

DTFT (4A)

- Discrete Time Fourier Transform

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DTFS and DTFT

Discrete Time Fourier Series

$$y_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)kn} \iff x[n] = \sum_{k=0}^N y_k e^{+j\left(\frac{2\pi}{N}\right)kn}$$

Discrete Frequency - Periodic

Periodic Continuous Time Signal

Discrete Time Fourier Transform

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n} \iff x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}}) e^{+j\hat{\omega}n} d\hat{\omega}$$

Continuous Frequency - Periodic

Aperiodic Continuous Time Signal

Aperiodic Signal Conversion $x[n]$



From Summation to Integration

DTFS

$$y_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)kn}$$



$$x[n] = \sum_{k=0}^{N-1} y_k e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$$N_0 \rightarrow \infty \quad \hat{\omega}_0 = \left(\frac{2\pi}{N_0}\right) \rightarrow 0$$

$$\hat{\omega}_0 \rightarrow d\hat{\omega}, \quad k\hat{\omega}_0 \rightarrow \hat{\omega}$$

$$x_{N_0}[n] \rightarrow x[n], \quad y_k N_0 \rightarrow X(e^{j\hat{\omega}})$$

$$x_{N_0}[n] = \sum_{k=0}^{N_0-1} y_k e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \frac{2\pi}{2\pi} \cdot \frac{N_0}{N_0}$$

$$x_{N_0}[n] = \frac{1}{2\pi} \sum_{k=0}^{N_0-1} y_k N_0 e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \frac{2\pi}{N_0}$$

DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$



$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}}) e^{+j\hat{\omega}n} d\hat{\omega}$$

From DTFS to DTFT

$$\begin{aligned}
 x_{N_0}[n] &= \sum_{k=0}^{N_0} y_k e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot 1 \\
 &= \sum_{k=0}^{N_0} y_k e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \left(\frac{N_0}{2\pi}\right) \cdot \left(\frac{2\pi}{N_0}\right) \\
 &= \frac{1}{2\pi} \sum_{k=0}^{N_0} y_k N_0 e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \left(\frac{2\pi}{N_0}\right)
 \end{aligned}$$

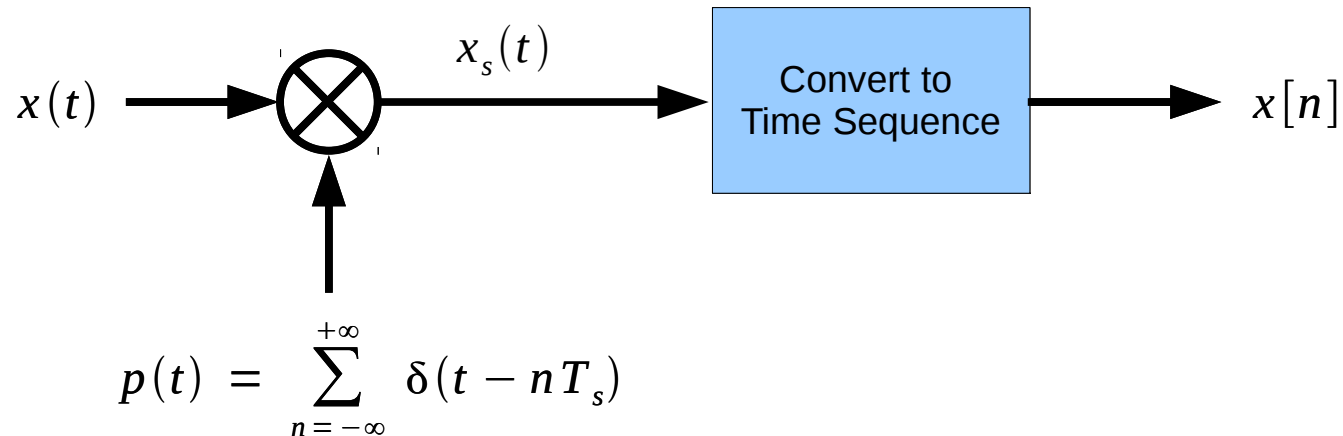
$$x_{N_0}[n] = \frac{1}{2\pi} \sum_{k=0}^{N_0} y_k N_0 e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \left(\frac{2\pi}{N_0}\right)$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}}) e^{+j\hat{\omega}n} d\hat{\omega}$$

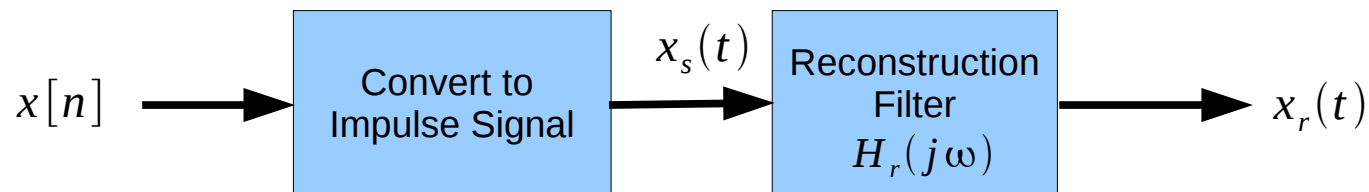
$$\begin{aligned}
 N_0 \rightarrow \infty \quad \hat{\omega}_0 &= \left(\frac{2\pi}{N_0}\right) \rightarrow 0 \\
 \hat{\omega}_0 \rightarrow d\hat{\omega}, \quad k\hat{\omega}_0 &\rightarrow \hat{\omega} \\
 x_{N_0}[n] \rightarrow x[n], \quad y_k N_0 &\rightarrow X(e^{j\hat{\omega}})
 \end{aligned}$$

Sampling and Reconstruction

Ideal Sampling

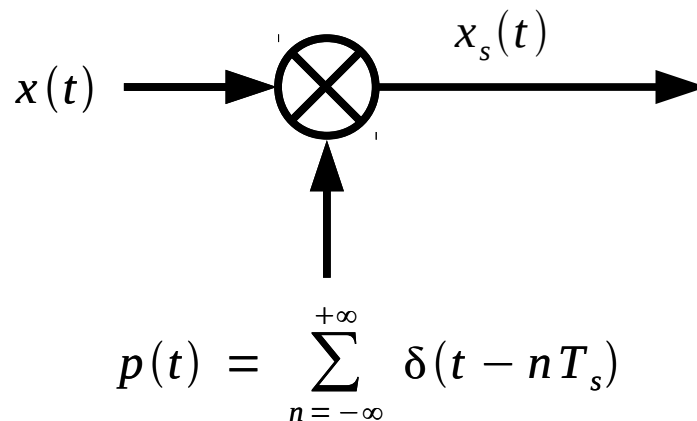


Ideal Reconstruction



Sampled Signal

Ideal Sampling



$$\begin{aligned} x_s(t) &= x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \\ &= \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s) \end{aligned}$$



$$\begin{aligned} x_s(t) &= x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t} \\ \omega_s &= \frac{2\pi}{T_s} \end{aligned}$$

CTFT Frequency Shift Property

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$x_c(t)$$



$$X_c(j\omega)$$

$$x_c(t) e^{jk\omega_s t}$$



$$X_c(j(\omega - k\omega_s))$$

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT Delay Property

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$\delta(t - t_d)$$



$$e^{j\omega t_d}$$

$$\delta(t - nT_s)$$



$$e^{-j\omega nT_s}$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$



$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s) e^{-j\omega nT_s}$$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$

CTFT of a Sampled Signal

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

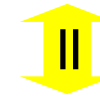
$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

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$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$



$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s) e^{-j\omega nT_s}$$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s) \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega nT_s}$$

z-Transform of a Sampled Signal

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s) e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT of a sampled signal

$$\sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

Z-Transform of a sampled signal

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad x[n] = x_c(nT_s)$$

$$X(z) \Big|_{z = e^{j\omega T_s}}$$

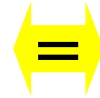
$$= X(e^{j\omega T_s})$$

evaluated at $z = e^{j\omega T_s}$

z-Transform and Normalized Frequency

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s) e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(z) \Big|_{z = e^{j\omega T_s}}$$

$$= X(e^{j\omega T_s}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}$$

z-Transform



$$\hat{\omega} = \omega T_s$$

Normalized Frequency

$$X(z) \Big|_{z = e^{j\hat{\omega}}}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$

Discrete Time Fourier Transform

DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s) e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n T_s}$$



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$

DTFT of a sampled signal

$$X(e^{j\hat{\omega}})$$

$$\hat{\omega} = \omega T_s$$

$$= X(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

CTFT of a sampled signal

CTFS and DTFS

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

Discrete Time Fourier Transform

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n} \quad \longleftrightarrow \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}}) e^{+j\hat{\omega}n} d\hat{\omega}$$

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