DTFT (4A)

- Discrete Time Fourier Transform

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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DTFS and DTFT

Discrete Time Fourier **Series**

$$\gamma_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \iff x[n] = \sum_{k=0}^{N} \gamma_k\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

Discrete Frequency – Periodic

Periodic Continuous Time Signal

Discrete Time Fourier **Transform**

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega}n} \iff x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}})\, e^{+j\hat{\omega}n}\, d\hat{\omega}$$

Continuous Frequency – Periodic

Aperiodic Continuous Time Signal

Aperiodic Signal Conversion $x[n]$

From Summation to Integration

DTFS

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \Longleftrightarrow \quad x[n] = \sum_{k=0}^{N} \gamma_k\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$N_0 \to \infty \qquad \hat{\omega}_0 = \left(\frac{2\pi}{N_0}\right) \to 0$

$\hat{\omega}_0 \to d\hat{\omega}, \quad k\hat{\omega}_0 \to \hat{\omega}$

$x_{N_0}[n] \to x[n], \quad \gamma_k N_0 \to X(e^{j\hat{\omega}})$

$$x_{N_0}[n] = \sum_{k=0}^{N_0} \gamma_k\, e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \frac{2\pi}{2\pi} \cdot \frac{N_0}{N_0}$$

$$x_{N_0}[n] = \frac{1}{2\pi}\sum_{k=0}^{N_0} \gamma_k N\, e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \frac{2\pi}{N_0}$$

DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega}n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\hat{\omega}})\, e^{+j\hat{\omega}n}\, d\hat{\omega}$$

From DTFS to DTFT

$$x_{N_0}[n] = \sum_{k=0}^{N_0} \gamma_k \, e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot 1$$

$$= \sum_{k=0}^{N_0} \gamma_k \, e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \left(\frac{N_0}{2\pi}\right) \cdot \left(\frac{2\pi}{N_0}\right)$$

$$= \frac{1}{2\pi} \sum_{k=0}^{N_0} \gamma_k N_0 \, e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \left(\frac{2\pi}{N_0}\right)$$

$$x_{N_0}[n] = \frac{1}{2\pi} \sum_{k=0}^{N_0} \gamma_k N_0 \, e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \left(\frac{2\pi}{N_0}\right)$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}}) \, e^{+j\hat{\omega} n} \, d\hat{\omega}$$

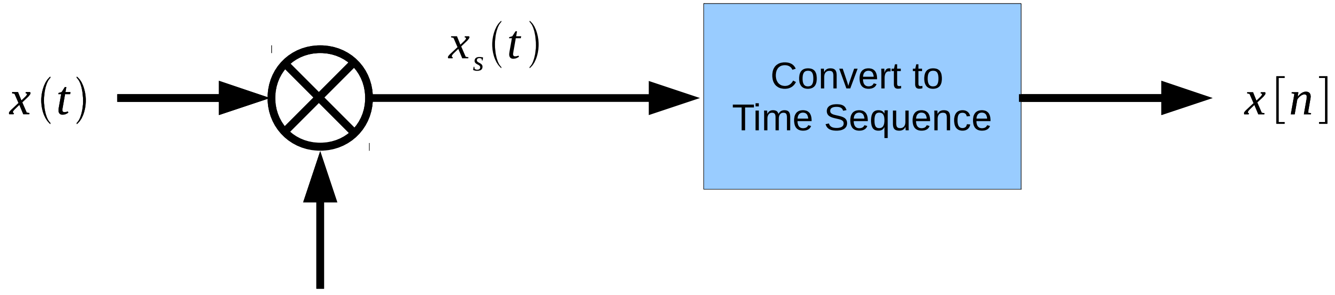
$N_0 \rightarrow \infty \qquad \hat{\omega}_0 = \left(\frac{2\pi}{N_0}\right) \rightarrow 0$

$\hat{\omega}_0 \rightarrow d\hat{\omega}, \quad k\hat{\omega}_0 \rightarrow \hat{\omega}$

$x_{N_0}[n] \rightarrow x[n], \quad \gamma_k N_0 \rightarrow X(e^{j\hat{\omega}})$

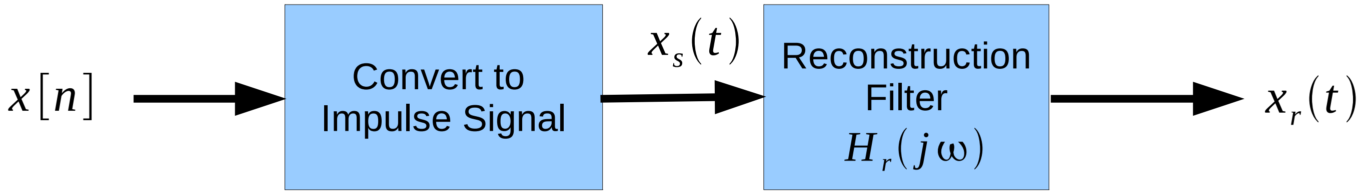
Sampling and Reconstruction

Ideal Sampling



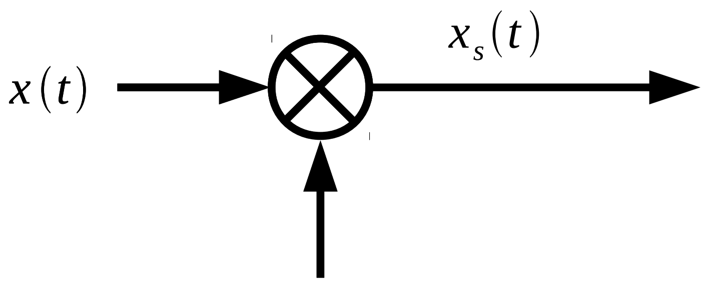
$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

Ideal Reconstruction



Sampled Signal

Ideal Sampling



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s)\delta(t - nT_s)$$

$\Leftrightarrow$

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT Frequency Shift Property

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \Leftrightarrow \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x_c(t) \quad \Leftrightarrow \quad X_c(j\omega)$$

$$x_c(t)e^{jk\omega_s t} \quad \Leftrightarrow \quad X_c(j(\omega - k\omega_s))$$

$$x_s(t) = x_c(t)\sum_{k=-\infty}^{+\infty}\frac{1}{T_s}e^{jk\omega_s t} \quad \Leftrightarrow \quad X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s} \qquad\qquad \omega_s = \frac{2\pi}{T_s}$$

CTFT Delay Property

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \Leftrightarrow \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\delta(t - t_d) \quad \Leftrightarrow \quad e^{j\omega t_d}$$

$$\delta(t - nT_s) \quad \Leftrightarrow \quad e^{-j\omega n T_s}$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\delta(t - nT_s)$$

$$\Leftrightarrow$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s)\, e^{-j\omega n T_s} = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

CTFT of a Sampled Signal

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \Longleftrightarrow \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x_s(t) = x_c(t)\sum_{k=-\infty}^{+\infty}\frac{1}{T_s}\, e^{jk\omega_s t} \quad \Longleftrightarrow \quad X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s} \qquad\qquad \omega_s = \frac{2\pi}{T_s}$$

$\Updownarrow$ || $\qquad\qquad$ $\Updownarrow$ ||

$$x_s(t) = x_c(t)\sum_{n=-\infty}^{+\infty}\delta(t - nT_s) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\,\delta(t - nT_s)$$

$\Longleftrightarrow$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s)\, e^{-j\omega nT_s} = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega nT_s}$$

z-Transform of a Sampled Signal

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s)\, e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$\Leftrightarrow$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT of a sampled signal

$$\sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

Z-Transform of a sampled signal

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad x[n] = x_c(nT_s)$$

$$X(z)\Big|_{z = e^{j\omega T_s}} = X(e^{j\omega T_s})$$

evaluated at $z = e^{j\omega T_s}$

z-Transform and Normalized Frequency

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s)\, e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(z)\Big|_{z=e^{j\omega T_s}} = X(e^{j\omega T_s}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

z-Transform

$$\hat{\omega} = \omega T_s$$

Normalized Frequency

$$X(z)\Big|_{z=e^{j\hat{\omega}}} \qquad X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

Discrete Time Fourier Transform

DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} X_c(nT_s)\, e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$\Leftrightarrow$ =

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) \quad \hat{\omega} = \omega T_s$$

$$= X(e^{j\omega T_s}) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c\left(j\left(\omega - \frac{2\pi k}{T_s}\right)\right)$$

CTFT of a sampled signal

CTFS and DTFS

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t} d\omega$$

Discrete Time Fourier Transform

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Leftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+pi} X(e^{j\hat{\omega}})\, e^{+j\hat{\omega} n}$$

References

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