulas (3)

Formulas (wffs)

Negation.

If φ is a formula,

then ¬φ is a formula.

Binary connectives.

If φ and ψ are formulas,

then (φ → ψ) is a formula.

Similar rules apply to other binary logical connectives.

Quantifiers.

If φ is a formula and x is a variable,

then ∀x φ (for all x, holds)

and ∃x φ (there exists x such that φ) are formulas.

$$\neg\, Q(x, y)$$

$$P(x) \wedge \neg Q(x, y)$$

$$\forall x, y \ \ (P(x) \wedge \neg Q(x, y))$$

https://en.wikipedia.org/wiki/First-order_logic#Formation_rules

Atoms and Compound Formulas

a formula that contains no logical connectives

a formula that has no strict subformulas

Atoms :

the simplest well-formed formulas of the logic. $P(x)$ $Q(x,y)$

Compound formulas :

formed by combining the atomic formulas using the logical connectives.

$$P(x) \wedge \neg Q(x,y)$$

$$\forall x,y \ (P(x) \wedge \neg Q(x,y))$$

https://en.wikipedia.org/wiki/Atomic_formula

Atomic Formula

for **propositional logic**
the atomic formulas are the propositional variables $\quad p \quad q$

for **predicate logic**
the atoms are predicate symbols together with their arguments, each argument being a term. $\quad P(x) \quad Q(x, f(y))$

In **model theory**
atomic formula are merely strings of symbols with a given signature which may or may not be satisfiable with respect to a given model

https://en.wikipedia.org/wiki/Atomic_formula

Basic Entities in FOL

propositional logic assumes world contains **facts**

first-order logic assumes the world contains **objects**, **relations**, and **functions**

- **Objects**: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries …

- **Relations**: red, round, bogus, prime, multistoried, is the brother of, is bigger than, is inside, is part of, has color, occurred after, owns, comes between, …

- **Functions**: father of, best friend, third inning of, one more than, end of

https://www.cs.umd.edu/~nau/cmsc421/chapter08.pdf

Types of Logic

Language	Ontological commitment* (what it talks about)	Epistemological commitment* (what it says about truth)
Prop. Logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

ontological commitment ≈ our assumptions about what things exist
epistemological commitment ≈ what we can know about those things

https://www.cs.umd.edu/~nau/cmsc421/chapter08.pdf

Model

A **model** is a pair **M**= (**D**,**I**),

D is a domain and

I is an interpretation

D contains

more than 1 **objects** (domain elements)

and **relations** among them

I specifies referents for

constant symbols → **objects** in the domain

predicate symbols → **relations** over objects in the domain

function symbols → **functional relations** over objects in the domain

- **Objects**: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries …
- **Relations**: red, round, bogus, prime, multistoried, Is the brother of, is bigger than, is inside, is part of, has color, occurred after, owns, comes between, …
- **Functions**: father of, best friend, third inning of, one more than, end of

Relation and Predicates

mathematically, a **relation** is a set of ordered n-tuples

An **atomic** sentence **predicate**($term_1$,...,$term_n$) is ***true*** in **M**

iff the **objects** referred to by $term_1$,...,$term_n$

are in the **relation** referred to by **predicate**

M is a **model** of a **sentence S**

If **S** is ***true*** in **M**

M= (**D**,**I**),

D is a domain and

I is an interpretation

Sentences

	P1( ) …	P2( ) …	S	T …
Interpretation I_1 ➡	T	T		
Interpretation I_2 ➡	T	F		
Interpretation I_3 ➡	F	T	T	
Interpretation I_4 ➡	F	F		
Interpretation I_5 ➡			T	
Interpretation I_6 ➡			T	
Interpretation I_7 ➡				
Interpretation I_8 ➡				

a model **M**

A Signature

First specify a **signature**

Constant **Symbols**	$\{c_1, c_2, \dots c_n\} = D$
Predicate **Symbols**	$\{\mathbf{P}_1, \mathbf{P}_2, \dots \mathbf{P}_m\}$
Function **Symbols**	$\{\mathbf{f}_1, \mathbf{f}_2, \dots \mathbf{f}_l\}$

A **model** is a pair **M**= (**D**,**I**),

D is a domain and

I is an interpretation

D contains

more than 1 **objects** (domain elements)

and **relations** among them

I specifies referents for

signature

constant symbols → **objects** in the domain

predicate symbols → **relations** over objects in the domain

function symbols → **functional** relations over objects in the domain

A Language

Determines the **language**

Given a language

A model is specified

A **domain of discourse**

a set of entities $\{entity_1, entity_2, \dots entity_n\}$

An **interpretation**

constant assignments $\{c_1, c_2, \dots c_n\} = D$

function assignments $\mathbf{f}_1(\ ), \mathbf{f}_2(\ ), \dots \mathbf{f}_l(\ )$

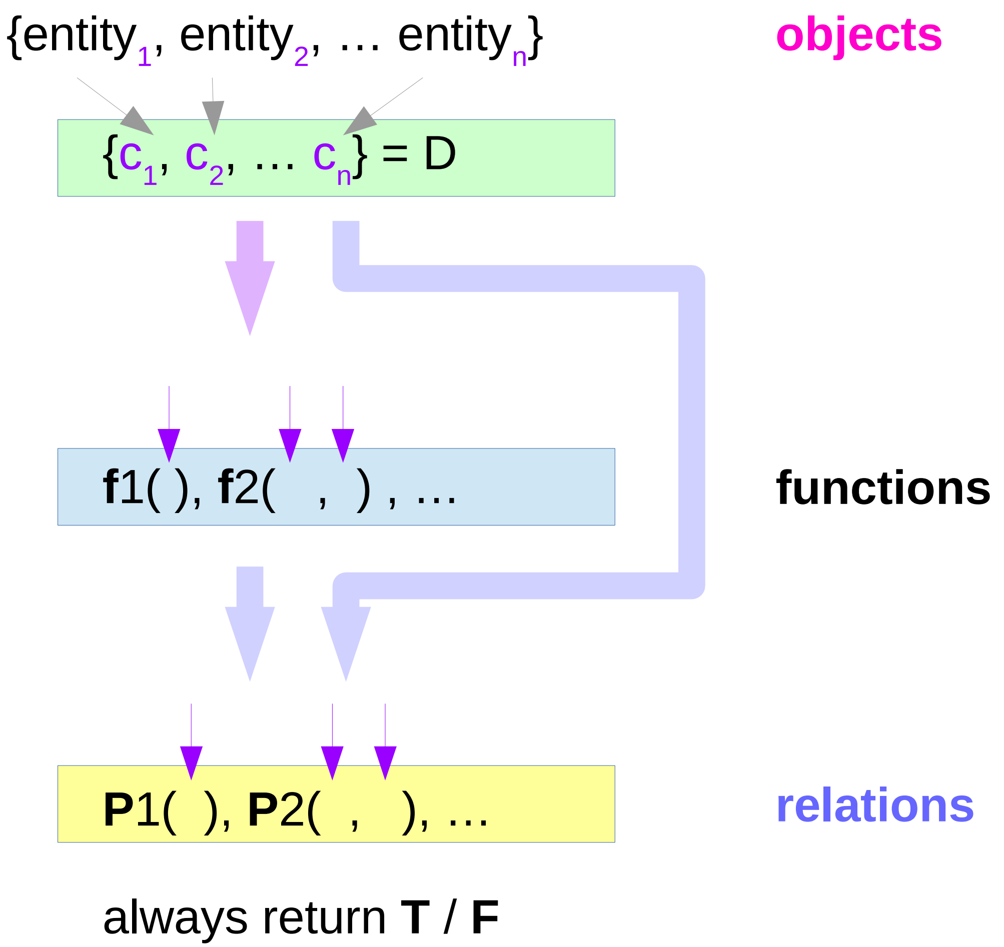
truth value assignments $\mathbf{P}_1(\ ), \mathbf{P}_2(\ ), \dots \mathbf{P}_m(\ )$

Interpretation – assigning the signature

Constant assignments

Function assignments

Truth value assignments



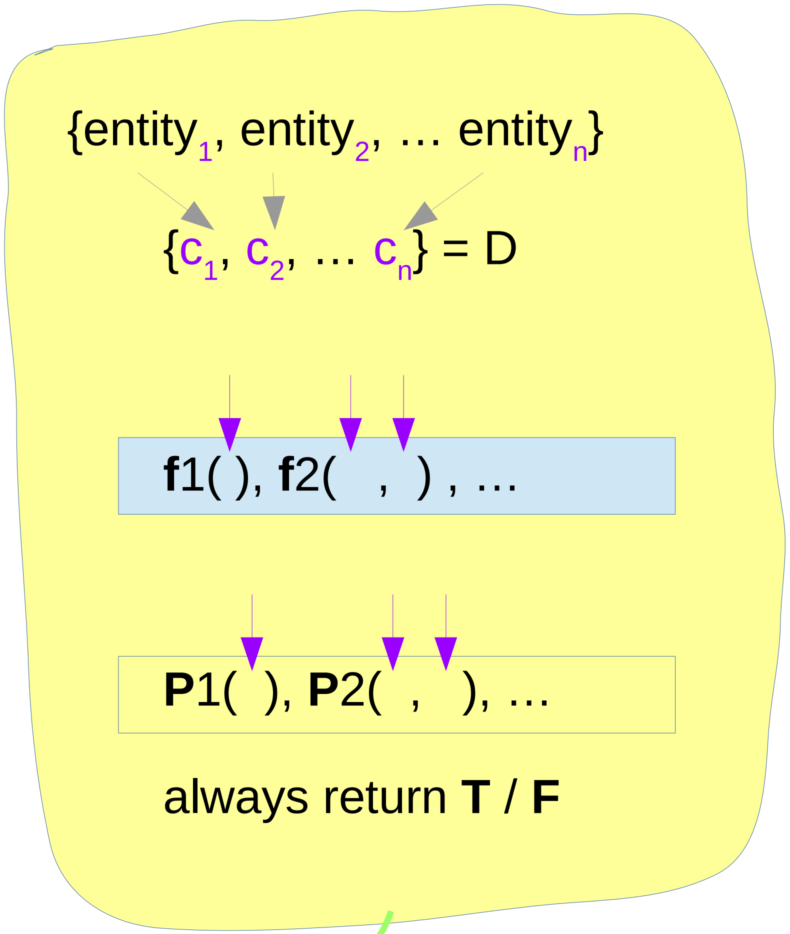
Interpretation – assigning atoms

Propositional Logic

	A	B	
Interpretation I_1 ➡	T	T	
Interpretation I_2 ➡	T	F	
Interpretation I_3 ➡	F	T	
Interpretation I_4 ➡	F	F	

First Order Logic

	P1( ) …	P2( ) …	Sentences S1	S2
Interpretation I_1 ➡	T	T		
Interpretation I_2 ➡	T	F		
Interpretation I_3 ➡	F	T		
Interpretation I_4 ➡	F	F		

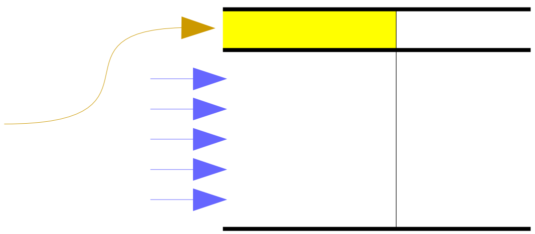


To assign truth values to predicates, constants and functions must be assigned

PL: A Model

A **model** or a **possible world**:

Every atomic proposition is assigned a value **T** or **F**

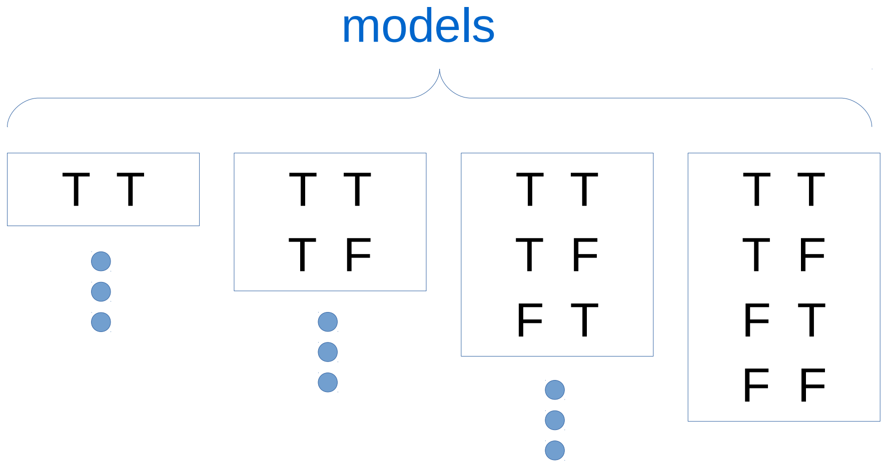


The set of **all** these assignments constitutes
A **model** or a **possible world**

All possible worlds (assignments) are permissible

A	B	A∧B	A∧B ⇒ A
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Every atomic proposition : A, B



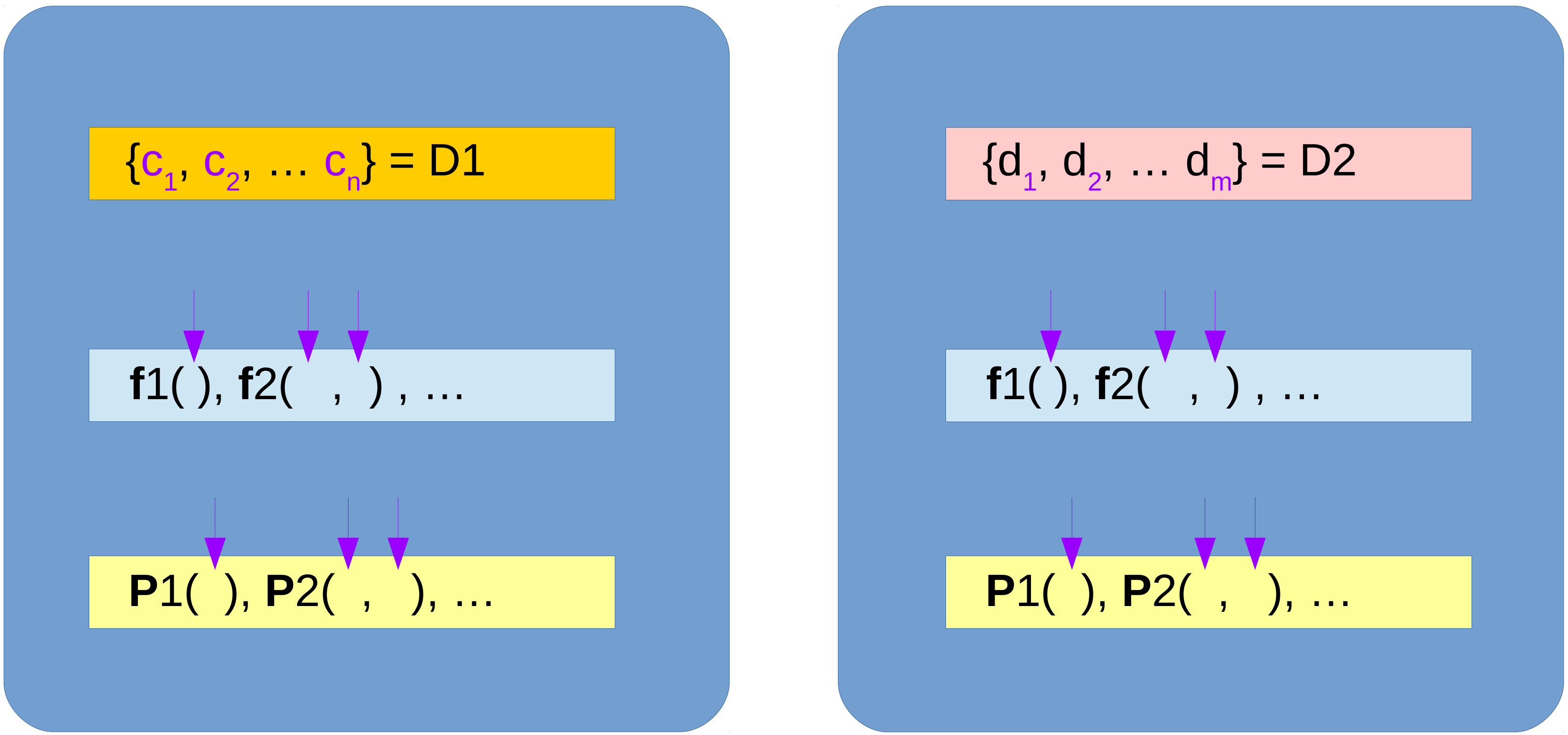
$$2^4 = 16$$

Models and Signatures

Different sets of constants (entities or objects)

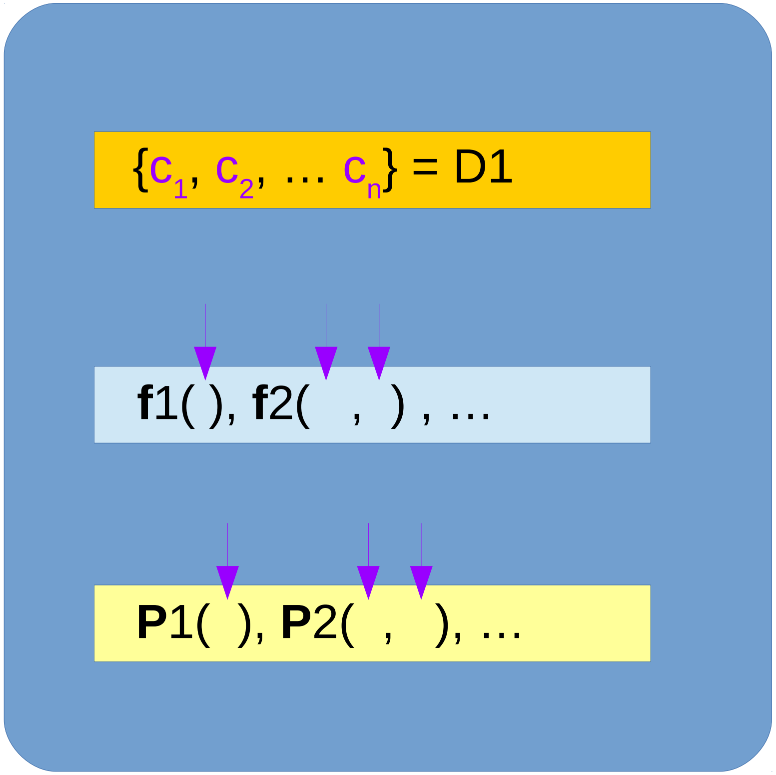
{John, Baker, …, Paul} = D1

{Mary, Jane, …, Elizabeth} = D2

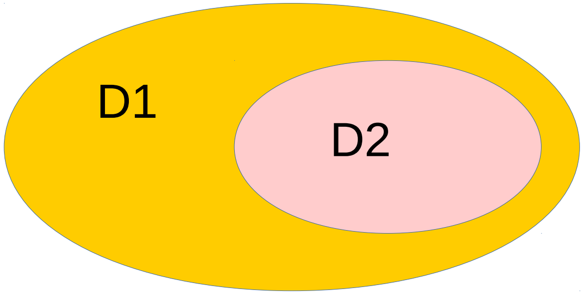


Models and Signatures

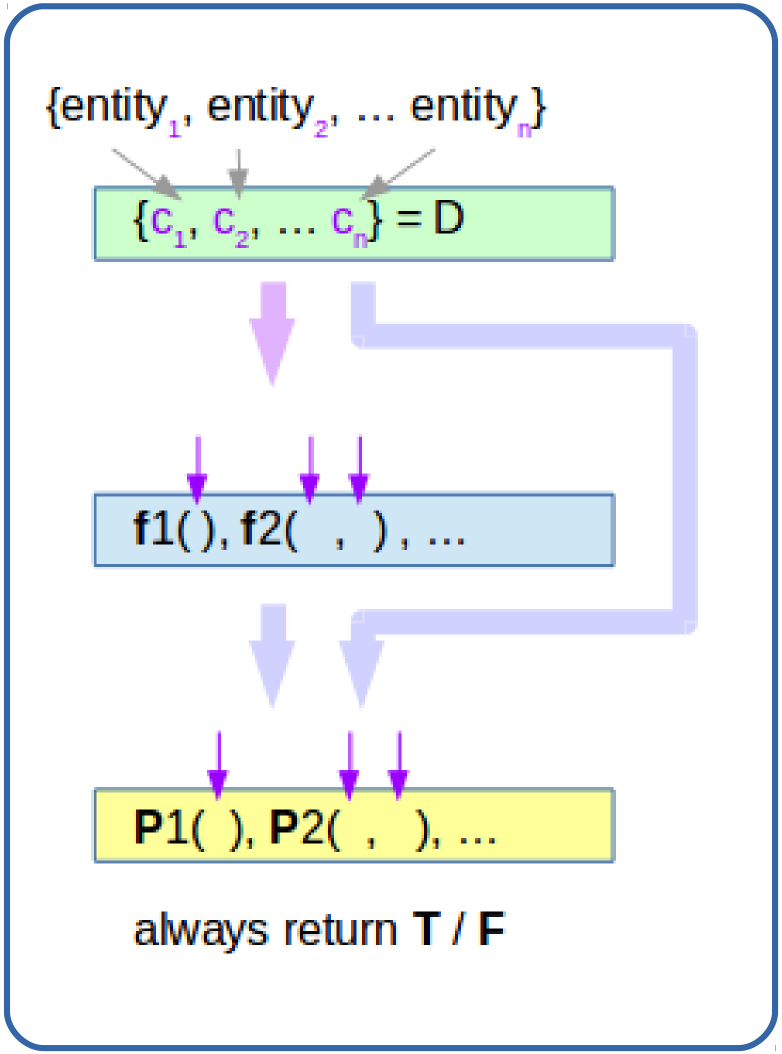
a subset of constants



{d_1, d_2, … d_m} = D2



Truth values of sentences



terms	$x \quad y \quad f(x) \quad g(x,y)$
atomic formulas	$P(x) \qquad Q(x, f(y))$
formulas / sentences	$\forall x, y \ (P(x) \wedge \neg Q(x,y))$

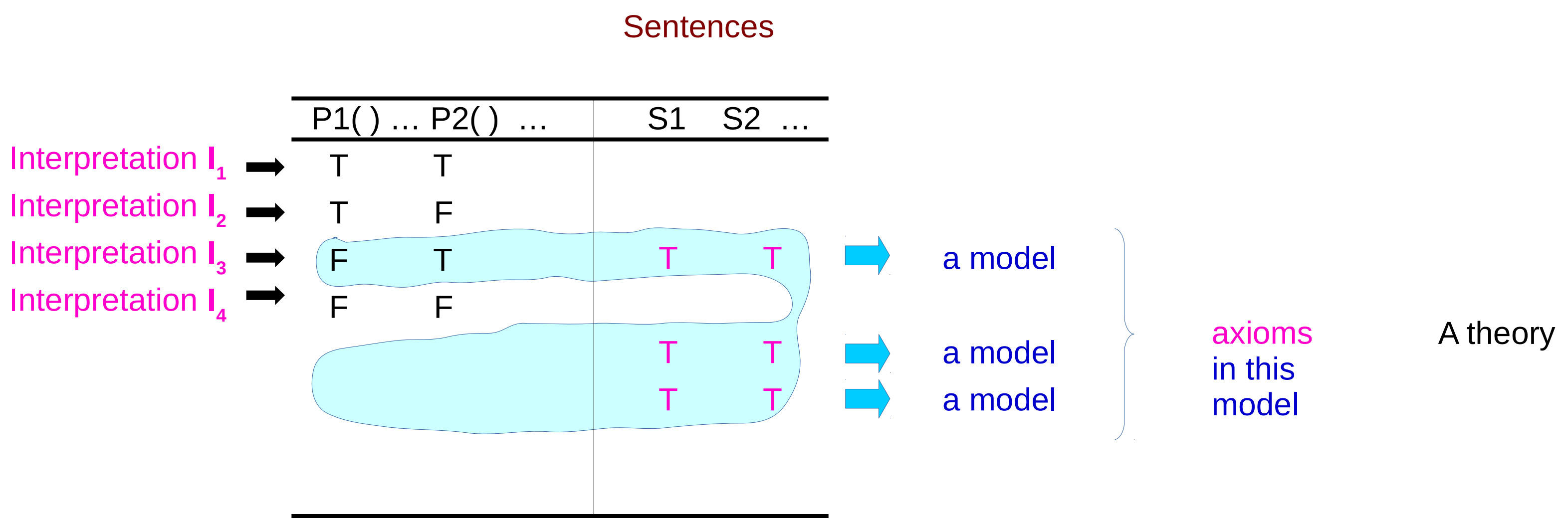
First Order Logic

Sentences

	P1( )	P2( ) …	S1	S2
Interpretation I_1 ➡	T	T		
Interpretation I_2 ➡	T	F		
Interpretation I_3 ➡	F	T		
Interpretation I_4 ➡	F	F		

Model Theory (1)

A first-order theory of a particular signature is
a set of axioms,
which are sentences
consisting of symbols from that signature.



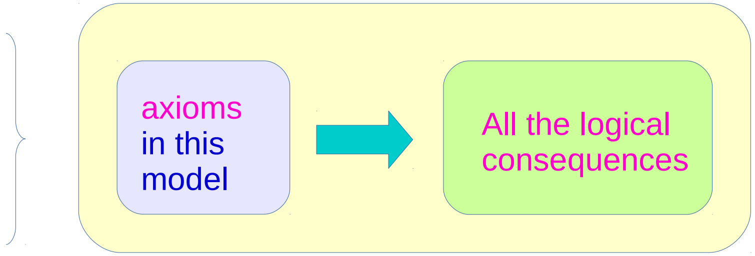
https://en.wikipedia.org/wiki/First-order_logic#First-order_theories.2C_models.2C_and_elementary_classes

Model Theory (2)

The set of axioms is
often **finite** or **recursively enumerable**,
in which case the theory is called **effective**.

Sometimes theories often include
all logical consequences of the axioms.

A theory



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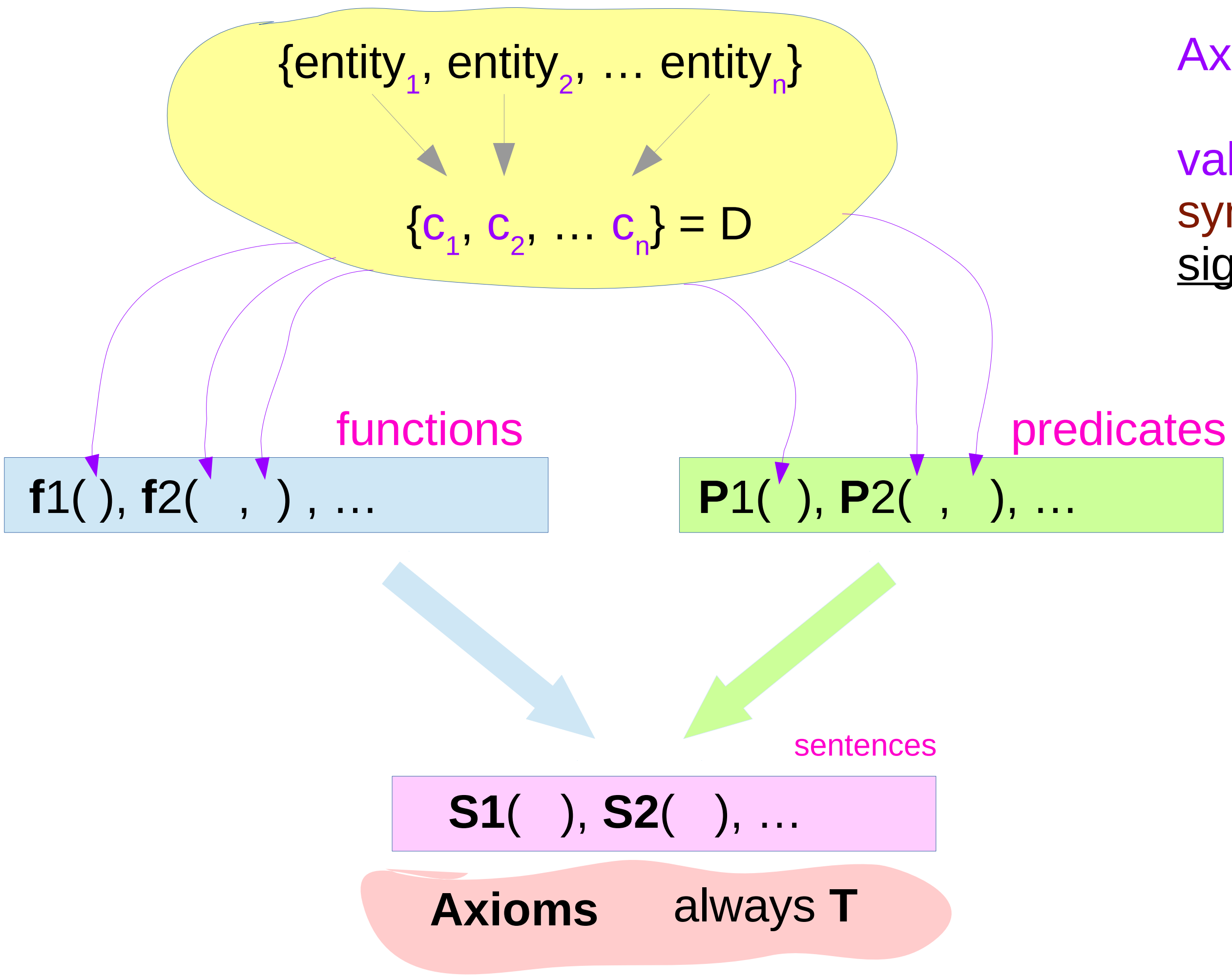
Axioms of a model theory



Axioms

valid sentences consisting of symbols from a particular signature.

Models

Propositional Logic

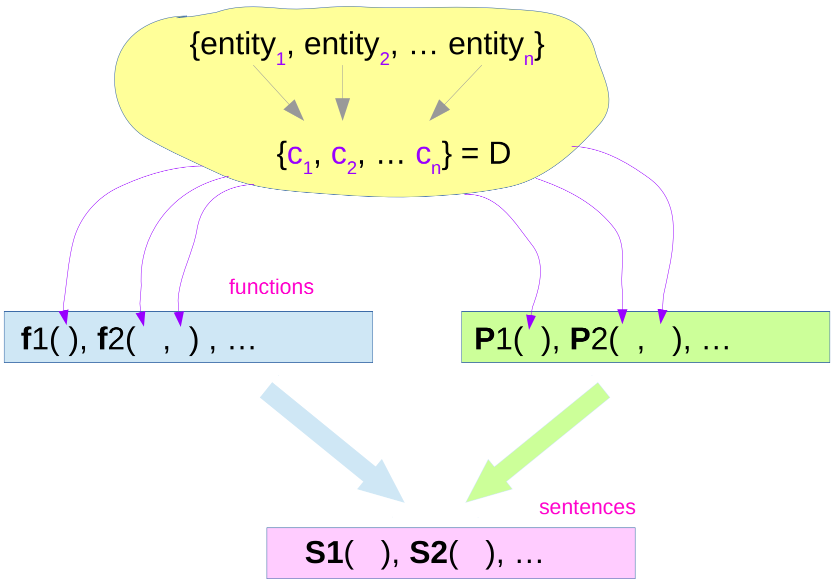
	A	B					
Interpretation $\mathbf{I}_1$ ➡	T	T					
Interpretation $\mathbf{I}_2$ ➡	T	F	T	T	T	➡	a model
Interpretation $\mathbf{I}_3$ ➡	F	T	T	T	T	➡	a model
Interpretation $\mathbf{I}_4$ ➡	F	F					

First Order Logic

Sentences

	P1( ) …	P2( ) …	S1	S2 …		
Interpretation $\mathbf{I}_1$ ➡	T	T				
Interpretation $\mathbf{I}_2$ ➡	T	F				
Interpretation $\mathbf{I}_3$ ➡	F	T	T	T	➡	a model
Interpretation $\mathbf{I}_4$ ➡	F	F				
			T	T	➡	a model
			T	T	➡	a model

Signature



Axioms always **T**

Axioms

Logical Axioms	- axioms
Non-logical Axioms	- postulate – deductive system

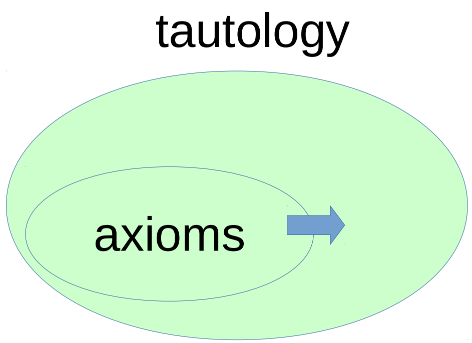
https://en.wikipedia.org/wiki/Axiom

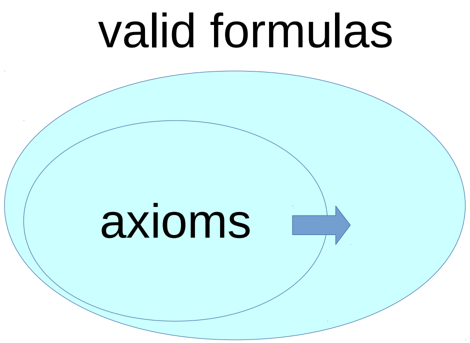
Logical Axioms

- formulas in a formal language that are universally valid
- formulas that are satisfied by every assignment of values (interpretations)

usually one takes as **logical axioms**
at least some minimal set of tautologies
that is sufficient for proving all tautologies in the language

in the case of predicate logic *more* **logical axioms** than that are required,
in order to prove logical truths that are not tautologies in the strict sense.





https://en.wikipedia.org/wiki/Axiom

Non-logical Axioms

formulas that play the role of theory-specific assumptions

reasoning about two different structures,
for example the natural numbers and the integers,
may involve the same **logical axioms**;

the purpose is to find out
what is special about *a particular structure*
(or set of structures, such as groups).

Thus non-logical axioms are not tautologies.

https://en.wikipedia.org/wiki/Axiom

Mathematical Discourse

Also called

- **postulate**
- **axioms in mathematical discourse**

this <u>does not mean</u> that it is claimed
that they are true in some absolute sense

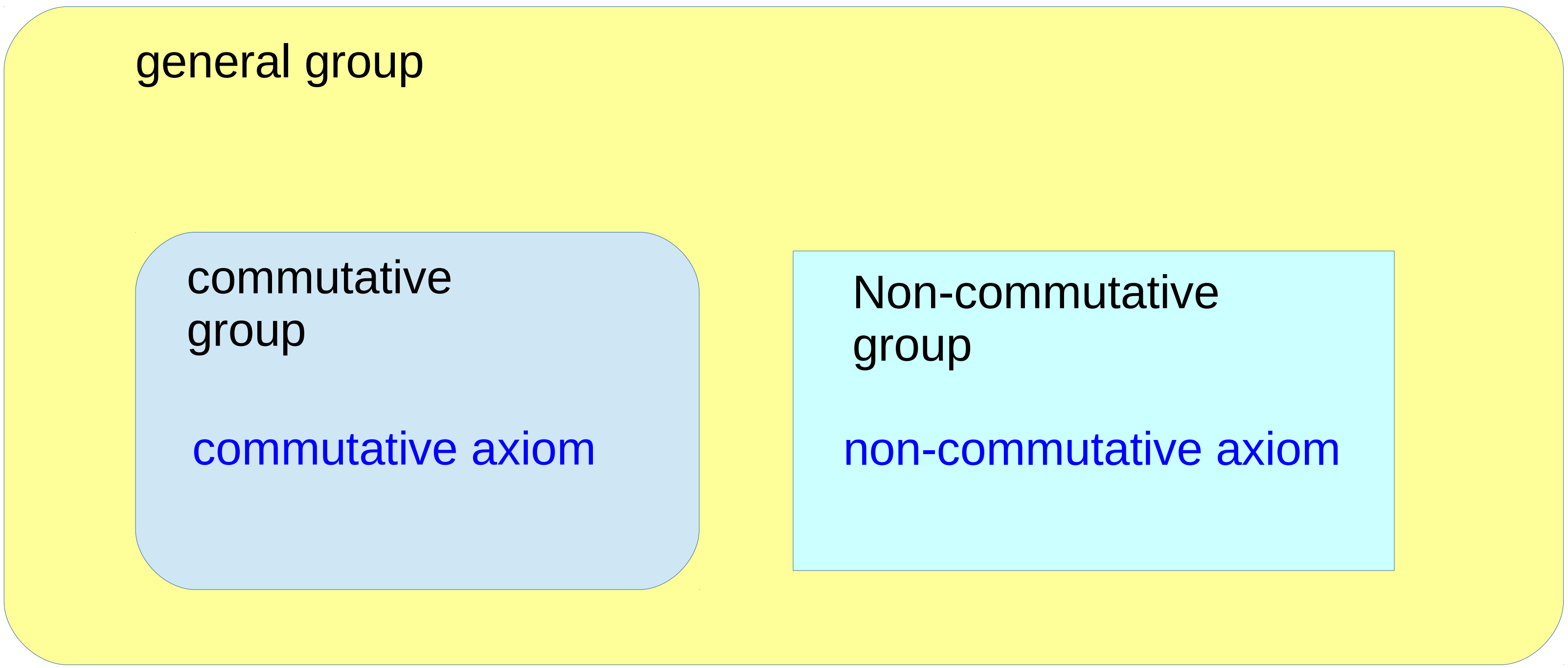
an elementary basis for a formal logic system

A deductive system

- **axioms** (non-logical)
- **rules of inference**

https://en.wikipedia.org/wiki/Axiom

Need not be tautologies



this <u>does not mean</u> that it is claimed

that they are true in some absolute sense

- Commutative axiom
- Non-commutative axiom

https://en.wikipedia.org/wiki/Axiom

Model Theory

The axioms are considered
to *hold* within the theory and

From axioms,
other sentences that *hold* within the theory
can be derived.

A first-order structure
that satisfies **all** sentences in a given theory
is said to be a **model** of the theory.

An elementary class is the set of **all** structures satisfying a particular theory.

These classes are a main subject of study in model theory.

https://en.wikipedia.org/wiki/First-order_logic#First-order_theories.2C_models.2C_and_elementary_classes

Truth values of sentences

Entailment in propositional logic can be computed
By **enumerating** the possible worlds (i.e. model checking)

How to **enumerate** possible worlds in FOL?

For each number of domain number n from 1 to infinity
For each k-ary predicate Pk in the vocabulary
For each possible k-ary relation on n objects
For each constant symbol C in the vocabulary
For each choice of referent for C from n objects. ..

Computing entailment in this way is not easy.

https://www.cs.umd.edu/~nau/cmsc421/chapter08.pdf

Truth values of sentences

domain number **n** [1, ∞)

k-ary predicate $\mathbf{P_k}$

k-ary relation $\mathbf{f_k}$ on n objects

constant symbol **C**

referent for C from n objects. ..

https://www.cs.umd.edu/~nau/cmsc421/chapter08.pdf

Model – domain of discourse

1. a nonempty set D of **entities** called a **domain of discourse**
   - this domain is a set
   - each element in the set : entity
   - each constant symbol : one entity in the domain

If we considering all individuals in a class,
The constant symbols might be
‘Mary’, - an entity
‘Fred’, - an entity
‘John’, - an entity
‘Tom’ - an entity

Model – interpretation

2. an **interpretation**

    (a) an entity in D is assigned to each of the constant symbols.

        Normally, every entity is assigned to a constant symbol.

    (b) for each **function**,

        an entity is assigned to each possible input of entities to the **function**

    (c) the predicate '**True**' is always assigned the value T

        The predicate '**False**' is always assigned the value F

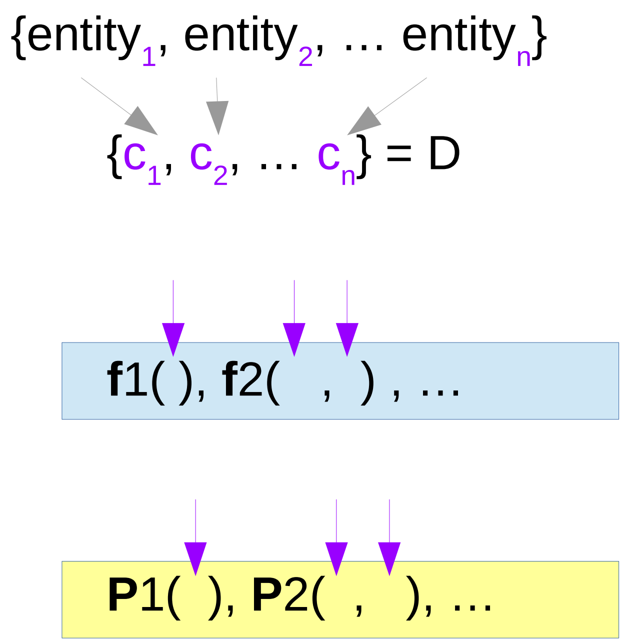
    (d) for every other **predicate**,

        the value T or F is assigned

        to each possible input of entities to the **predicate**

Each possible input of entities

Arity one: C(n, 1)
Arity two: C(n, 2)
Arity three: C(n, 3)
…



Arity one functions & predicates: C(n, 1)
Arity two: C(n, 2)
Arity three: C(n, 3)
…

always return **T** / **F**

Interpretation

Constant assignments

(a) an entity → the constant symbols.

Function assignments

(b) an entity → each possible input of entities to the **function**

Truth value assignments

(c) the value T →the predicate ‘**True**’
the value F → the predicate ‘**False**’

(d) for every other **predicate**,
the value T or F is assigned → every other predicate
to each possible input of entities to the **predicate**

Signature Model Examples A – (1)

Signature

1. constant symbols = { Mary, Fred, Sam }
2. predicate symbols = { married, young }
   married(x, y) : arity two
   young(x) : arity one

Model

1. domain of discourse D : the set of three particular *individuals*

   - this domain is a set
   - each element in the set : entity (= individuals)
   - each constant symbol : one entity in the domain (= one individual)

2. interpretation
   (a) a different individual is assigned to each of the constant symbols

   (a) an entity in D is assigned to each of the constant symbols.
   Normally, every entity is assigned to a constant symbol.

Signature Model Examples A – (2)

(b) for each **function**,
an entity is assigned to each possible input of entities to the **function**

(c) the predicate '**True**' is always assigned the value T
The predicate '**False**' is always assigned the value F

(d) the truth value assignments for every predicate

young(Mary) = F, young(Fred) = F, young(Sam) = T

married(Mary, Mary) = F, married(Mary, Fred) = T, married(Mary, Sam) = F
married(Fred, Mary) = T, married(Fred, Fred) = F, married(Fred, Sam) = F
married(Sam, Mary) = F, married(Sam, Fred) = F, married(Sam, Sam) = F

(d) for every other **predicate**,
the value T or F is assigned
to each possible input of entities to the **predicate**

(Mary, Mary), (Mary, Fred), (Mary, Sam)
(Fred, Mary), (Fred, Fred), (Fred, Sam)
(Sam, Mary), (Sam, Fred), (Sam, Sam)

Signature Model Examples B – (1)

Signature

1. constant symbols = { Fred, Mary, Sam }
2. predicate symbols = { love }   love(x, y) : arity two
3. function symbols = { mother }   mother(x) : arity one

Model

1. domain of discourse D : the set of three particular individuals
2. interpretation
   (a) a different individual is assigned to each of the constant symbols

   (b) the truth value assignments for every predicate
      love(Fred, Fred) = F, love(Fred, Mary) = F, love(Fred, Ann) = F
      love(Mary, Fred) = T, love(Mary, Mary) = F, love(Mary, Ann) = T
      love(Ann, Fred) = T, love(Ann, Mary) = T, love(Ann, Ann) = F

   (c) the function assignments
      mother(Fred) = Mary, mother(Mary) = Ann, mother(Ann) = - (no assignment)

Signature Model Examples B – (2)

2. <u>interpretation</u>

(a) a different individual is assigned to each of the constant symbols

(a) an <u>entity</u> in D is assigned to each of the <u>constant symbols</u>.
Normally, every entity is assigned to a constant symbol.

(b) the truth value assignments

(b) for each **function**,
an <u>entity</u> is assigned to each possible <u>input of entities</u> to the **function**

love(Fred, Fred) = F, love(Fred, Mary) = F, love(Fred, Ann) = F
love(Mary, Fred) = T, love(Mary, Mary) = F, love(Mary, Ann) = T
love(Ann, Fred) = T, love(Ann, Mary) = T, love(Ann, Ann) = F

(c) the function assignments

(d) for every other **predicate**,
the value T or F is assigned
to each possible <u>input of entities</u> to the **predicate**

mother(Fred) = Mary, mother(Mary) = Ann, mother(Ann) = - (no assignment)

The truth value of sentences

The truth values of **all sentences** are assigned :

1. the truth values for **sentences** developed with the symbols ¬, ∧, ∨, ⇒, ⇔ are assigned as in propositional logic.

2. the truth values for two terms connected by the = symbol is **T** if both terms refer to the same entity; otherwise it is **F**

3. the truth values for ∀x p(x) has value **T** if p(x) has value **T** for **every assignment** to x of an entity in the domain D; otherwise it has value **F**

4. the truth values for ∃x p(x) has value **T** if p(x) has value **T** for **at least one assignment** to x of an entity in the domain D; otherwise it has value **F**

5. the operator **precedence** is as follows ¬, =, ∧, ∨, ⇒, ⇔

6. the **quantifiers** have precedence over the operators

7. **parentheses** change the order of the precedence

Formulas and Sentences

An **formula**

- A **atomic formula**
- The operator ¬ followed by a **formula**
- Two formulas separated by ∧, ∨, ⇒, ⇔
- A quantifier following by a variable followed by a formula

A **sentence**

- A **formula** with no free variables

∀x love(x,y)	: free variable y	: not a sentence
∀x tall(x)	: no free variable	: a sentence

Finding the truth value

Find the truth values of **all sentences**

1. ¬, ∧, ∨, ⇒, ⇔

2. = symbol

3. ∀x p(x)

4. ∃x p(x)

5. the **operator precedence** is as follows ¬, =, ∧, ∨, ⇒, ⇔

6. the **quantifiers** (∀, ∃) have precedence over the **operators**

7. **parentheses** change the order of the precedence

Sentence Examples (1)

Signature

Constant Symbols = {Socrates, Plato, Zeus, Fido}
Predicate Symbols = {human, mortal, legs} all arity one

Model
D: the set of these four particular individuals

Interpretation

(a) a different individual is assigned to each of the constant symbols

(b) the truth value assignment

human(Socrates)=T, human(Plato)=T, human(Zeus)=F, human(Fido)=F
mortal(Socrates)=T, mortal(Plato)=T, mortal(Zeus)=F, mortal(Fido)=T
legs(Socrates)=T, legs(Plato)=T, legs(Zeus)=T, legs(Fido)=T

Sentence Examples (2)

Sentence 1: human(Zeus) ∧ human(Fido) ∨ human(Socrates) = T
F ∧ F ∨ T

Sentence 2: human(Zeus) ∧ (human(Fido) ∨ human(Socrates)) = F
F ∧ (F ∨ T)

Sentence 3: ∀x human(x) = F
human(Zeus)=F, human(Fido)=F

Sentence 4: ∀x mortal(x) = F
mortal(Zeus)=F

Sentence 5: ∀x legs(x) = T
legs(Socrates)=T, legs(Plato)=T, legs(Zeus)=T, legs(Fido)=T

Sentence 6: ∃x human(x) = T
human(Socrates)=T, human(Plato)=T

Sentence 7: ∀x (human(x) ⇒ mortal(x)) = T

Sentence Examples (3)

Sentence 7: $\forall x$ (human(x) $\Rightarrow$ mortal(x)) = T

human(Socrates)=T,	mortal(Socrates)=T,	T $\Rightarrow$ T : T
human(Plato)=T,	mortal(Plato)=T,	T $\Rightarrow$ T : T
human(Zeus)=F,	mortal(Zeus)=F,	F $\Rightarrow$ F : T
human(Fido)=F	mortal(Fido)=T	F $\Rightarrow$ T : T

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