Vector Calculus Identities (H.1)

20160426

Based on
https://en.wikipedia.org/wiki/Dyadics

Gradient of a vector [edit]

See also: covariant derivative

Since the total derivative of a vector field is a linear mapping from vectors to vectors, it is a tensor quantity.

In rectangular coordinates, the gradient of a vector field **f** = (f_1, f_2, f_3) is defined by

$$\nabla \mathbf{f} = g^{jk} \frac{\partial f^i}{\partial x_j} \mathbf{e}_i \mathbf{e}_k$$

where the Einstein summation notation is used and the product of the vectors $\mathbf{e}_i$, $\mathbf{e}_k$ is a dyadic tensor of type (2,0), or the Jacobian matrix

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial (f_1, f_2, f_3)}{\partial (x_1, x_2, x_3)}.$$

In curvilinear coordinates, or more generally on a curved manifold, the gradient involves Christoffel symbols:

$$\nabla \mathbf{f} = g^{jk} \left(\frac{\partial f^i}{\partial x_j} + \Gamma^i{}_{jl} f^l \right) \mathbf{e}_i \mathbf{e}_k$$

where g^{jk} are the components of the metric tensor and the $\mathbf{e}_i$ are the coordinate vectors.

Expressed more invariantly, the gradient of a vector field **f** can be defined by the Levi-Civita connection and metric tensor:[1]

$$\nabla^a \mathbf{f}^b = g^{ac} \nabla_c \mathbf{f}^b$$

where ∇_c is the connection.

Jacobian Matrix

$$\vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$J = \frac{d\vec{f}}{d\vec{x}} = \begin{bmatrix} \frac{d\vec{f}}{dx_1} & \frac{d\vec{f}}{dx_2} & \frac{d\vec{f}}{dx_3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \frac{df_1}{dx_3} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \frac{df_2}{dx_3} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_2} & \frac{df_3}{dx_3} \end{bmatrix}$$

$$J_i^j = \frac{\partial f_i}{\partial x_j} \qquad \begin{array}{cc} & j \\ & \downarrow \\ i \rightarrow & \begin{bmatrix} & & \\ & \square & \\ & & \end{bmatrix} \end{array}$$

$$J_i^j = \frac{\partial f_i}{\partial x_j}$$

$$\begin{array}{c} i=1 \rightarrow \\[2ex] i=2 \rightarrow \\[2ex] i=3 \rightarrow \end{array} \begin{bmatrix} \frac{d f_1}{d x_1} & \frac{d f_1}{d x_2} & \frac{d f_1}{d x_3} \\[1ex] \frac{d f_2}{d x_1} & \frac{d f_2}{d x_2} & \frac{d f_2}{d x_3} \\[1ex] \frac{d f_3}{d x_1} & \frac{d f_3}{d x_2} & \frac{d f_3}{d x_3} \end{bmatrix}$$

$$J_i^j = \frac{\partial f_i}{\partial x_j}$$

$$\begin{array}{c} \begin{array}{ccc} j=1 \downarrow & j=2 \downarrow & j=3 \downarrow \end{array} \\ \begin{bmatrix} \frac{d f_1}{d x_1} & \frac{d f_1}{d x_2} & \frac{d f_1}{d x_3} \\[1ex] \frac{d f_2}{d x_1} & \frac{d f_2}{d x_2} & \frac{d f_2}{d x_3} \\[1ex] \frac{d f_3}{d x_1} & \frac{d f_3}{d x_2} & \frac{d f_3}{d x_3} \end{bmatrix} \end{array}$$

https://en.wikipedia.org/wiki/Vector_calculus_identities

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \\
&= (\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla)\mathbf{A} - (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla)\mathbf{B} \\
&= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}}) - \nabla \cdot (\mathbf{A}\mathbf{B}^{\mathrm{T}}) \\
&= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{B}^{\mathrm{T}})
\end{aligned}$$

.

- The scalar triple product is invariant under a circular shift of its three operands (**a**, **b**, **c**):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

- Swapping the positions of the operators without re-ordering the operands leaves the triple product unchanged. This follows from the preceding property and the commutative property of the dot product.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

- Swapping any two of the three operands negates the triple product. This follows from the circular-shift property and the anticommutativity of the cross product.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$$
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$$
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$$

- The scalar triple product can also be understood as the determinant of the 3 × 3 matrix (thus also its inverse) having the three vectors either as its rows or its columns (a matrix has the same determinant as its transpose):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

https://en.wikipedia.org/wiki/Triple_product

- If the scalar triple product is equal to zero, then the three vectors **a**, **b**, and **c** are coplanar, since the "parallelepiped" defined by them would be flat and have no volume.
- If any two vectors in the triple scalar product are equal, then its value is zero:

$$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{a})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{b})=\mathbf{a}\cdot(\mathbf{a}\times\mathbf{a})=0$$

- Moreover,

$$[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\mathbf{a}=(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})$$

- The simple product of two triple products (or the square of a triple product), may be expanded in terms of dot products:[1]

$$((\mathbf{a}\times\mathbf{b})\cdot\mathbf{c})\;((\mathbf{d}\times\mathbf{e})\cdot\mathbf{f})=\det\left[\begin{pmatrix}\mathbf{a}\\\mathbf{b}\\\mathbf{c}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{d}&\mathbf{e}&\mathbf{f}\end{pmatrix}\right]=\det\Bigg[$$

https://en.wikipedia.org/wiki/Triple_product

The **vector triple product** is defined as the cross product of one vector with the cross product of the other two. The following relationship holds:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

This is known as **triple product expansion**, or **Lagrange's formula**,[2][3] although the latter name is also used for several other formulae. Its right hand side can be remembered by using the mnemonic "BAC − CAB", provided one keeps in mind which vectors are dotted together. A proof is provided below.

Since the cross product is anticommutative, this formula may also be written (up to permutation of the letters) as:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -(\mathbf{c} \cdot \mathbf{b})\mathbf{a} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

From Lagrange's formula it follows that the vector triple product satisfies:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \underline{\mathbf{c} \times (\mathbf{a} \times \mathbf{b})} = 0$$

which is the Jacobi identity for the cross product. Another useful formula follows:

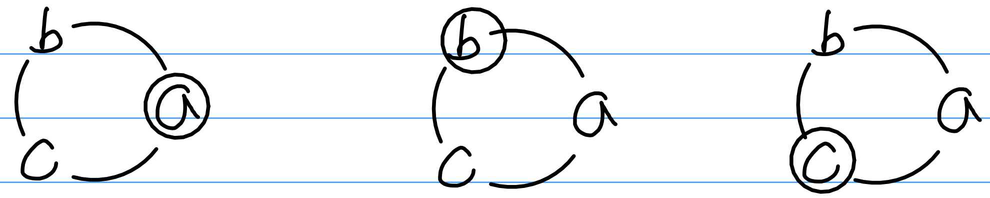
$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \times (\mathbf{a} \times \mathbf{c})$$

These formulas are very useful in simplifying vector calculations in physics. A related identity regarding gradients and useful in vector calculus is Lagrange's formula of vector cross-product identity:[4]

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{f}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{f}$$

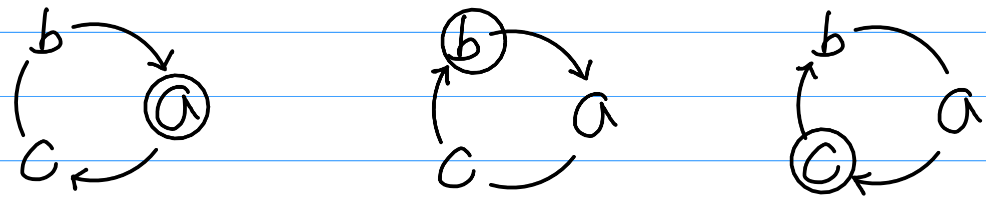
$$(a \times b) \times c = a \times (b \times c) - b \times (a \times c)$$

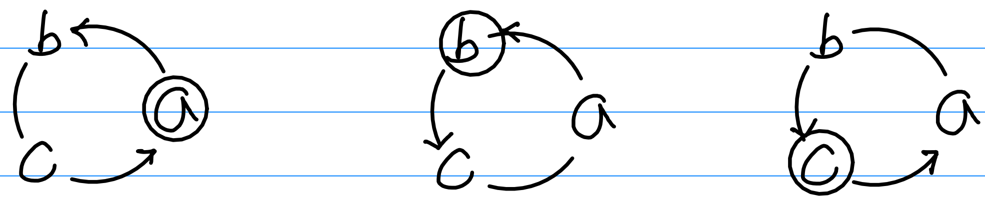
https://en.wikipedia.org/wiki/Triple_product



$\vec{a} \times (\vec{b} \times \vec{c})$ $\quad$ $\vec{b} \times (\vec{c} \times \vec{a})$ $\quad$ $\vec{c} \times (\vec{a} \times \vec{b})$

$b(a \cdot c)$ $\quad$ $c(b \cdot a)$ $\quad$ $a(c \cdot b)$

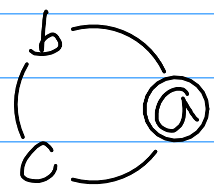


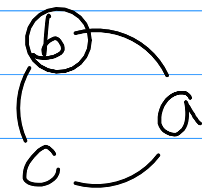


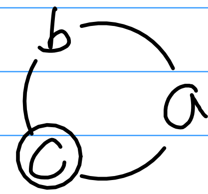
$-c(a \cdot b)$ $\quad$ $-a(b \cdot c)$ $\quad$ $-b(c \cdot a)$

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

$$\begin{array}{ccc} \parallel & \parallel & \parallel \\ b(a \cdot c) & c(b \cdot a) & a(c \cdot b) \\ -c(a \cdot b) & -a(b \cdot c) & -b(c \cdot a) \end{array}$$







$\vec{a}\times(\vec{b}\times\vec{c})$ $\vec{b}\times(\vec{c}\times\vec{a})$ $\vec{c}\times(\vec{a}\times\vec{b})$

a b c   b c a   c a b

a b c   b c a   c a b

$b(a\cdot c)$ $c(b\cdot a)$ $a(c\cdot b)$

$-c(a\cdot b)$ $-a(b\cdot c)$ $-b(c\cdot a)$

$$\boxed{\vec{a}\times(\vec{b}\times\vec{c})} + \boxed{\vec{b}\times(\vec{c}\times\vec{a})} + \boxed{\vec{c}\times(\vec{a}\times\vec{b})} = 0$$

$$\boxed{\begin{matrix} b\,(a\cdot c) \\ -c\,(a\cdot b)\end{matrix}} \qquad \boxed{\begin{matrix} c\,(b\cdot a) \\ -a\,(b\cdot c)\end{matrix}} \qquad \boxed{\begin{matrix} a\,(c\cdot b) \\ -b\,(c\cdot a)\end{matrix}}$$

$$-\boxed{\vec{a}\times(\vec{b}\times\vec{c})} - \boxed{\vec{b}\times(\vec{c}\times\vec{a})} = \boxed{\vec{c}\times(\vec{a}\times\vec{b})}$$

$$-\boxed{\begin{matrix} b\,(a\cdot c) \\ -c\,(a\cdot b)\end{matrix}} - \boxed{\begin{matrix} c\,(b\cdot a) \\ -a\,(b\cdot c)\end{matrix}} \qquad \boxed{\begin{matrix} a\,(c\cdot b) \\ -b\,(c\cdot a)\end{matrix}}$$

$$\begin{aligned}\nabla\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \\ &= (\nabla\cdot\mathbf{B} + \mathbf{B}\cdot\nabla)\mathbf{A} - (\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla)\mathbf{B} \\ &= \nabla\cdot(\mathbf{B}\mathbf{A}^{\mathrm{T}}) - \nabla\cdot(\mathbf{A}\mathbf{B}^{\mathrm{T}}) \\ &= \nabla\cdot(\mathbf{B}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{B}^{\mathrm{T}})\end{aligned}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\ -\mathbf{b}\times(\mathbf{a}\times\mathbf{c})$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=-\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=-(\mathbf{c}\cdot\mathbf{b})\mathbf{a}+(\mathbf{c}\cdot\mathbf{a})\mathbf{b}$$

$$\begin{aligned}\nabla\times(\mathbf{A}\times\mathbf{B})&=\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}\\&=(\nabla\cdot\mathbf{B}+\mathbf{B}\cdot\nabla)\mathbf{A}-(\nabla\cdot\mathbf{A}+\mathbf{A}\cdot\nabla)\mathbf{B}\\&=\nabla\cdot(\mathbf{B}\mathbf{A}^{\mathrm{T}})-\nabla\cdot(\mathbf{A}\mathbf{B}^{\mathrm{T}})\\&=\nabla\cdot(\mathbf{B}\mathbf{A}^{\mathrm{T}}-\mathbf{A}\mathbf{B}^{\mathrm{T}})\end{aligned}$$

$$\nabla\times(A\times B)=\nabla\times(A\times B)+\nabla\times(A\times B)$$

$$\begin{aligned}\nabla\times(A\times B)&=A(\nabla\cdot B)-B(\nabla\cdot A)\\&=(B\cdot\nabla)A-B(\nabla\cdot A)\end{aligned}$$

$$\begin{aligned}\nabla\times(A\times B)&=A(\nabla\cdot B)-B(\nabla\cdot A)\\&\quad(B\cdot\nabla)A-(A\cdot\nabla)B\end{aligned}$$

$$\begin{aligned}\nabla\times(A\times B)=&\ A(\nabla\cdot B)-B(\nabla\cdot A)\\&+(B\cdot\nabla)A-(A\cdot\nabla)B\end{aligned}$$

http://math.stackexchange.com/questions/836041/curl-of-cross-product-of-two-vectors

Directional Derivatives

$$B \cdot \nabla = \langle B_x, B_y, B_z \rangle \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

$$= B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}$$

$$\cancel{=} \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$(B \cdot \nabla) f = \langle B_x, B_y, B_z \rangle \cdot \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f$$

$$= B_x \frac{\partial f}{\partial x} + B_y \frac{\partial f}{\partial y} + B_z \frac{\partial f}{\partial z}$$

$$= B \cdot (\underbrace{\nabla f}_{\text{grad}}) \qquad f_B$$

$$\vec{u} = \langle a, b, c \rangle$$

$$D_{\vec{u}} f(x,y,z) = a \cdot f_x(x,y,z) + b \cdot f_y(x,y,z) + c \cdot f_z(x,y,z)$$

$$= \langle f_x, f_y, f_z \rangle \cdot \langle a, b, c \rangle$$

$$\mathbf{B}\cdot\nabla = B_x,\ B_y,\ B_z$$

$$= B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}$$

$$\cancel{=}\ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$\nabla\blacksquare = \frac{\partial\blacksquare}{\partial x}\vec{i} + \frac{\partial\blacksquare}{\partial y}\vec{j} + \frac{\partial\blacksquare}{\partial z}\vec{k}$$

Op

$$\mathbf{B}\cdot\nabla = \left(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}\right)\cdot\left(\frac{\partial\blacksquare}{\partial x}\vec{i} + \frac{\partial\blacksquare}{\partial y}\vec{j} + \frac{\partial\blacksquare}{\partial z}\vec{k}\right)$$

$$= B_x\frac{\partial\blacksquare}{\partial x} + B_y\frac{\partial\blacksquare}{\partial y} + B_z\frac{\partial\blacksquare}{\partial z}$$

Scalar

$$\nabla\cdot\mathbf{B} = \left(\frac{\partial\blacksquare}{\partial x}\vec{i} + \frac{\partial\blacksquare}{\partial y}\vec{j} + \frac{\partial\blacksquare}{\partial z}\vec{k}\right)\cdot\left(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}\right)$$

$$= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

Dot Del Operator $\dot{\nabla}$

Op $\quad \mathbf{B}\cdot\nabla = B_x \frac{\partial \square}{\partial x} + B_y \frac{\partial \square}{\partial y} + B_z \frac{\partial \square}{\partial z}$

Scalar $\quad \nabla\cdot\mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$

$$\mathbf{B}\cdot\nabla \neq \nabla\cdot\mathbf{B}$$

$$\underset{\text{dot}}{\mathbf{B}\cdot}\underset{\text{del}}{\nabla} = \dot{\nabla}\cdot\mathbf{B}$$