Variable Block Adder (1B)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Delay model

n : the number of bits in a carry skip adder
m : the number of groups into which the bits are divided
$x_1, \dots, x_m$: the sizes of the groups beginning with the most significant bit

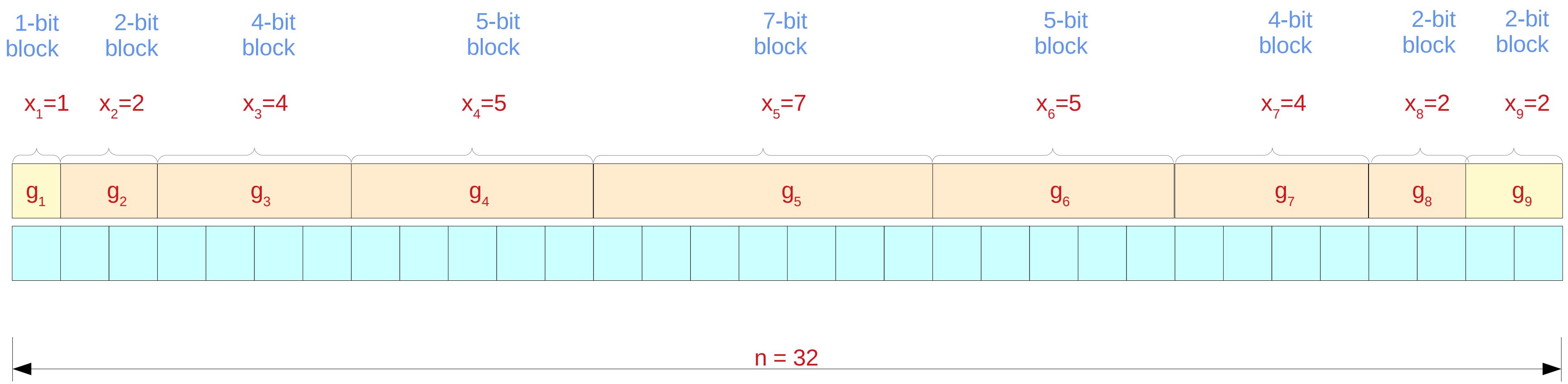
T : the time required for a carry signal to skip over a group of bits

To be precise we should write $T = T(x)$ to indicate that
T depends on the size x of the group over which the carry is skipped
However, T changes very slowly with x over the range of group sizes
So we assume that T is constant

For a given n, the following three step procedure gives
An optimal way of dividing an n bit adder into groups of bits

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Variable Block



- total n = 32 bits
- m = 9 groups
- i-th group has x_i bits (size)
- constant skip delay $T = T(x_i)$

Maximum propagation time P

Lemma 1 When the bits of a carry skip adder are grouped according to the scheme (i)-(iii), the maximum propagation time of a carry signal is mT

- ***n*** bits
- ***m*** groups

The carry generated at the 2^{nd} bit position and terminating at the $(n\text{-}1)^{th}$ bit position clearly has propagation time mT.

We must show that *any other* carry signal has propagation time smaller than or equal to mT

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum propagation time P

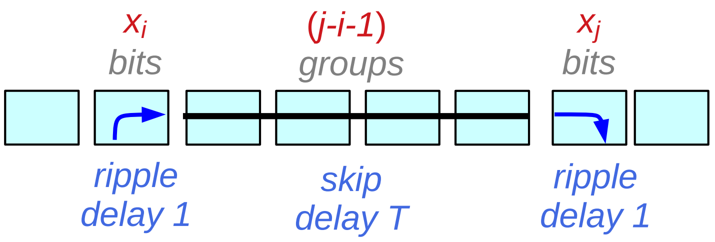
- ***n*** bits
- ***m*** groups

Consider a carry signal originating in the *i-th* group and terminating in the *j-th* group i < j.

Denote its propagation time by P.

$$P \le (x_i - 1) + (j - i - 1)T + (x_j - 1) \le mT$$

$$\forall i, \forall j \quad 1 \le i, j \le m$$



P: the propagation time of any delay path is less than or equal to mT

*generated in the **i-th** group*
*terminated in the **j-th** group*

delay of a carry [Gi, Gj] for all i , j

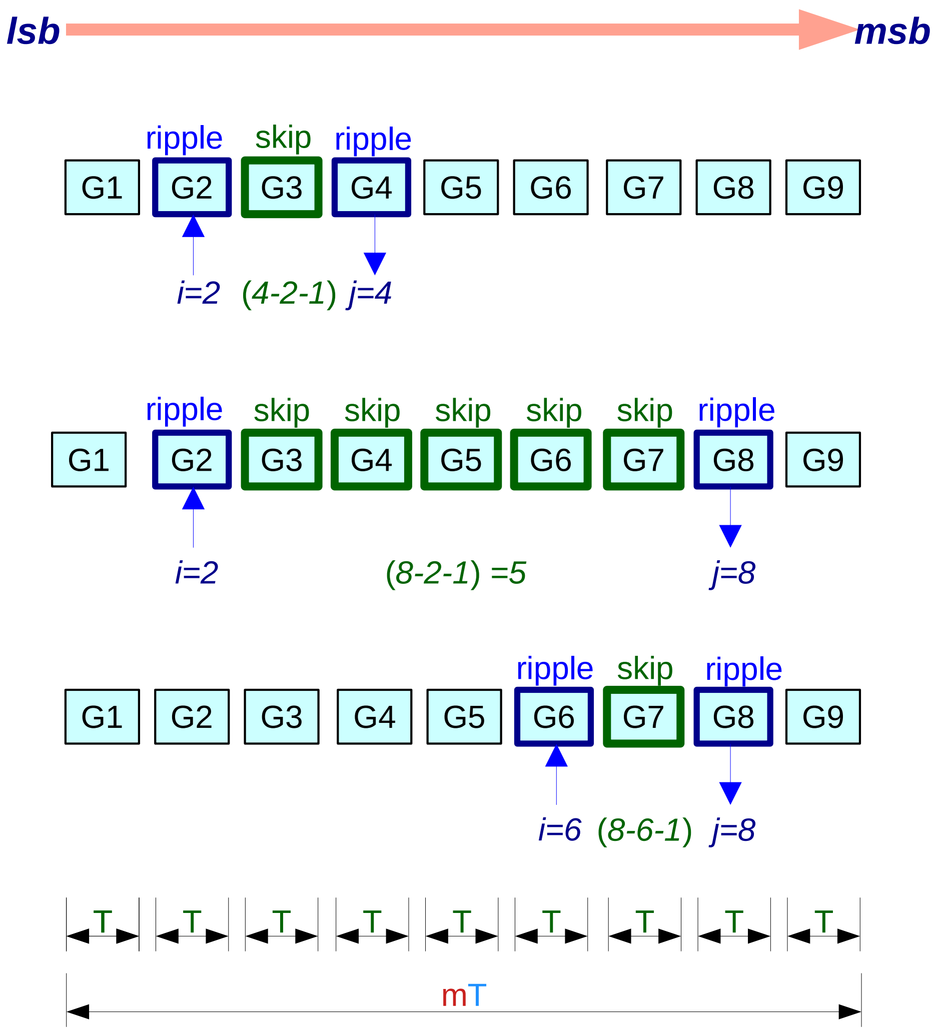
$P_{i,j} \le mT$

$$P = P_{i,j} \le mT$$

$$\forall i, \forall j \quad 1 \le i, j \le m$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

A carry signal from the ith group to the jth group



- ***n*** bits
- ***m*** groups

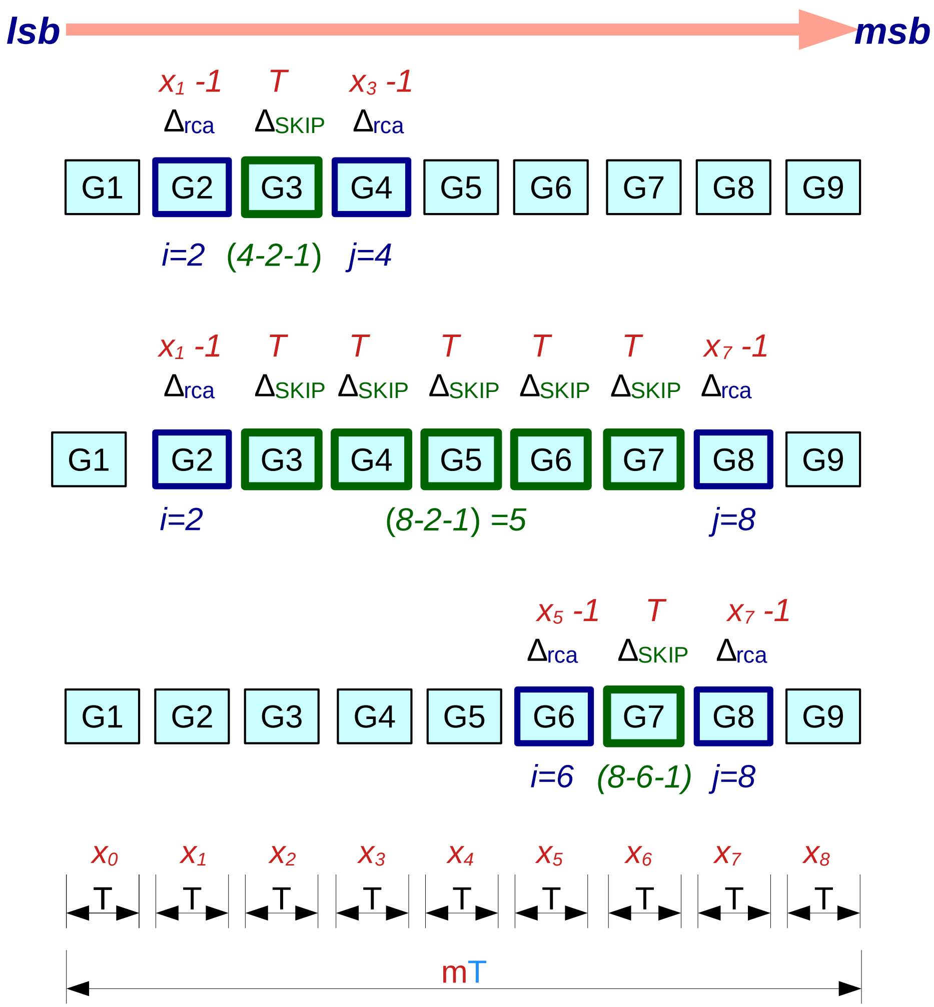
The propagation time of any delay path (three cases) **≤ *mT***

a carry is generated in the *i-th* group
and terminated in the *j-th* group $i < j$.

$$P \le (x_i - 1)\cdot 1 + (j - i - 1)\cdot T + (x_j - 1)\cdot 1$$
$$\le mT$$

$\Delta_{rca} = 1$	$\Delta_{SKIP} = T$	$\Delta_{rca} = 1$
ripple delay	skip delay	ripple delay

Three cases



- ***n*** bits
- ***m*** groups

The propagation time of any delay path is less than or equal to mT

$$P \le (x_i - 1)\cdot 1 + (j - i - 1)\cdot T + (x_j - 1)\cdot 1$$

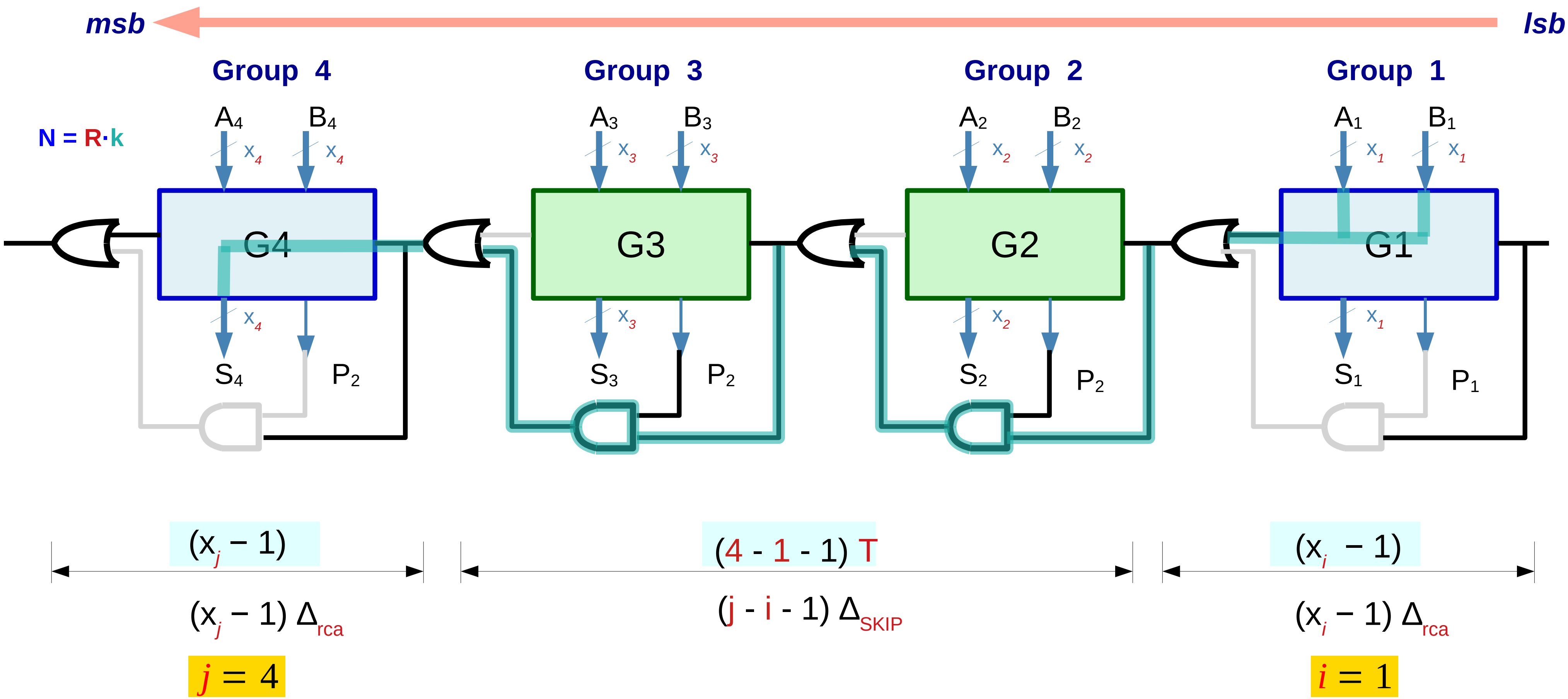
$$\le mT$$

x_i the bit size of the i-th group optimally chosen

$$n = \sum_{i=1}^{m} x_i$$

$$n \le \sum_{i=1}^{m} y_i$$

Propagation delay P example

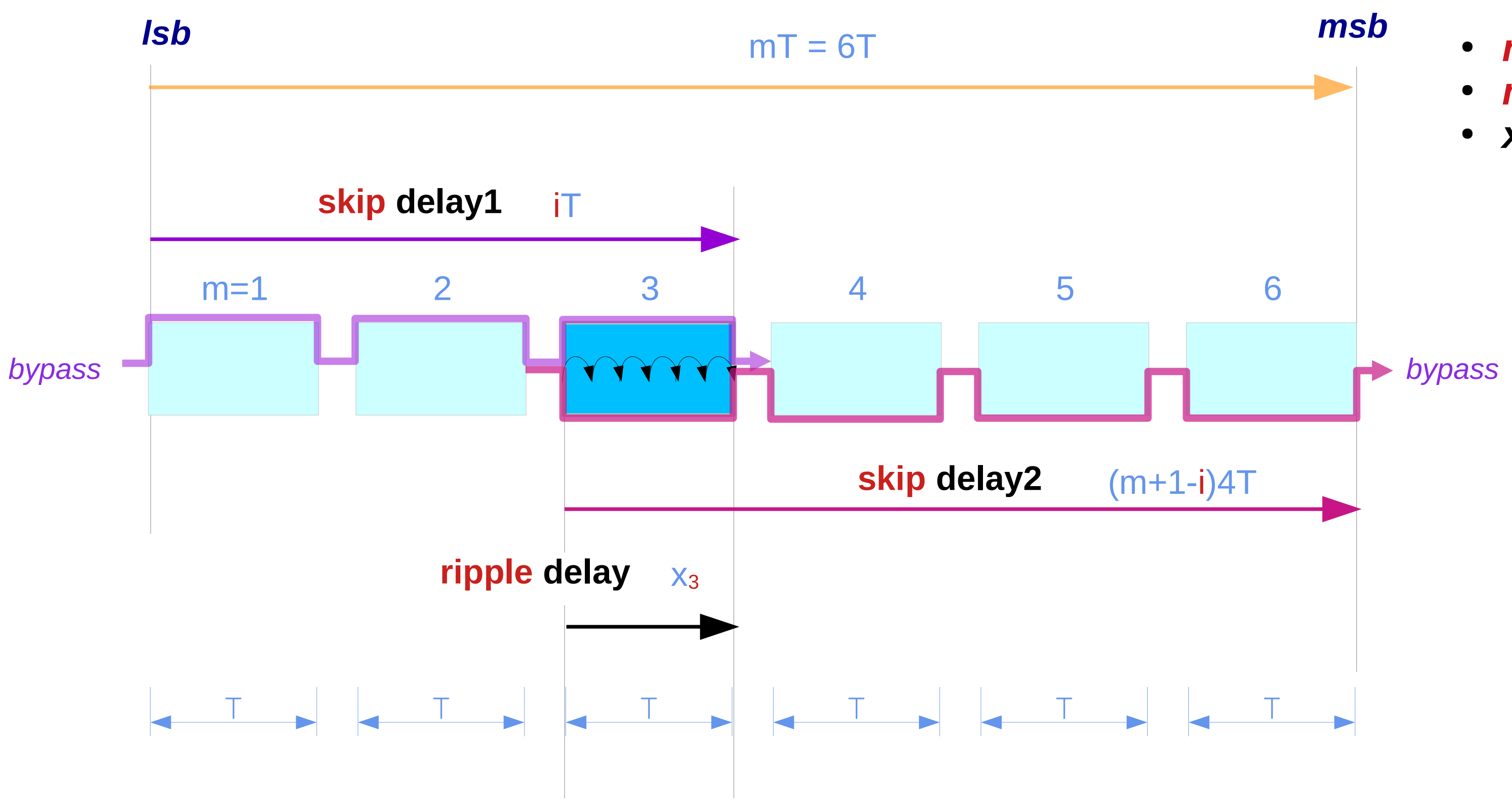


$$P \le (x_0 - 1) + (4 - 1 - 1)T + (x_3 - 1) \le mT$$

$\Delta_{rca} = 1$ ripple delay over a bit

$\Delta_{SKIP} = T$ skip delay over a group

Minimum skip path delay y_i of the i^{th} group

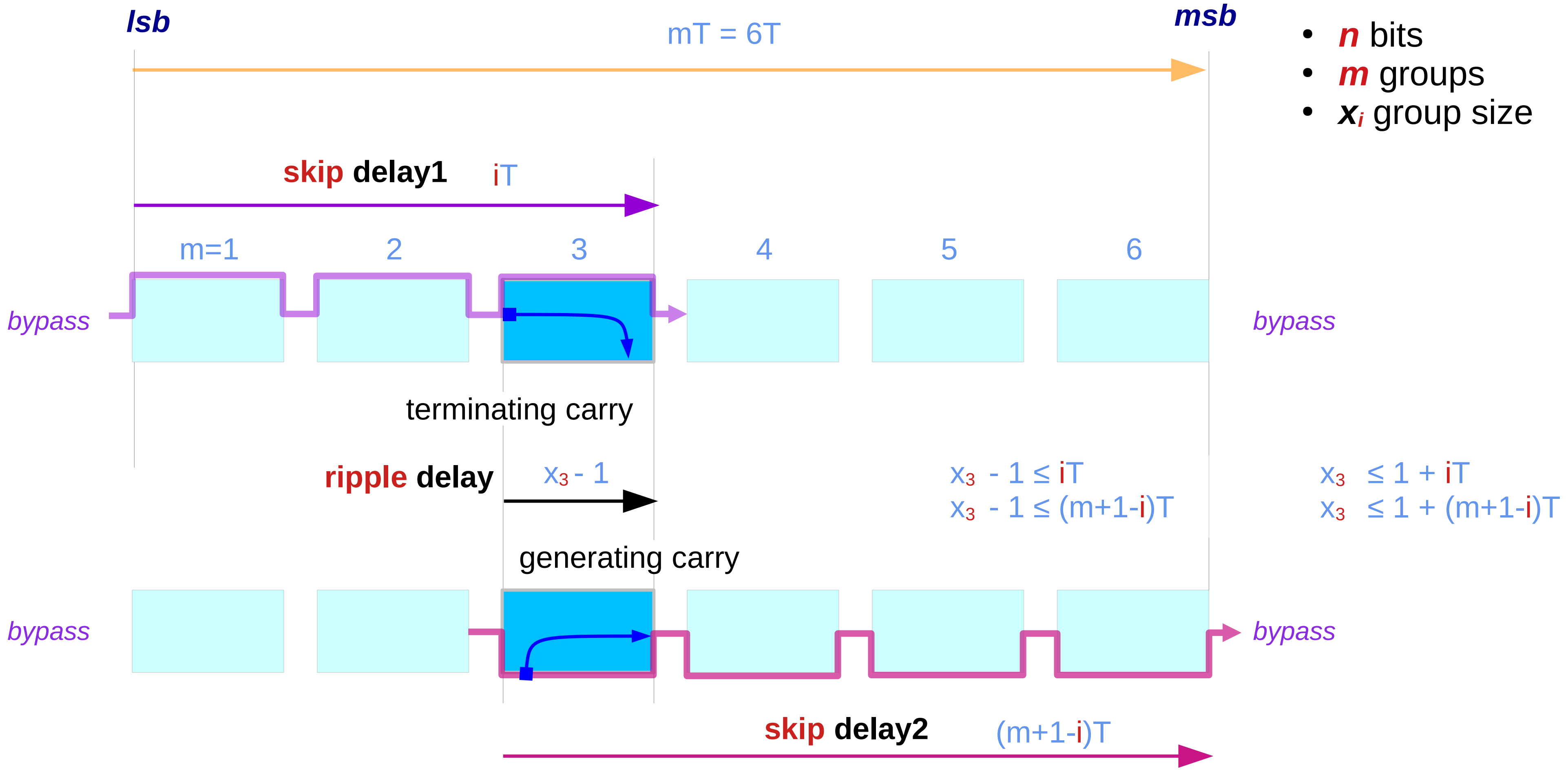


- ***n*** bits
- ***m*** groups
- x_i group size

terminating carry	**ripple delay ≤ skip delay1**
generating carry	**ripple delay ≤ skip delay2**

Minimum skip path delay y_i of the i^{th} group

- ***n*** bits
- ***m*** groups
- x_i group size



Ripple delay x_i constraints of the i^{th} group

x_i delay$_{ripple}$ ripple delay of a group

delay1$_{skip}$
delay2$_{skip}$ skip delay over a group

delay$_{ripple}$ ≤ delay1$_{skip}$
delay$_{ripple}$ ≤ delay2$_{skip}$

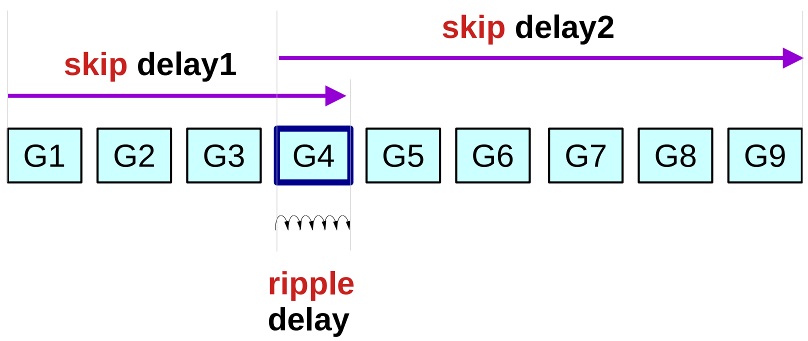
$$y_i = min\{1+iT,\ 1+(m+1-i)T\}$$

min {delay1$_{skip}$, delay2$_{skip}$}

minimum skip path delay

$$0 \le x_i \le y_i, \quad i=1,\dots,m$$

beginning skip path delay **ending skip path delay**

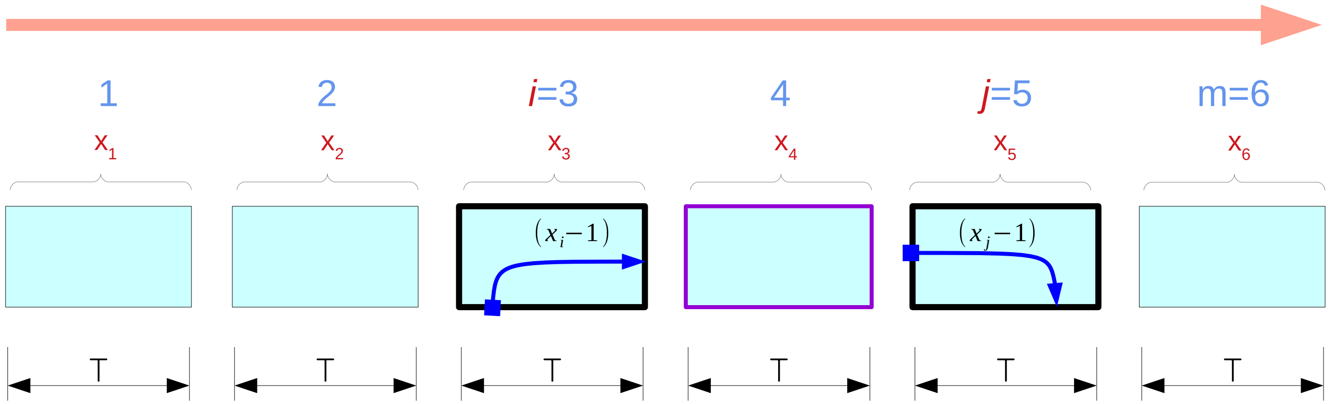


ripple delay ≤ skip delay1
ripple delay ≤ skip delay2

ripple delay
≤ min { **skip delay1**, **skip delay2** }

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Applying x_i constraints to the propagation delay P (1)



- n bits
- m groups

j-i-1 = 5-3-1 = 1

$\Delta_{rca} = 1$

$\Delta_{SKIP} = T$

the *i-th* group : carry generating group

the *j-th* group : carry terminating group

$$x_i \le y_i = min\{1 + i\,T,\ 1 + (m + 1 - i)\,T\}$$

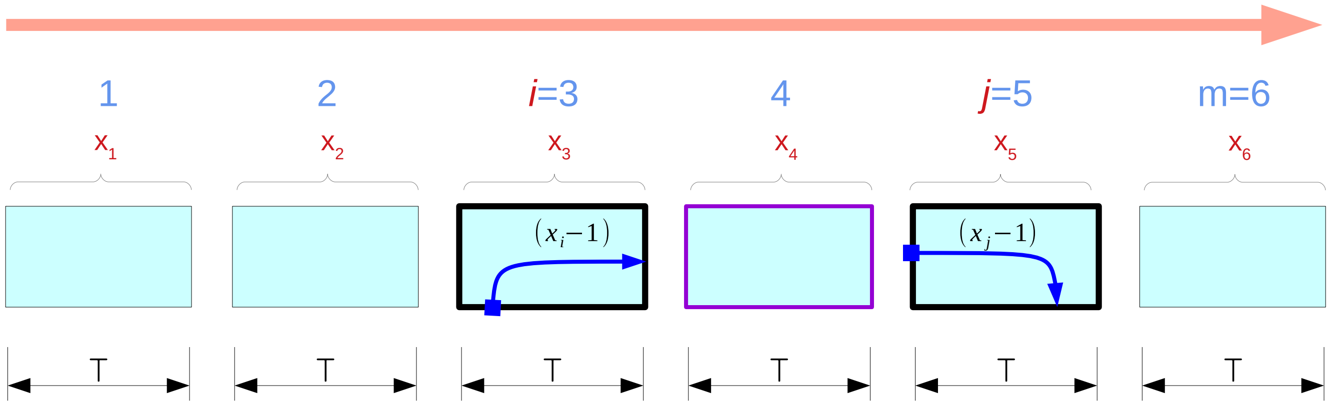
$$x_j \le y_j = min\{1 + j\,T,\ 1 + (m + 1 - j)\,T\}$$

beginning skip path delay

ending skip path delay

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Applying x_i constraints to the propagation delay P (2)



- ***n*** bits
- ***m*** groups

j-i-1 = 5-3-1 = 1

$\Delta_{rca} = 1$

$\Delta_{SKIP} = T$

$$P \le (x_i - 1) + (j - i - 1)T + (x_j - 1) \le mT$$

$$x_i \le \min\{1 + iT, 1 + (m+1-i)T\}$$

$$x_j \le \min\{1 + jT, 1 + (m+1-j)T\}$$

Assume a carry signal is generated in the *i-th* group and terminated in the *j-th*, i < j.

P denotes propagation time

$$P \le \min\{1 + iT, 1 + (m+1-i)T\} + \min\{1 + jT, 1 + (m+1-j)T\} + (j - i - 1)T - 2$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Three cases of y_i and y_j

Case 1: both groups are in the left half

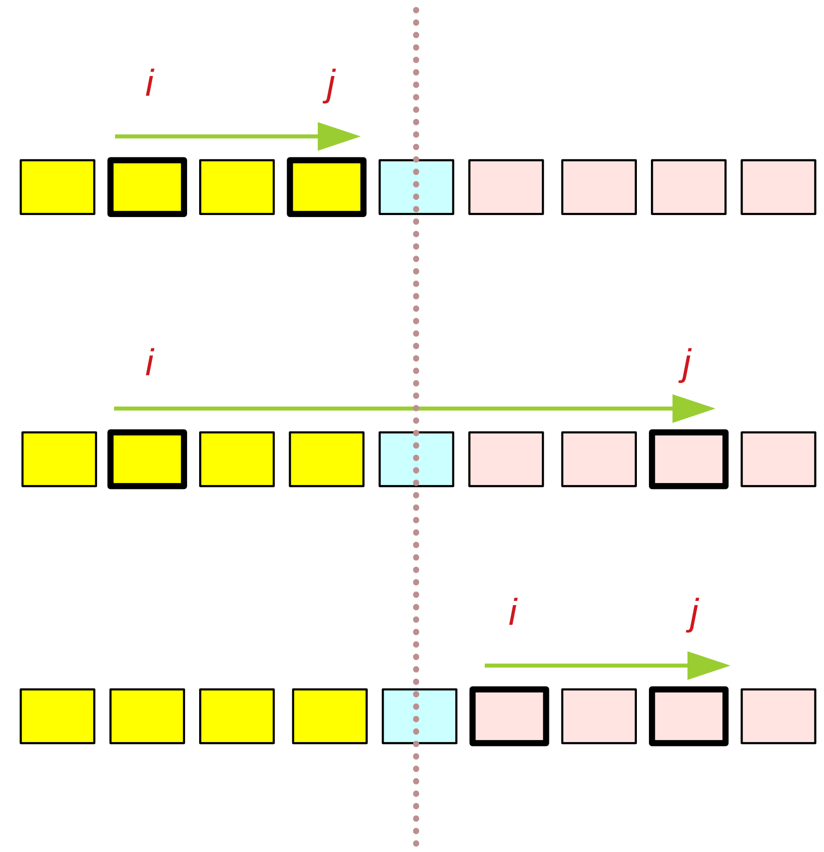
$$y_i = min\{1+iT, 1+(m+1-i)T\} = 1+iT$$
$$y_j = min\{1+jT, 1+(m+1-j)T\} = 1+jT$$

Case 2: one group in the left and the other in the right

$$y_i = min\{1+iT, 1+(m+1-i)T\} = 1+iT$$
$$y_j = min\{1+jT, 1+(m+1-j)T\} = 1+(m+1-j)T$$

Case 3: both groups are in the right half

$$y_i = min\{1+iT, 1+(m+1-i)T\} = 1+(m+1-i)T$$
$$y_j = min\{1+jT, 1+(m+1-j)T\} = 1+(m+1-j)T$$



Beginning Skip Paths

$1+iT$

$1+jT$

Ending Skip Paths

$1+(m+1-i)T$

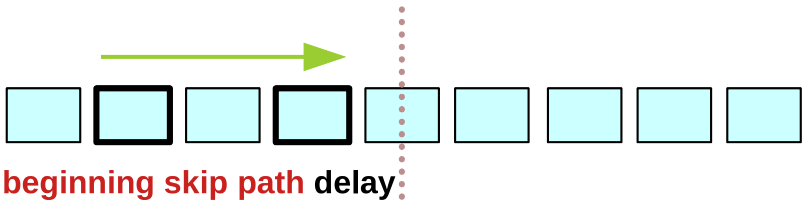
$1+(m+1-j)T$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Case 1

Case 1: both groups are in the left half

$$min\{1+iT, 1+(m+1-i)T\} = 1+iT$$
$$min\{1+jT, 1+(m+1-j)T\} = 1+jT$$



$$P \leq min\{1+iT, 1+(m+1-i)T\} + min\{1+jT, 1+(m+1-j)T\} + (j-i-1)T - 2$$
$$= 1+iT + 1+jT + (j-i-1)T - 2$$
$$= 2jT - T \leq mT$$

$$min\{1+jT, 1+(m+1-j)T\} = 1+jT$$
$$1+jT \leq 1+(m+1-j)T \Rightarrow 2jT \leq (m+1)T \Rightarrow 2jT - T \leq mT$$

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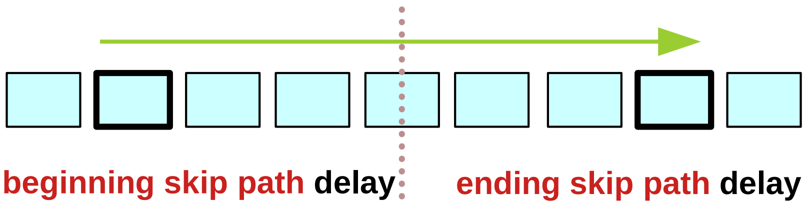
Case 2

Case 2: one group in the left and the other in the right

$$\min\{1+iT, 1+(m+1-i)T\} = 1+iT$$

$$\min\{1+jT, 1+(m+1-j)T\} = 1+(m+1-j)T$$



$$P \le \min\{1+iT, 1+(m+1-i)T\} + \min\{1+jT, 1+(m+1-j)T\} + (j-i-1)T - 2$$

$$= 1+iT + 1+(m+1-j)T + (j-i-1)T - 2 = mT$$

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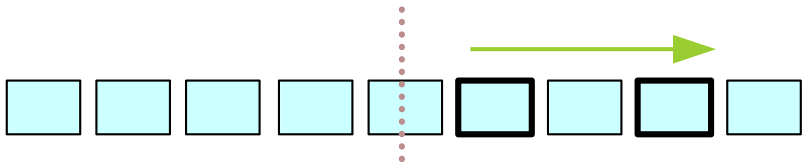
Case 3

Case 3: both groups are in the right half

$$\min\{1+iT,\ 1+(m+1-i)T\} = 1+(m+1-i)T$$

$$\min\{1+jT,\ 1+(m+1-j)T\} = 1+(m+1-j)T$$



$$P \le \min\{1+iT,\ 1+(m+1-i)T\} + \min\{1+jT,\ 1+(m+1-j)T\} + (j-i-1)T - 2$$

$$= 1+(m+1-i)T + 1+(m+1-j)T + (j-i-1)T - 2$$

$$= 2(m+1-i)T - T = 2mT - (2iT - T) = mT$$

$$1+iT \ge 1+(m+1-i)T \Rightarrow 2iT \ge (m+1)T \Rightarrow 2iT - T \ge mT$$

$$-(2iT - T) \le -mT$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum delay of a carry signal

Lemma 2 Let *D* denote the maximum delay of a carry signal in a *n* bit carry skip adder with group sizes chosen optimally. Then

$$(m-1)T \le D \le mT$$

- ***n*** bits
- ***r*** groups

Since we have exhibited a division of the carry chain into groups
In such a way that the maximum delay of a carry signal is *mT*
We clearly have $D \le mT$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum delay of a carry signal

- ***n*** bits
- ***r*** groups

$$D \leq mT$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum delay of a carry signal

D : the maximum delay of a carry signal
in a *n* bit carry skip adder of *m* groups
with group sizes x_i's chosen optimally.

- ***n*** bits
- ***m*** groups

$$(m-1)T \le D \le mT$$

$$P = P_{i,j} \le mT$$

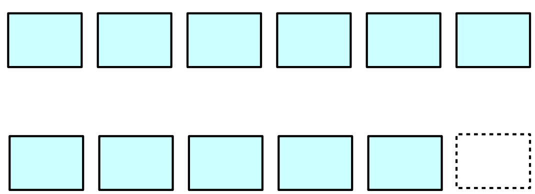
$$\forall i, \forall j \quad 1 \le i, j \le m$$

the carry chain is divided into ***m*** groups
the maximum delay of a carry signal is *mT*

$$D = max\{P_{i,j}\} \le mT$$

m groups

m-1 groups



mT

(m-1)T

$$D \le mT$$
$$(m-1)T \le D$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum delay of a carry signal

$$(m-1)T \;\le\; D \;\le\; mT$$

*Assume there are **r** groups*

*then 2 cases : **even r**, **odd r***

for each of these 2 cases

prove $mT - D < T + 1$

$mT - D \le T$

then $(m-1)T \le D$

P: ***the propagation delay of any carry signal path ≤ mT***

upper bound

D: ***the max of them***

diff (mT, D) ≤ T

diff (mT,** max **P) ≤ T

tight upper bound

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Determining *m*

Method 1 – using a histogram

Let *m* be the <u>smallest</u> positive integer such that

$$n \le \sum_{i=1}^{m} y_i$$

$$0 \le x_i \le y_i, \quad i=1,\ldots,m$$

$$y_i = min\{1+iT, 1+(m+1-i)T\}, \quad i = 1,\ldots,m$$

Method 2 – using a closed formula

Let *m* be the <u>smallest</u> positive integer such that

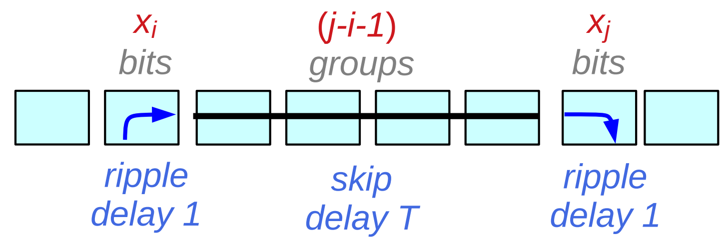
$$n \le m + \frac{1}{2}mT + \frac{1}{4}m^2T + (1-(-1)^m)\frac{1}{8}T$$

Determining $\mathbf{x_i}$ for an even number of groups **r = 2k**

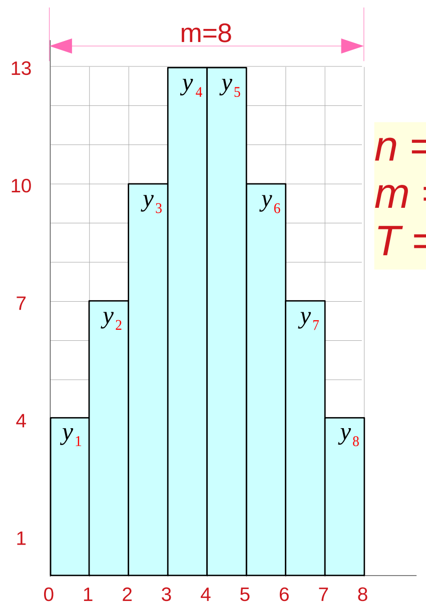
$r = 2k$

$$P \le (x_i - 1) + (j - i - 1)T + (x_j - 1) \le mT$$

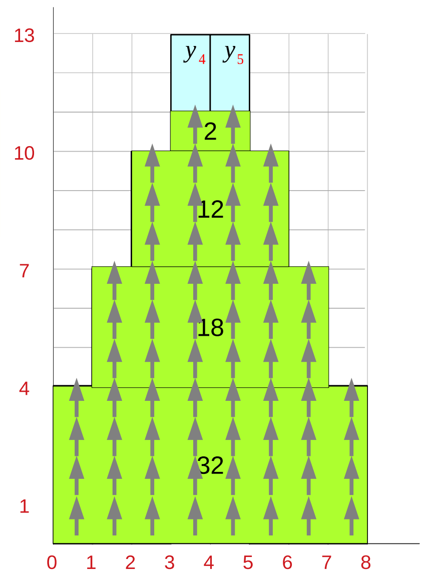
$$(x_i - 1) + (x_j - 1) + (j - i - 1)T$$



- ***n*** bits
- ***r* = *m* = 2k** groups



n =64
m =8
T =3



The x_i's can be computed iteratively as follows:

Initially take $x_1 = x_m = 0$

At each iteration,
increase as many of the x_i's as possible by one unit, without violating the constraints

$$0 \le x_i \le y_i, \quad i = 1, \ldots, m$$

Thus, at some iteration, we have $\sum_{i=1}^{m} x_i = n$ and the algorithm terminates

x_1=4, x_2=7, x_3=10, x_4=11, x_5=11, x_6=10, x_7=7, x_8=4

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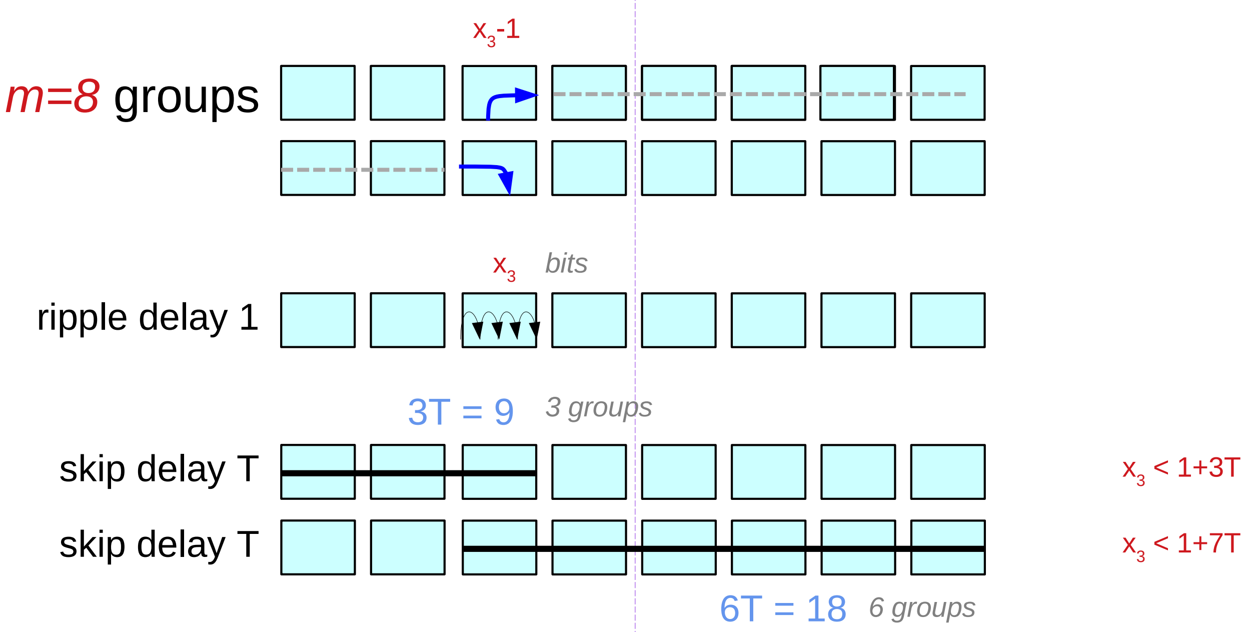
An example of $x_3 \le y_3$ example (r = 2k)

overlapping latencies



$$y_3 = min\{1+3T, 1+(8+1-3)T\} = 1+3T$$

$$x_3 - 1 \le min\{3T, (8+1-3)T\} = 3T$$

$$x_3 \le min\{1+3T, 1+(8+1-3)T\} = 1+3T$$

$$y_i = min\{1+iT, 1+(m+1-i)T\}$$
$$i = 1, \dots, m$$

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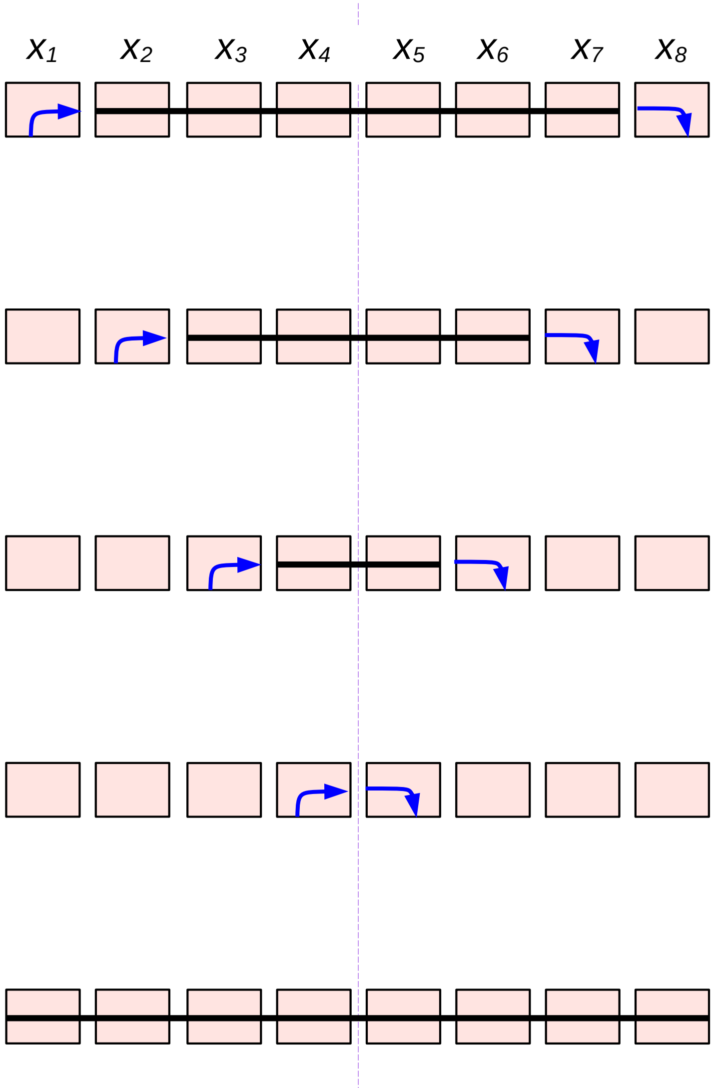
Maximum delays of carry signals (**r = 2k**)



The maximum delay of carry signals generated in the 1st group or terminated in the 8th group $\leq D$

The maximum delay of carry signals generated in the 2nd group or terminated in the 7th group $\leq D$

The maximum delay of carry signals generated in the 3rd group or terminated in the 6th group $\leq D$

The maximum delay of carry signals generated in the 4th group or terminated in the 5th group $\leq D$

All skip delay $\leq D$

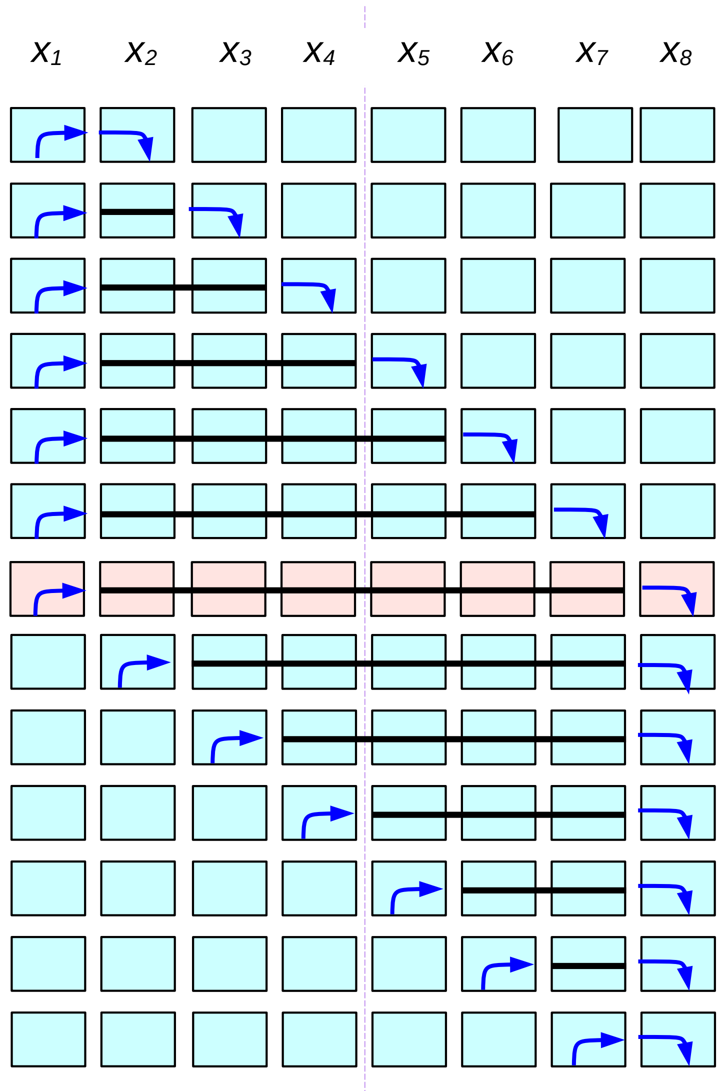
Max delay of all carry signals

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Delays of carry signals [G1, G8] (r = 2·4)

carry signals generating in the 1st groups

carry signals terminating in the 8th groups



x_1=4, x_2=7, x_3=10, x_4=11, x_5=11, x_6=10, x_7=7, x_8=4

$x_1 + x_2 - 2$	$= 4 + 7 - 2$	$= 9$	$= 9$
$x_1 + T + x_3 - 2$	$= 4 + T + 10 - 2$	$= 12 + T$	$= 15$
$x_1 + 2T + x_4 - 2$	$= 4 + 2T + 11 - 2$	$= 13 + 2T$	$= 19$
$x_1 + 3T + x_5 - 2$	$= 4 + 3T + 11 - 2$	$= 13 + 3T$	$= 22$
$x_1 + 4T + x_6 - 2$	$= 4 + 4T + 10 - 2$	$= 12 + 4T$	$= 24$
$x_1 + 5T + x_7 - 2$	$= 4 + 5T + 7 - 2$	$= 9 + 5T$	$= 24$
$x_1 + 6T + x_8 - 2$	$= 4 + 6T + 4 - 2$	$= 6 + 6T$	$= 24$
$x_2 + 5T + x_8 - 2$	$= 7 + 5T + 4 - 2$	$= 9 + 5T$	$= 24$
$x_3 + 4T + x_8 - 2$	$= 10 + 4T + 4 - 2$	$= 12 + 4T$	$= 24$
$x_4 + 3T + x_8 - 2$	$= 11 + 3T + 4 - 2$	$= 13 + 3T$	$= 22$
$x_5 + 2T + x_8 - 2$	$= 11 + 2T + 4 - 2$	$= 13 + 2T$	$= 19$
$x_6 + T + x_8 - 2$	$= 10 + T + 4 - 2$	$= 12 + T$	$= 15$
$x_7 + x_8 - 2$	$= 7 + 4 - 2$	$= 9$	$= 9$

Assume T=3

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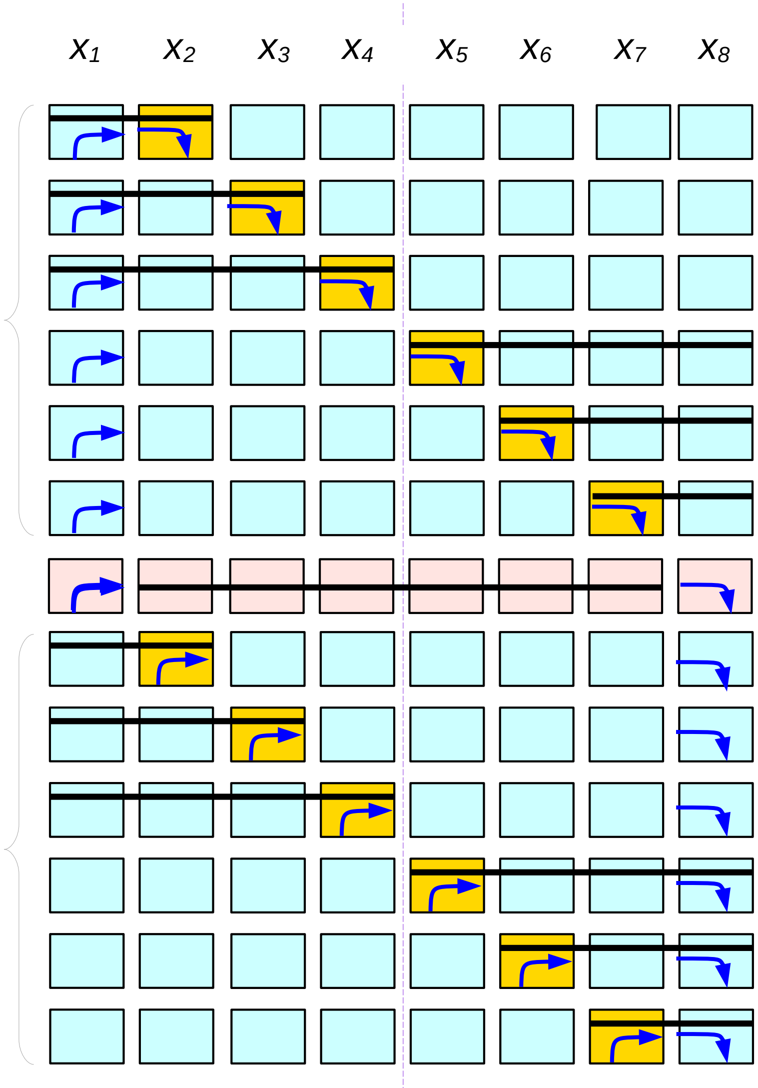
Verifying x_i constraints (r = 2·4)

carry signals generating in the 1st groups

carry signals terminating in the 8th groups



$x_1=4$, $x_2=7$, $x_3=10$, $x_4=11$, $x_5=11$, $x_6=10$, $x_7=7$, $x_8=4$

$x_2 \le 2T+1$	$x_2 = 7$	≤ 7
$x_3 \le 3T+1$	$x_3 = 10$	≤ 10
$x_4 \le 4T+1$	$x_4 = 11$	≤ 13
$x_5 \le 4T+1$	$x_5 = 11$	≤ 13
$x_6 \le 3T+1$	$x_6 = 10$	≤ 10
$x_7 \le 2T+1$	$x_7 = 7$	≤ 7
$x_8, x_1 \le 1T+1$	$x_8, x_1 = 4$	≤ 4
$x_2 \le 2T+1$	$x_2 = 7$	≤ 7
$x_3 \le 3T+1$	$x_3 = 10$	≤ 10
$x_4 \le 4T+1$	$x_4 = 11$	≤ 13
$x_5 \le 4T+1$	$x_5 = 11$	≤ 13
$x_6 \le 3T+1$	$x_6 = 10$	≤ 10
$x_7 \le 2T+1$	$x_7 = 7$	≤ 7

$$x_i \le y_i = \min\{1+iT, 1+(m+1-i)T\}$$

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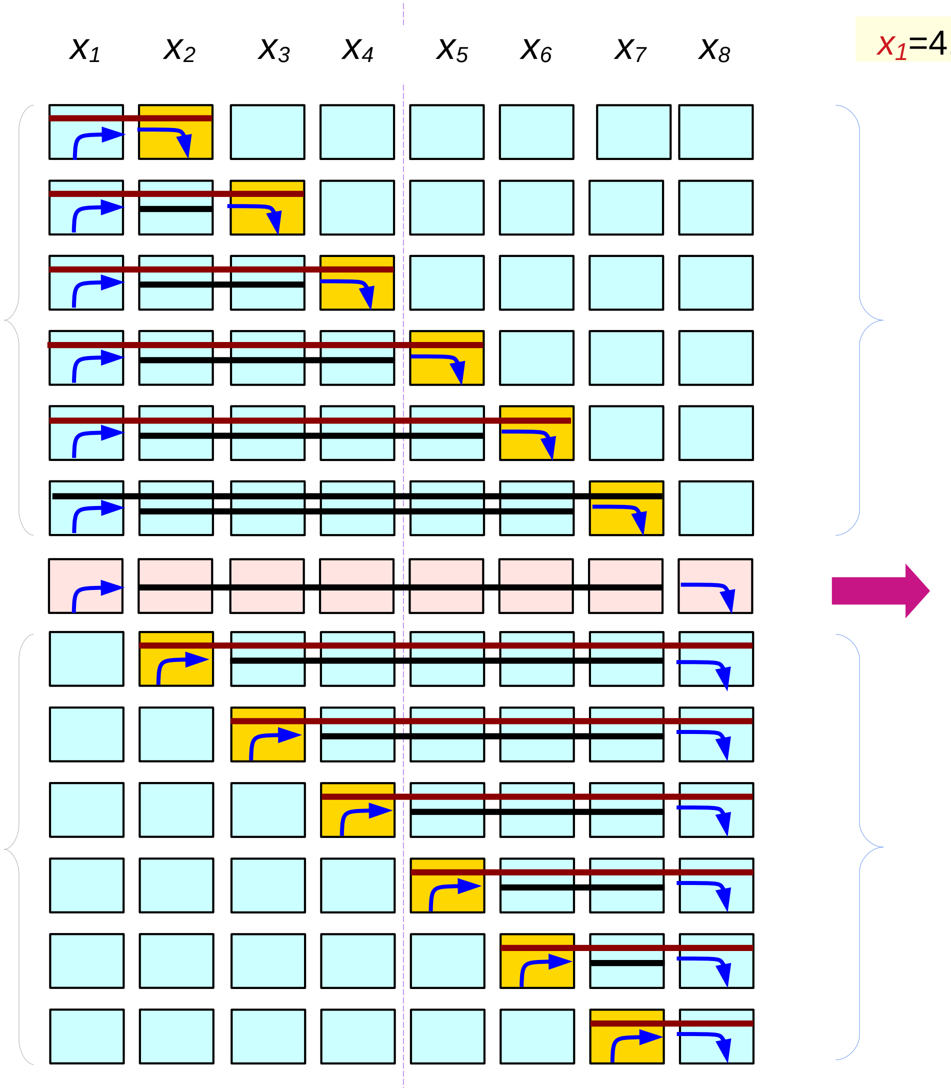
Loose upper bound (r = 2k)



carry signals generating in the 1st groups

carry signals terminating in the 8th groups

$x_1=4$, $x_2=7$, $x_3=10$, $x_4=11$, $x_5=11$, $x_6=10$, $x_7=7$, $x_8=4$

$$x_1 \leq 1+1T$$

$$x_i \leq 1+iT$$

loose upper bound

the maximum delay of carry signals generated in the 1st group or terminated in the 8th group

loose upper bound

$$x_i \leq 1+(m+1-i)T$$

$$x_8 \leq 1+1T$$

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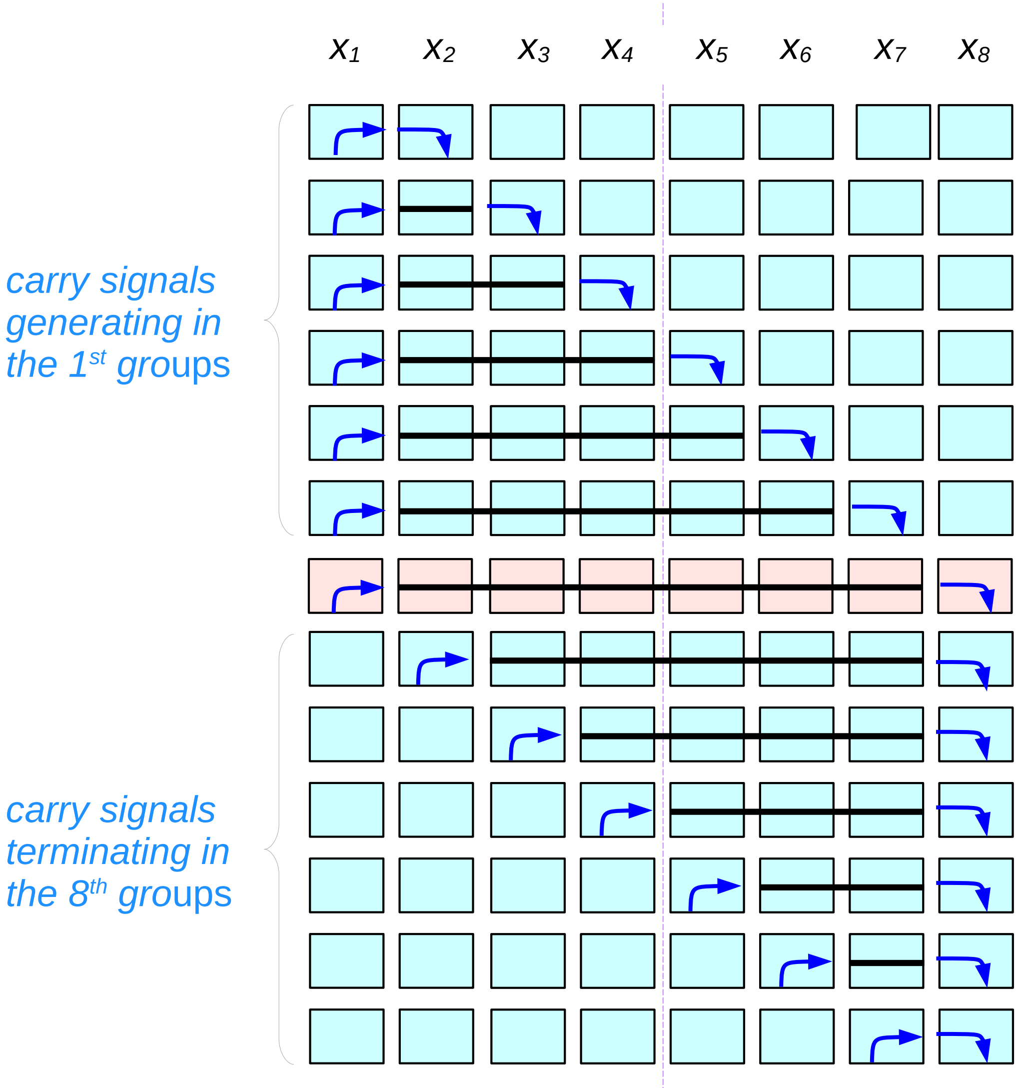
Maximum delays using x_i constraints (r = 2·4)



x_1=4, x_2=7, x_3=10, x_4=11, x_5=11, x_6=10, x_7=7, x_8=4

$x_1 + x_2$	$\leq$	$1T +1 + 0T + 2T +1$	$= 3T +2$
$x_1 + T + x_3$	$\leq$	$1T +1 + 1T + 3T +1$	$= 5T +2$
$x_1 + 2T + x_4$	$\leq$	$1T +1 + 2T + 4T +1$	$= 7T +2$
$x_1 + 3T + x_5$	$\leq$	$1T +1 + 3T + 4T +1$	$= 8T +2$
$x_1 + 4T + x_6$	$\leq$	$1T +1 + 4T + 3T +1$	$= 8T +2$
$x_1 + 5T + x_7$	$\leq$	$1T +1 + 5T + 2T +1$	$= 8T +2$
$x_1 + 6T + x_8$	$\leq$	$1T +1 + 6T + 1T +1$	$= 8T +2$
$x_2 + 5T + x_8$	$\leq$	$2T +1 + 5T + 1T +1$	$= 8T +2$
$x_3 + 4T + x_8$	$\leq$	$3T +1 + 4T + 1T +1$	$= 8T +2$
$x_4 + 3T + x_8$	$\leq$	$4T +1 + 3T + 1T +1$	$= 8T +2$
$x_5 + 2T + x_8$	$\leq$	$4T +1 + 2T + 1T +1$	$= 7T +2$
$x_6 + T + x_8$	$\leq$	$3T +1 +1T + 1T +1$	$= 5T +2$
$x_7 + x_8$	$\leq$	$2T +1 + 0T + 1T +1$	$= 3T +2$

Max delay $24 \leq 8T +2 = 26$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Assume T=3

r = 2k groups (1) generated and terminated group

Let $x_{1,} x_{2,} \ldots, x_r$ denote the optimal group sizes corresponding to the maximum delay *D.*

Given the maximum delay D,
the optimal group sizes are $x_{1,} x_{2,} \ldots, x_r$

the number of groups = ***r***

assume that $r = 2k$ is even.

By considering carries
- generated in group i
- terminated in group $r - i + 1$
- $i = 1, \ldots, k$

- ***n*** bits
- ***r*** = *2k* groups

$prove \quad (m-1)T \leq D$

Starting group i

terminating group r-i+1 $\quad r = 2k$

$1,$	$2k$	$= r-(1-1),$	$i = 1$
$2,$	$2k-1$	$= r-(2-1),$	$i = 2$
$k,$	$k+1$	$= r-(k-1),$	$i = k$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

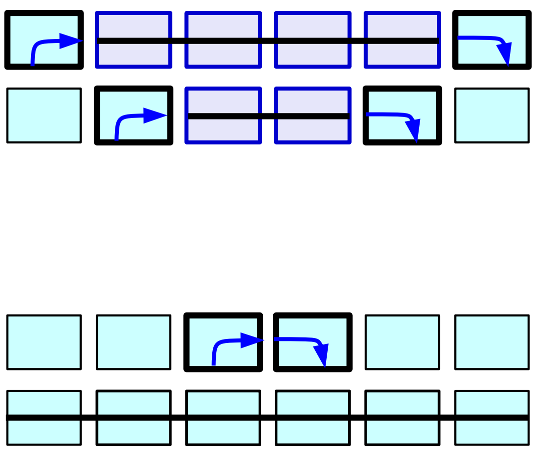
r = 2k groups (2) ***(j–i–1)*** = ***2(k–i)***

$r = 2k$

$P \le (x_i - 1) + (j - i - 1)T + (x_j - 1) \le rT$

- ***n*** bits
- ***r*** groups

$i = 1$	$j = r$
$i = 2$	$j = r - 1$
$i = 3$	$j = r - 2$
$i = k$	$j = r - k + 1$



$(j - i - 1) = r - 2 \quad = 2(k - 1)$

$(j - i - 1) = r - 4 \quad = 2(k - 2)$

$(j - i - 1) = r - 6 \quad = 2(k - 3)$

$(j - i - 1) = r - 2k \quad = 2(k - k)$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r = 2k groups (3) propagation time P, max delay D

$r = 2k$

- ***n*** bits
- ***r*** groups

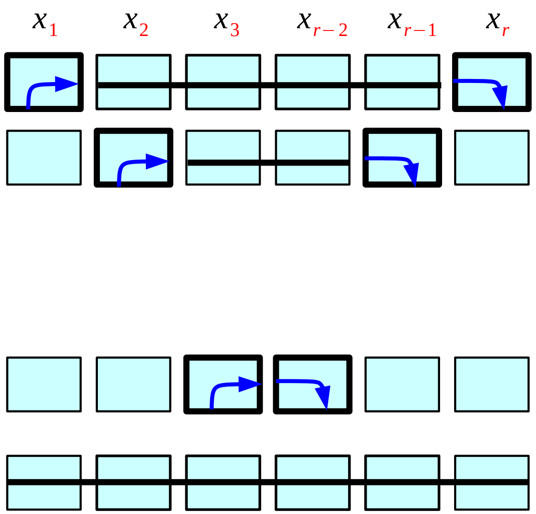
$$P \le (x_i - 1) + (j - i - 1)T + (x_j - 1) \le rT$$

Consider a carry signal
originating in the *i-th* group
terminating in the *j-th* group i < j.

Denote its propagation time by P.

$$(x_1 - 1) + (r-2)T + (x_r - 1) \le D$$
$$(x_2 - 1) + (r-4)T + (x_{r-1} - 1) \le D$$
$$(x_3 - 1) + (r-6)T + (x_{r-2} - 1) \le D$$
$$(x_k - 1) + (r-2k)T + (x_{k+1} - 1) \le D$$
$$rT \le D$$

Let *D* denote the maximum delay of
a carry signal (max of all *P*)
in a *n* bit carry skip adder
with group sizes chosen optimally.



Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r = 2k groups (4) max delay constraints

$r = 2k$

$$P \le (x_i - 1) + (j - i - 1)T + (x_j - 1) \le rT$$

- ***n*** bits
- ***r*** groups

$$(x_1 - 1) + (r-2)T + (x_r - 1) \le D$$
$$(x_2 - 1) + (r-4)T + (x_{r-1} - 1) \le D$$
$$(x_3 - 1) + (r-6)T + (x_{r-2} - 1) \le D$$
$$(x_k - 1) + (r-2k)T + (x_{k+1} - 1) \le D$$

$$rT \le D$$

$$(x_1 - 1) + (2k - 2 \cdot 1)T + (x_{2k+1-1} - 1) \le D$$
$$(x_2 - 1) + (2k - 2 \cdot 2)T + (x_{2k+1-2} - 1) \le D$$
$$(x_3 - 1) + (2k - 2 \cdot 3)T + (x_{2k+1-3} - 1) \le D$$
$$(x_k - 1) + (2k - 2 \cdot k)T + (x_{2k+1-k} - 1) \le D$$

$$2kT \le D$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r = 2k groups (5) sum of all the inequalities

$r = 2k$

$$P \le (x_i - 1) + (j - i - 1)T + (x_j - 1) \le rT$$

- *n* bits
- *r* groups

$$\sum_{i=1}^{r} x_i = n$$

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

$$2k = r$$

$$(x_1 - 1) + 2(k-1)T + (x_{2k} - 1) \le D \quad i = 1$$

$$(x_2 - 1) + 2(k-2)T + (x_{2k-1} - 1) \le D \quad i = 2$$

$$(x_3 - 1) + 2(k-3)T + (x_{2k-2} - 1) \le D \quad i = 3$$

$$(x_k - 1) + 2(k-k)T + (x_{k+1} - 1) \le D \quad i = k$$

$$rT \qquad \le D$$

$$n - 2k + (k+1)rT - k(k+1)T \le (k+1)D$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r = 2k groups (6) arithmetic and geometric means

$r = 2k$

$$n - 2k + (k+1)rT - k(k+1)T \le (k+1)D$$

$$\frac{n-2k}{k+1} + rT - kT \le D$$

$$\frac{n-2k}{k+1} + 2kT - kT \le D$$

$$\frac{n-2(k+1)+2}{k+1} + kT \le D$$

$$\frac{n+2}{k+1} + (k+1)T - (T+2) \le D$$

$$2\sqrt{(n+2)T} - (T+2) \le D$$

$$\sqrt{4nT+8T} - (T+2) \le D$$

arith mean $\ge$ *geo mean*

$$\frac{n+2}{k+1} + (k+1)T \ge 2 \cdot \sqrt{\frac{n+2}{k+1} \cdot (k+1)T}$$

$$\frac{n+2}{k+1} + (k+1)T \ge 2\sqrt{(n+2)T}$$

min when $\frac{n+2}{k+1} = (k+1)T$

$$\frac{n+2}{T} = (k+1)^2$$

$$(k+1) = \sqrt{\frac{n+2}{T}}$$

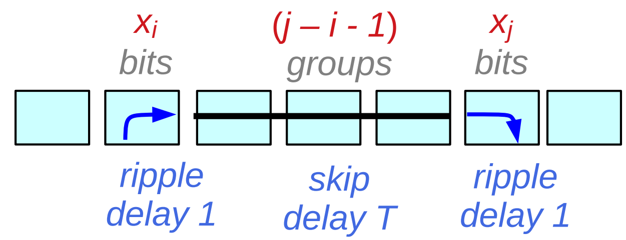
Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Determining x_i for an odd number of groups $r = 2k+1$

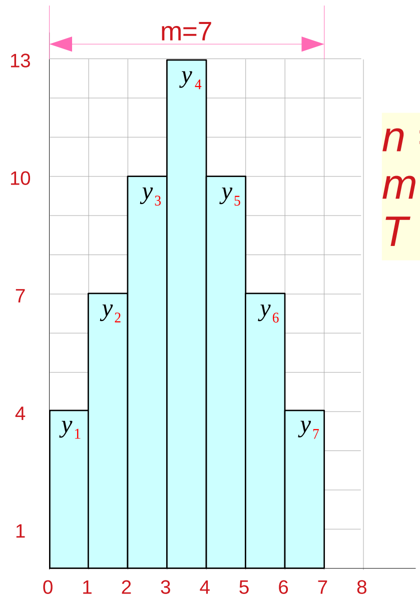
$r = 2k+1$

$$P \le (x_i - 1) + (j - i - 1)T + (x_j - 1) \le mT$$

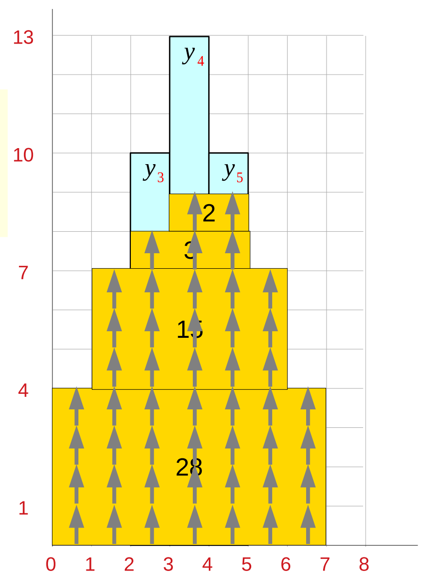
$$(x_i - 1) + (x_j - 1) + (j - i - 1)T$$



- ***n*** bits
- ***r* = *m* = 2k+1** groups



n =48
m =7
T =3



The x_i's can be computed iteratively as follows:

Initially take $x_1 = x_m = 0$

At each iteration,
increase as many of the x_i's as possible by one unit,
without violating the constraints

$$0 \le x_i \le y_i, \quad i = 1, \ldots, m$$

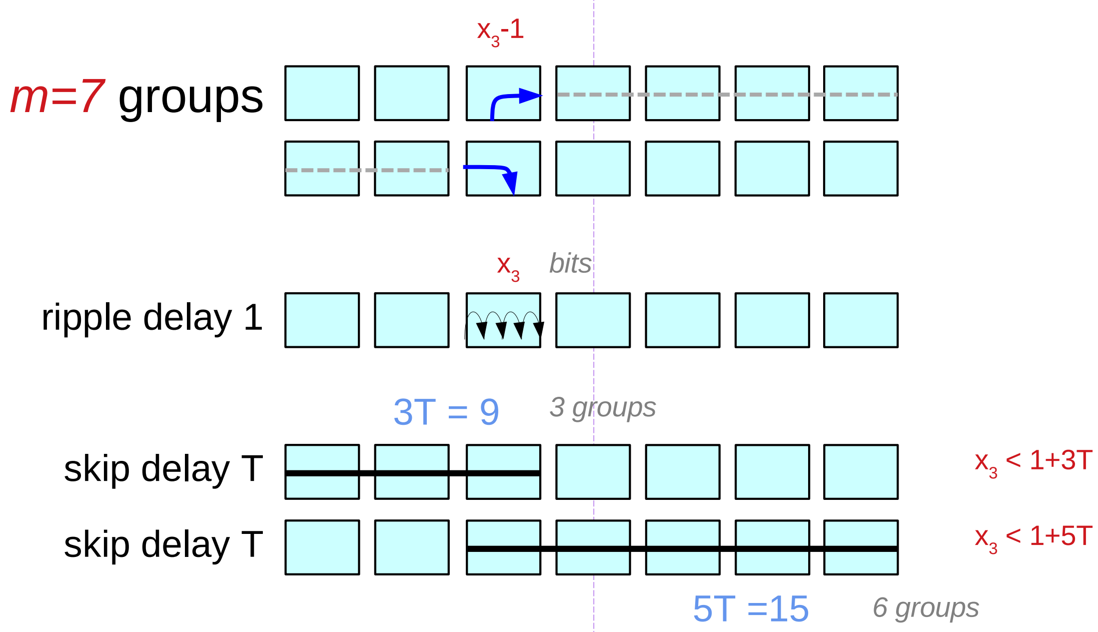
Thus, at some iteration, we have $\sum_{i=1}^{m} x_i = n$ and
the algorithm terminates

x_1=4, x_2=7, x_3=8, x_4=9, x_5=9, x_6=7, x_7=4

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

An example of $\mathbf{x_3 \le y_3}$ example ($\mathbf{r = 2k+1}$)

overlapping latencies



$$y_3 = min\{1+3T, 1+(7+1-3)T\} = 1+3T$$

$$x_3 - 1 \le min\{3T, (7+1-3)T\} = 3T$$

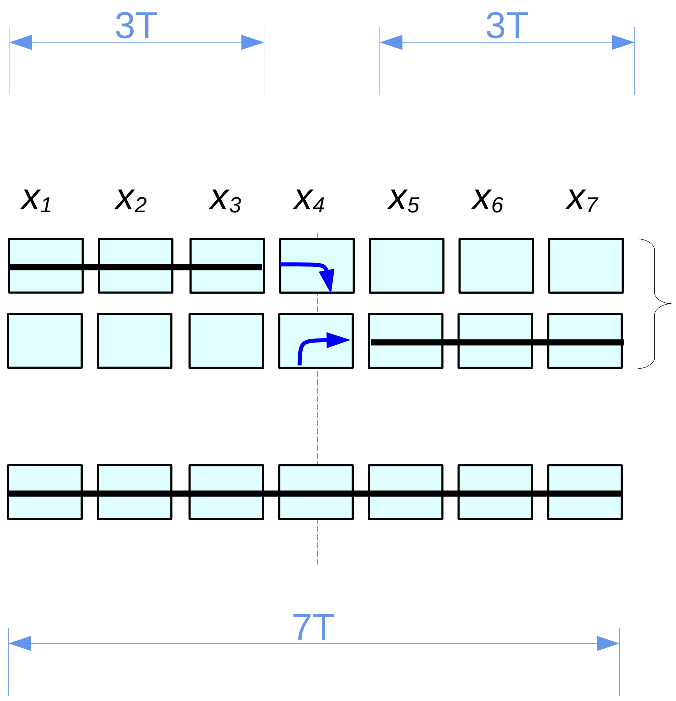
$$x_3 \le min\{1+3T, 1+(7+1-3)T\} = 1+3T$$

$$y_i = min\{1+iT, 1+(m+1-i)T\}$$
$$i = 1, \dots, m$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

All skip delay constraints (r = 2k+1)

$r = 2k+1$



$$3T + x_4 \le 1 + 7T \quad \Leftarrow \quad x_4 \le 1 + 4T$$

$$3T + x_4 - 1 \le 7T \le D$$

$$kT + (x_{k+1} - 1) \le (2k+1)T \le D$$

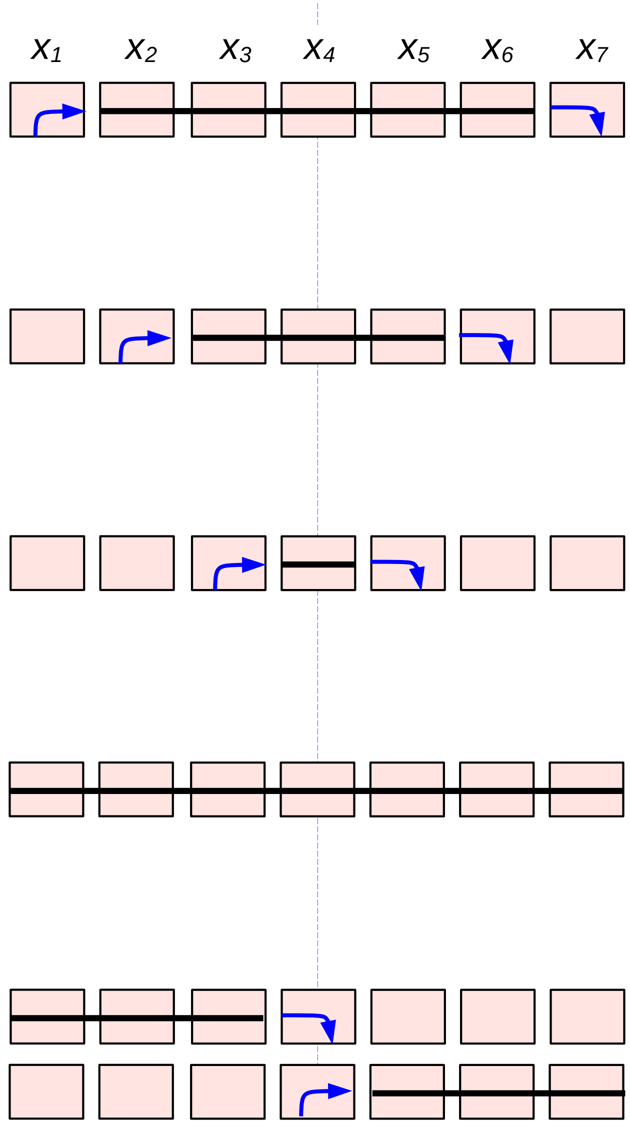
$$kT + (x_{k+1} - 1) \le D$$

Use this delay to find the lower bound of D,

instead of all skip delay

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum delays of carry signals (**r = 2k+1**)



The maximum delay of carry signals generated in the 1st group or terminated in the 7th group ≤ D

The maximum delay of carry signals generated in the 2nd group or terminated in the 6th group ≤ D

The maximum delay of carry signals generated in the 3rd group or terminated in the 5th group ≤ D

All skip delay ≤ D

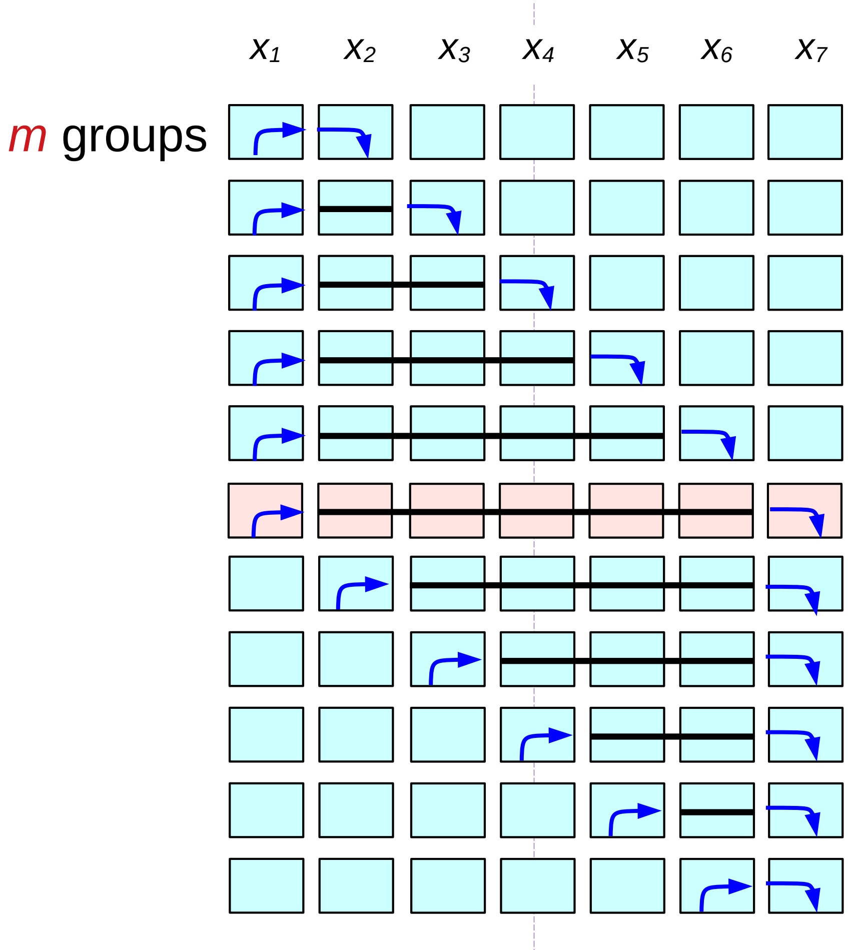
Comparable to all skip delay ≤ D

Use this delay to find the lower bound of D

Max delay of all carry signals

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Delays of carry signals [G1, G7] (r = 2·3+1)



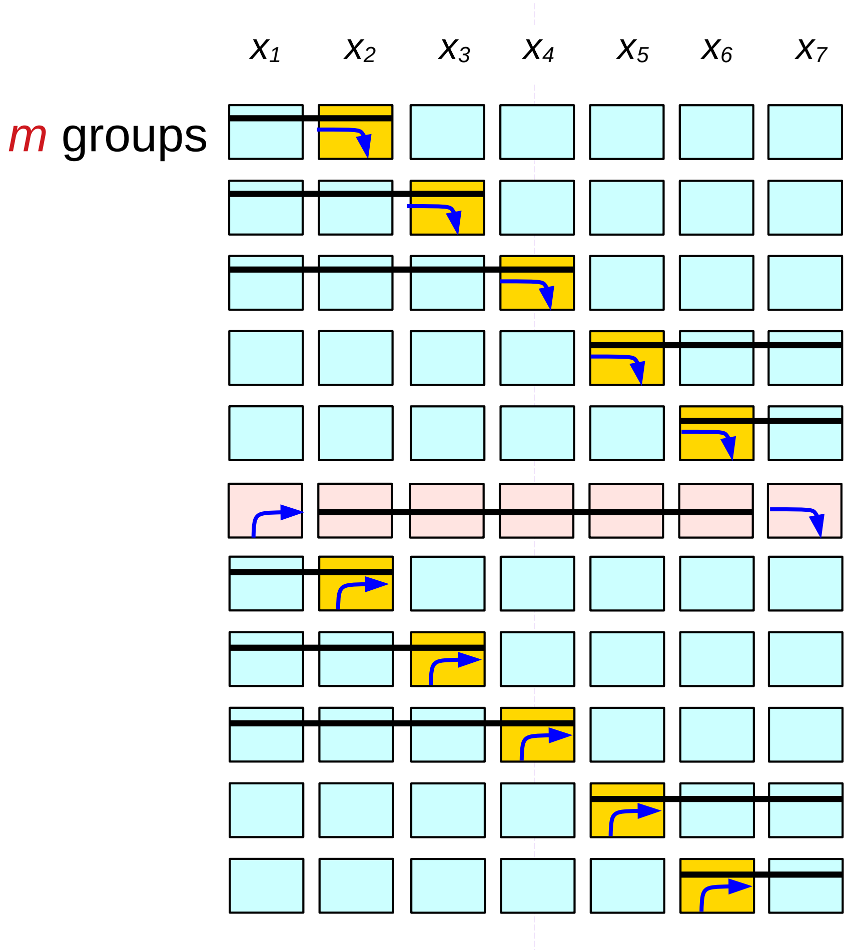
x_1=4, x_2=7, x_3=8, x_4=9, x_5=9, x_6=7, x_7=4

$x_1 + x_2$	$= 4 + 7$	$= 11$	$= 11$
$x_1 + T + x_3$	$= 4 + T + 8$	$= 12 + T$	$= 15$
$x_1 + 2T + x_4$	$= 4 + 2T + 9$	$= 13 + 2T$	$= 19$
$x_1 + 3T + x_5$	$= 4 + 3T + 9$	$= 13 + 3T$	$= 21$
$x_1 + 4T + x_6$	$= 4 + 4T + 7$	$= 11 + 4T$	$= 23$
$x_1 + 5T + x_7$	$= 4 + 5T + 4$	$= 8 + 5*3$	$= 23$
$x_2 + 4T + x_7$	$= 7 + 4T + 4$	$= 11 + 4T$	$= 23$
$x_3 + 3T + x_7$	$= 8 + 3T + 4$	$= 12 + 3T$	$= 21$
$x_4 + 2T + x_7$	$= 9 + 2T + 4$	$= 13 + 2T$	$= 19$
$x_5 + T + x_7$	$= 9 + T + 4$	$= 13 + T$	$= 16$
$x_6 + x_7$	$= 7 + 4$	$= 11$	$= 11$

Assume T=3

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Verifying x_i constraints (r = 2·3+1)



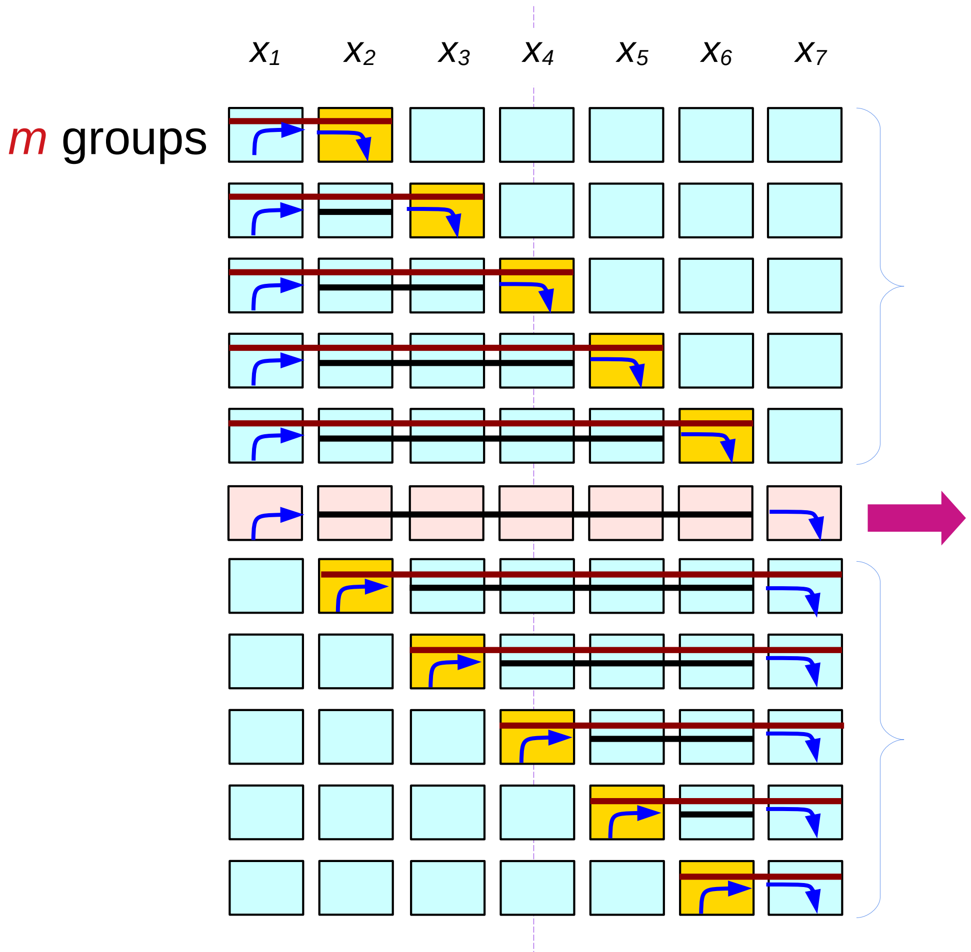
$x_1=4$, $x_2=7$, $x_3=8$, $x_4=9$, $x_5=9$, $x_6=7$, $x_7=4$

$x_2 \le 2T+1$	$x_2 = 7$	≤ 7
$x_3 \le 3T+1$	$x_3 = 8$	≤ 10
$x_4 \le 4T+1$	$x_4 = 9$	≤ 13
$x_5 \le 3T+1$	$x_5 = 9$	≤ 13
$x_6 \le 2T+1$	$x_6 = 7$	≤ 7
$x_7, x_1 \le 1T+1$	$x_7, x_1 = 4$	≤ 4
$x_2 \le 2T+1$	$x_2 = 7$	≤ 7
$x_3 \le 3T+1$	$x_3 = 8$	≤ 10
$x_4 \le 4T+1$	$x_4 = 9$	≤ 13
$x_5 \le 3T+1$	$x_5 = 9$	≤ 10
$x_6 \le 2T+1$	$x_6 = 7$	≤ 7

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

$$x_i \le y_i = min\{1+iT, 1+(m+1-i)T\}$$

Loose upper bound



x_1=4, x_2=7, x_3=8, x_4=9, x_5=9, x_6=7, x_7=4

$$x_1 \le 1 + 1T$$

$$x_i \le 1 + iT$$

loose upper bound

the maximum delay of carry signals generated in the 1st group or terminated in the 8th group

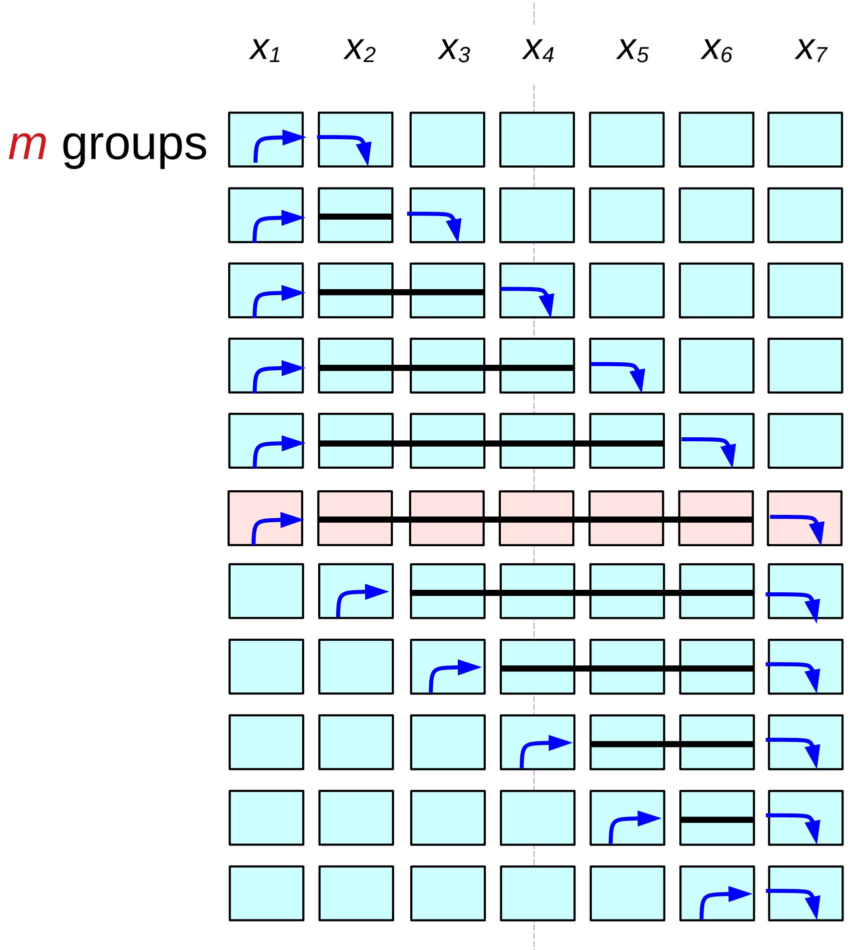
loose upper bound

$$x_i \le 1 + (m + 1 - i)T$$

$$x_7 \le 1 + 1T$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Maximum delays using x_i constraints (r = 2·3+1)



x_1=4, x_2=7, x_3=8, x_4=9, x_5=9, x_6=7, x_7=4

$x_1 + x_2$	$\le 1T +1 + 0T + 2T +1$	$= 3T +2$
$x_1 + T + x_3$	$\le 1T +1 + 1T + 3T +1$	$= 5T +2$
$x_1 + 2T + x_4$	$\le 1T +1 + 2T + 4T +1$	$= 7T +2$
$x_1 + 3T + x_5$	$\le 1T +1 + 3T + 4T +1$	$= 8T +2$
$x_1 + 4T + x_6$	$\le 1T +1 + 4T + 3T +1$	$= 8T +2$
$x_1 + 5T + x_7$	$\le 1T +1 + 5T + 2T +1$	$= 8T +2$
$x_2 + 4T + x_7$	$\le 3T +1 + 4T + 1T +1$	$= 8T +2$
$x_3 + 3T + x_7$	$\le 4T +1 + 3T + 1T +1$	$= 8T +2$
$x_4 + 2T + x_7$	$\le 4T +1 + 2T + 1T +1$	$= 7T +2$
$x_5 + T + x_7$	$\le 3T +1 +1T + 1T +1$	$= 5T +2$
$x_6 + x_7$	$\le 2T +1 + 0T + 1T +1$	$= 3T +2$

Max delay $23 \le 8T +2 = 26$

Assume T=3

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r = 2k+1 groups (1) generated and terminated group

Let $x_{1,} x_{2,} \ldots, x_r$ denote the optimal group sizes corresponding to the maximum delay *D.*

- ***n*** bits
- ***r*** = *2k* groups

Given the maximum delay D,
the optimal group sizes are $x_{1,} x_{2,} \ldots, x_r$

$prove \quad (m-1)T \leq D$

the number of groups = ***r***

assume that $r = 2k+1$ is even.

By considering carries
generated in group i
terminated in group $r-i+1$
$i = 1, \ldots, k$

Starting group i

terminating group r-i+1

$r = 2k+1$

Starting group i	terminating group $r-i+1$		
$1,$	$2k+1$	$= r-(1-1),$	$i = 1$
$2,$	$2k$	$= r-(2-1),$	$i = 2$
$k,$	$k+2$	$= r-(k-1),$	$i = k$

$k+1$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

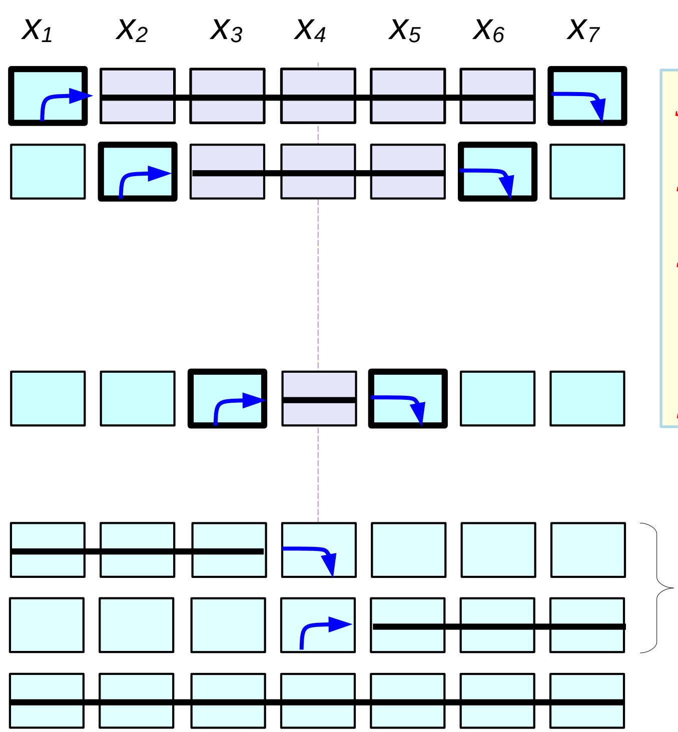
r = 2k+1 groups (2) (*j–i–1*) = *2*(*k–i*)

$r = 2k+1$

$$P \le (x_i - 1) + (j - i - 1)T + (x_j - 1) \le mT$$

- ***n*** bits
- ***r*** groups

i		j	
$i = 1$		$j = r$	$(j - i - 1) = r - 2 \quad = 2(k-1) + 1$
$i = 2$		$j = r-1$	$(j - i - 1) = r - 4 \quad = 2(k-2) + 1$
$i = 3$		$j = r-2$	$(j - i - 1) = r - 6 \quad = 2(k-3) + 1$
$i = k$		$j = r-k+1$	$(j - i - 1) = r - 2k \quad = 2(k-k) + 1$



$$3T + x_4 \le 1 + 7T \quad \Leftarrow \quad x_4 \le 1 + 4T$$

$$3T + x_4 < 7T \quad < D$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r = 2k+1 groups (3) propagation time P, max delay D

$r = 2k + 1$

- ***n*** bits
- ***r*** groups

$$P \le (x_i - 1) + (j - i - 1)T + (x_j - 1) \le rT$$

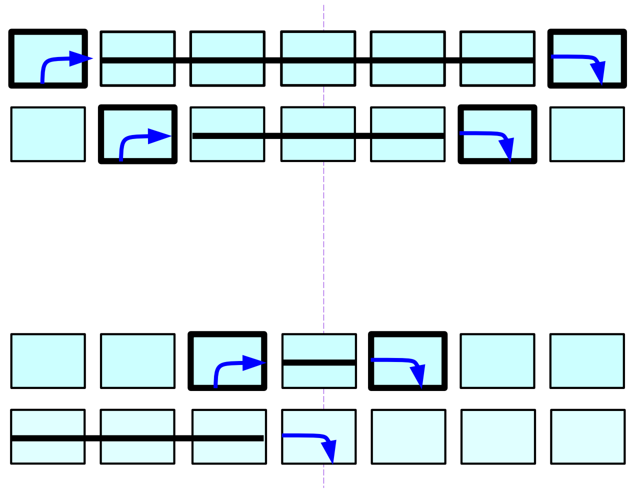
Consider a carry signal
originating in the *i-th* group
terminating in the *j-th* group i < j.

Denote its propagation time by P.

Let *D* denote the maximum delay of a carry signal (max of all *P*) in a *n* bit carry skip adder with group sizes chosen optimally.

$$(x_1 - 1) + (r - 2)T + (x_r - 1) \le D$$
$$(x_2 - 1) + (r - 4)T + (x_{r-1} - 1) \le D$$
$$(x_3 - 1) + (r - 6)T + (x_{r-2} - 1) \le D$$

$$(x_k - 1) + (r - 2k)T + (x_{r-k+1} - 1) \le D$$
$$kT + (x_{k+1} - 1) \le D$$



Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r = 2k+1 groups (4) max delay constraints

$r = 2k$

$$P \le (x_i - 1) + (j - i - 1)T + (x_j - 1) \le mT$$

- ***n*** bits
- ***r*** groups

$$
\begin{aligned}
(x_1 - 1) &+ (r-2)T + (x_r - 1) \le D \\
(x_2 - 1) &+ (r-4)T + (x_{r-1} - 1) \le D \\
(x_3 - 1) &+ (r-6)T + (x_{r-2} - 1) \le D \\
\\
(x_k - 1) &+ (r-2k)T + (x_{k+1} - 1) \le D
\end{aligned}
$$

$$rT \le D$$

$$
\begin{aligned}
(x_1 - 1) &+ (2k - 2\cdot 1)T + (x_{2k+1-1} - 1) \le D \\
(x_2 - 1) &+ (2k - 2\cdot 2)T + (x_{2k+1-2} - 1) \le D \\
(x_3 - 1) &+ (2k - 2\cdot 3)T + (x_{2k+1-3} - 1) \le D \\
\\
(x_k - 1) &+ (2k - 2\cdot k)T + (x_{2k+1-k} - 1) \le D
\end{aligned}
$$

$$2kT \le D$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r = 2k+1 groups (5) sum of all the inequalities

$r = 2k+1$

$$P \le (x_i - 1) + (j - i - 1)T + (x_j - 1) \le mT$$

- ***n*** bits
- ***r*** groups

$$(x_1 - 1) + 2(k-1)T + T + (x_{2k+1} - 1) \le D$$
$$(x_2 - 1) + 2(k-2)T + T + (x_{2k} - 1) \le D$$
$$(x_3 - 1) + 2(k-3)T + T + (x_{2k-1} - 1) \le D$$
$$(x_k - 1) + 2(k-k)T + T + (x_{k+2} - 1) \le D$$
$$kT + (x_{k+1} - 1) \le D$$

$$n - 2k - 1 + (r+1)kT - k(k+1)T \le (k+1)D$$

$$\sum_{i=1}^{r} x_i = n$$

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

$$2k+1 = r$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r = 2k+1 groups (6) arithmetic and geometric means

$r = 2k+1$

$$n - 2k - 1 + (r+1)kT - k(k+1)T \leq (k+1)D$$

$$\frac{n-2k-1}{(k+1)} + \frac{(r+1)kT}{(k+1)} - kT \leq D$$

$$\frac{n-2k-1}{(k+1)} + \frac{(2(k+1))kT}{(k+1)} - kT \leq D$$

$$\frac{n-2k-1}{(k+1)} + kT \leq D$$

$$\frac{n-2(k+1)+1}{(k+1)} + (k+1)T - T \leq D$$

$$\frac{(n+1)}{(k+1)} + (k+1)T - (T+2) \leq D$$

$$2 \cdot \sqrt{(n+1)T} - (T+2) \leq D$$

$$\sqrt{4nT+4T} - (T+2) \leq D$$

arith mean $\geq$ *geo mean*

$$\frac{(n+1)}{(k+1)} + (k+1)T \geq 2 \cdot \sqrt{(n+1)T}$$

min when $\quad \frac{(n+1)}{(k+1)} = (k+1)T$

$$\frac{n+1}{T} = (k+1)^2$$

$$(k+1) = \sqrt{\frac{n+1}{T}}$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Closed formula for $y_1 + y_2 + \ldots + y_m$

$$\sum_{i=1}^{m} y_i = m + \frac{1}{2} m T + \frac{1}{4} m^2 T + \left(1 - (-1)^m\right) \frac{1}{8} T$$

$$\sum_{i=1}^{m} y_i = m + \frac{1}{2} m T + \frac{1}{4} m^2 T \qquad (even \quad m)$$

$$\sum_{i=1}^{m} y_i = m + \frac{1}{2} m T + \frac{1}{4} m^2 T + \frac{1}{4} T \qquad (odd \quad m)$$

$$y_i = min\{1 + i T, 1 + (m + 1 - i) T\}, \quad i = 1, \ldots, m$$

$$\begin{aligned} y_1 &= 1 + 1 \cdot T & \qquad y_m &= 1 + 1 \cdot T \\ y_2 &= 1 + 2 \cdot T & \qquad y_{m-1} &= 1 + 2 \cdot T \\ y_3 &= 1 + 3 \cdot T & \qquad y_{m-2} &= 1 + 3 \cdot T \\ &\vdots & &\vdots \end{aligned}$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

When $m = 2k$, find a closed formula for $\Sigma\, y_i$ (1)

$m = 2k$

$\frac{m}{2} = k$

$$y_i = \min\{1 + iT,\ \ 1 + (m+1-i)T\}, \quad i = 1, \dots, m$$

$m \quad \frac{1}{2}\cdot k(k+1)$

$$
\begin{array}{llllllll}
y_1 = 1 + \min\{1\cdot T, & (m-0)\cdot T\} & & 0 \le x_1 & \le 1 + 1\cdot T \\
y_2 = 1 + \min\{2\cdot T, & (m-1)\cdot T\} & & 0 \le x_2 & \le 1 + 2\cdot T \\
y_3 = 1 + \min\{3\cdot T, & (m-2)\cdot T\} & & 0 \le x_3 & \le 1 + 3\cdot T \\
y_k = 1 + \min\{k\cdot T, & (k+1)\cdot T\} & & 0 \le x_k & \le 1 + k\cdot T \\
y_{k+1} = 1 + \min\{(k+1)\cdot T, & k\cdot T\} & & 0 \le x_{k+1} & \le 1 + k\cdot T \\
y_{m-2} = 1 + \min\{(m-2)\cdot T, & 3\cdot T\} & & 0 \le x_{m-2} & \le 1 + 3\cdot T \\
y_{m-1} = 1 + \min\{(m-1)\cdot T, & 2\cdot T\} & & 0 \le x_{m-1} & \le 1 + 2\cdot T \\
y_{m-0} = 1 + \min\{(m-0)\cdot T, & 1\cdot T\} & & 0 \le x_{m-0} & \le 1 + 1\cdot T
\end{array}
$$

$\frac{1}{2}\cdot k(k+1)$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

$$0 \le x_i \le y_i, i = 1, \dots, m$$

When $m = 2k$, find a closed formula for $\Sigma\, y_i$ (2)

$$1 + 2 + \cdots + k = \frac{1}{2}k(k+1)$$

$$\frac{n(a+l)}{2}$$

$$m + 2\cdot\frac{1}{2}k(k+1)T$$

$$= m + k(k+1)T$$

$$= m + \frac{m}{2}\left(\frac{m}{2}+1\right)T$$

$$= m + \frac{m}{2}T + \frac{m^2}{4}T$$

$$m = 2k$$

$$\frac{m}{2} = k$$

$$even\ m : m + \frac{1}{2}mT + \frac{1}{4}m^2 T$$

$$odd\ m : m + \frac{1}{2}mT + \frac{1}{4}m^2 T + \frac{1}{4}T$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

When *m = 2k+1*, find a closed formula for Σ y_i (1)

$m = 2k+1$

$m+1 = 2k+2$

$\frac{m+1}{2} = k+1$

$y_i = \min\{1+iT, \quad 1+(m+1-i)T\}, \quad i = 1,\ldots,m$ m $\frac{1}{2}\cdot k(k+1)$

$$
\begin{array}{llll}
y_1 = 1 + \min\{1\cdot T, & (m-0)\cdot T\} & 0 \le x_1 & \le 1 + 1\cdot T \\
y_2 = 1 + \min\{2\cdot T, & (m-1)\cdot T\} & 0 \le x_2 & \le 1 + 2\cdot T \\
y_3 = 1 + \min\{3\cdot T, & (m-2)\cdot T\} & 0 \le x_3 & \le 1 + 3\cdot T \\
y_k = 1 + \min\{(k)\cdot T, & (k+3)\cdot T\} & 0 \le x_k & \le 1 + k\cdot T \\
y_{k+1} = 1 + \min\{(k+1)\cdot T, & (k+2)\cdot T\} & 0 \le x_{k+1} & \le 1 + (k+1)\cdot T \quad (k+1) \\
y_{k+2} = 1 + \min\{(k+2)\cdot T, & (k+1)\cdot T\} & 0 \le x_{k+2} & \le 1 + k\cdot T \\
y_{m-2} = 1 + \min\{(m-2)\cdot T, & 3\cdot T\} & 0 \le x_{m-2} & \le 1 + 3\cdot T \\
y_{m-1} = 1 + \min\{(m-1)\cdot T, & 2\cdot T\} & 0 \le x_{m-1} & \le 1 + 2\cdot T \\
y_{m-0} = 1 + \min\{(m-0)\cdot T, & 1\cdot T\} & 0 \le x_{m-0} & \le 1 + 1\cdot T
\end{array}
$$

$\frac{1}{2}\cdot k(k+1)$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

$0 \le x_i \le y_i, i=1,\ldots,m$

When m = 2k+1, find a closed formula for Σ y_i (2)

$$1 + 2 + \cdots + k = \frac{1}{2}k(k+1) \qquad \frac{n(a+l)}{2}$$

$$m + 2 \cdot \frac{1}{2}k(k+1)T + (k+1)T$$

$$= m + k(k+1)T + (k+1)T$$

$$= m + (k+1)^2 T$$

$$= m + \left(\frac{m+1}{2}\right)^2 T$$

$$= m + \frac{m^2}{4}T + \frac{m}{2}T + \frac{1}{4}T$$

$$m = 2k+1$$

$$m+1 = 2k+2$$

$$\frac{m+1}{2} = k+1$$

$$even\ m : m + \frac{1}{2}mT + \frac{1}{4}m^2 T$$

$$odd\ m : m + \frac{1}{2}mT + \frac{1}{4}m^2 T + \frac{1}{4}T$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Combining two cases (odd & even **r**) (1-1)

$r=2k$	$\sqrt{4nT+8T} - (T+2) \le D$
$r=2k+1$	$\sqrt{4nT+4T} - (T+2) \le D$

$$\sqrt{4nT+4T} - (T+2) \le \sqrt{4nT+8T} - (T+2)$$

$$\Rightarrow \sqrt{4nT+4T} - (T+2) \le D$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Combining two cases (odd & even **r**) (1-2)

We will not produce an upper bound on ***mT***.

Since *m* is the smallest positive integer satisfying

$$\sum_{i=1}^{m-1} y_i < n \le \sum_{i=1}^{m} y_i$$

$$\sum_{i=1}^{m} y_i = m + \frac{1}{2} m T + \frac{1}{4} m^2 T + \left(1 - (-1)^m\right) \frac{1}{8} T$$

$$(m-1) + \frac{1}{2}(m-1)T + \frac{1}{4}(m-1)^2 T + \left(1 - (-1)^{m-1}\right) \frac{T}{8} < n$$

$$n \le m + \frac{1}{2} m T + \frac{1}{4} m^2 T + \left(1 - (-1)^m\right) \frac{1}{8} T$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Combining two cases (odd & even **r**) (2-1)

$$(m-1) + \frac{1}{2}(m-1)T + \frac{1}{4}(m-1)^2 T + (1-(-1)^{m-1})\frac{T}{8} < n$$

$$\sum_{i=1}^{m-1} y_i < n$$

$$(m-1) + \frac{1}{2}(m-1)T + \frac{1}{4}(m-1)^2 T + (1-(-1)^{m-1})\frac{T}{8} + 1 \le n$$

$$\sum_{i=1}^{m-1} y_i + 1 \le n$$

$$(m) + \frac{1}{2}(mT - T) + \frac{1}{4}(m^2 T - 2mT + T) + (1-(-1)^{m-1})\frac{T}{8} \le n$$

$$m - \frac{1}{4}T + \frac{1}{4}m^2 T + (1-(-1)^{m-1})\frac{T}{8} \le n$$

$$\frac{1}{4}m^2 T \le n - m + \frac{1}{4}T - (1-(-1)^{m-1})\frac{T}{8}$$

$$\left(\frac{1}{4}m^2 T\right)\cdot 4T \le \left(n - m + \frac{1}{4}T - (1-(-1)^{m-1})\frac{T}{8}\right)\cdot 4T$$

$$m^2 T^2 \le 4nT - 4mT + T^2 - (1-(-1)^{m-1})\frac{T^2}{2}$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Combining two cases (odd & even **r**) (2-2)

$$m^2 T^2 \le 4nT - 4mT + T^2 - (1-(-1)^{m-1})\frac{T^2}{2}$$

$$m^2 T^2 + 4mT \le 4nT + T^2 - (1-(-1)^{m-1})\frac{T^2}{2}$$

$$m^2 T^2 + 4mT + 4 \le 4 + 4nT + T^2 - (1-(-1)^{m-1})\frac{T^2}{2}$$

$$(mT + 2)^2 \le 4 + 4nT + T^2 - (1-(-1)^{m-1})\frac{T^2}{2}$$

$$(mT + 2) \le \sqrt{4 + 4nT + T^2 - (1-(-1)^{m-1})\frac{T^2}{2}}$$

$$mT \le -2 + \sqrt{4 + 4nT + T^2 - (1-(-1)^{m-1})\frac{T^2}{2}}$$

$$\sum_{i=1}^{m-1} y_i < n$$

$$\sum_{i=1}^{m} y_i = m + \frac{1}{2}mT + \frac{1}{4}m^2 T + (1-(-1)^m)\frac{1}{8}T$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r=2k even r

$r = 2k$

$$mT \le -2 + \sqrt{4nT + T^2 + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}$$

$$-D \le -\sqrt{4nT + 8T} + (T + 2)$$

$$mT - D \le T - \sqrt{4nT+8T} + \sqrt{4nT + T^2 + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}$$

$$mT - D \le T + \frac{T^2 - 8T + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}{\sqrt{4nT+8T} + \sqrt{4nT + T^2 + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}}$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r=2k even r

$r = 2k$

$X \stackrel{\text{def}}{=} 4 - (1 - (-1)^{m-1}) \dfrac{T^2}{2}$

$$mT \le -2 + \sqrt{4nT + T^2 + X}$$

$$-D \le -\sqrt{4nT + 8T} + (T + 2)$$

$$mT - D \le T - \sqrt{4nT + 8T} + \sqrt{4nT + T^2 + X}$$

$$(-\sqrt{a} + \sqrt{b}) \cdot \frac{(+\sqrt{a} + \sqrt{b})}{(+\sqrt{a} + \sqrt{b})} = \frac{(-a + b)}{(+\sqrt{a} + \sqrt{b})}$$

$$\left(-\sqrt{4nT + 8T} + \sqrt{4nT + T^2 + X}\right) \cdot \frac{\left(\sqrt{4nT + 8T} + \sqrt{4nT + T^2 + X}\right)}{\left(\sqrt{4nT + 8T} + \sqrt{4nT + T^2 + X}\right)}$$

$$\{-\sqrt{4nT + 8T} + \sqrt{4nT + T^2 + X}\} \cdot \{+\sqrt{4nT + 8T} + \sqrt{4nT + T^2 + X}\}$$

$$= -(4nT + 8T) + (4nT + T^2 + X) = T^2 - 8T + X$$

$$mT - D \le T + \frac{T^2 - 8T + X}{\sqrt{4nT + 8T} + \sqrt{4nT + T^2 + X}}$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r=2k+1 odd r

$r = 2k + 1$

$$\sqrt{4nT + 4T} - (T + 2) \le D$$

$$-\sqrt{4nT + 4T} + (T + 2) \ge -D$$

$$mT \le -2 + \sqrt{4nT + T^2 + 4 - (1 - (-1)^{m-1})\frac{T^2}{2}}$$

$$-D \le -\sqrt{4nT + 4T} + (T + 2)$$

$$mT - D \le T - \sqrt{4nT + 8T} + \sqrt{4nT + T^2 + 4 - (1 - (-1)^{m-1})\frac{T^2}{2}}$$

$$mT - D \le T + \frac{T^2 - 4T + 4 - (1 - (-1)^{m-1})\frac{T^2}{2}}{\sqrt{4nT + 4T} + \sqrt{4nT + T^2 + 4 - (1 - (-1)^{m-1})\frac{T^2}{2}}}$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r=2k+1 odd r

$$r = 2k+1$$

$$X \stackrel{\text{def}}{=} 4-\left(1-(-1)^{m-1}\right)\frac{T^2}{2}$$

$$mT \le -2 + \sqrt{4nT + T^2 + X}$$

$$-D \le -\sqrt{4nT+4T} + (T+2)$$

$$mT - D \le T - \sqrt{4nT+4T} + \sqrt{4nT+T^2+X}$$

$$(-\sqrt{a} + \sqrt{b}) \cdot \frac{(+\sqrt{a} + \sqrt{b})}{(+\sqrt{a} + \sqrt{b})} = \frac{(-a + b)}{(+\sqrt{a} + \sqrt{b})}$$

$$\left(-\sqrt{4nT+4T} + \sqrt{4nT+T^2+X}\right) \cdot \frac{\left(\sqrt{4nT+4T} + \sqrt{4nT+T^2+X}\right)}{\left(\sqrt{4nT+4T} + \sqrt{4nT+T^2+X}\right)}$$

$$\left\{-\sqrt{4nT+4T} + \sqrt{4nT+T^2+X}\right\} \cdot \left\{+\sqrt{4nT+4T} + \sqrt{4nT+T^2+X}\right\}$$

$$= -(4nT+4T) + (4nT+T^2+X) = T^2 - 4T + X$$

$$mT - D \le T + \frac{T^2 - 4T + X}{\sqrt{4nT+4T} + \sqrt{4nT+T^2+X}}$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Combining two cases (odd & even **r**) (3)

$r = 2k$ $\quad \sqrt{4nT + 8T} - (T+2) \le D$

$r = 2k+1$ $\quad \sqrt{4nT + 4T} - (T+2) \le D$

$$mT \le -2 + \sqrt{4 + 4nT + T^2 - (1 - (-1)^{m-1})\frac{T^2}{2}}$$

$$mT - D \le T + \frac{T^2 - 8T + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}{\sqrt{4nT + 8T} + \sqrt{4nT + T^2 + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}} \qquad r = 2k$$

$$mT - D \le T + \frac{T^2 - 4T + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}{\sqrt{4nT + 4T} + \sqrt{4nT + T^2 + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}} \qquad r = 2k+1$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

r=2k even r

$$\sqrt{4nT + T^2 + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}$$

$$X \stackrel{\text{def}}{=} 4 - (1-(-1)^{m-1})\frac{T^2}{2}$$

$X = 4$ odd m

$X = 4 - T^2$ even m

$$\sqrt{4nT + T^2 + 4 - (1-(-1)^{m-1})\frac{T^2}{2}} = \sqrt{4nT + T^2 + X}$$

- odd m

$(-1)^{m-1} = 1$

$(1-(-1)^{m-1}) = 0$

$\sqrt{4nT + 4 + T^2}$

- even m

$(-1)^{m-1} = -1$

$(1-(-1)^{m-1}) = 2$

$\sqrt{4nT + T^2 + 4 - T^2}$

$= \sqrt{4nT + 4}$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Combining two cases (odd & even **r**) (3)

$r = 2k$

$$mT - D \le T + \frac{T^2 - 8T + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}{\sqrt{4nT + 8T} + \sqrt{4nT + T^2 + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}}$$

$X = 4 - T^2$ even m

$$mT - D \le T + \frac{T^2 - 8T + X}{\sqrt{4nT + 8T} + \sqrt{4nT + T^2 + X}}$$

$$\le T + \frac{T^2 - 8T + 4 - T^2}{\sqrt{4nT + 8T} + \sqrt{4nT + T^2 + 4}}$$

$$\le T + \frac{-8T + 4}{\sqrt{4nT + 8T} + \sqrt{4nT + 4}}$$

$r = 2k + 1$

$$mT - D \le T + \frac{T^2 - 4T + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}{\sqrt{4nT + 4T} + \sqrt{4nT + T^2 + 4 - (1-(-1)^{m-1})\frac{T^2}{2}}}$$

$X = 4$ odd m

$$mT - D \le T + \frac{T^2 - 4T + X}{\sqrt{4nT + 4T} + \sqrt{4nT + T^2 + X}}$$

$$\le T + \frac{T^2 - 4T + 4}{\sqrt{4nT + 4T} + \sqrt{4nT + T^2 + 4}}$$

$$\le T + \frac{(T - 2)^2}{\sqrt{4nT + 4T} + \sqrt{4nT + T^2 + 4}}$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Combining two cases (odd & even **r**) (3)

$r = 2k$

$X = 4 - T^2$

$$mT - D \le T + \frac{T^2 - 8T + X}{\sqrt{4nT + 8T} + \sqrt{4nT + T^2 + X}}$$

$$mT - D \le T + \frac{-8T + 4}{\sqrt{4nT + 8T} + \sqrt{4nT + 4}}$$

$$\frac{-8 \cdot 3 + 4}{\sqrt{4 \cdot 32 \cdot 3 + 8 \cdot 3} + \sqrt{4 \cdot 32 \cdot 3 + 4}} = -\frac{28}{20.2 + 19.7} = -0.702$$

$r = 2k + 1$

$X = 4$

$$mT - D \le T + \frac{T^2 - 4T + X}{\sqrt{4nT + 4T} + \sqrt{4nT + T^2 + X}}$$

$$mT - D \le T + \frac{(T - 2)^2}{\sqrt{4nT + 4T} + \sqrt{4nT + T^2 + 4}}$$

$$\frac{(3 - 2)^2}{\sqrt{4 \cdot 32 \cdot 3 + 4 \cdot 3} + \sqrt{4 \cdot 32 \cdot 3 + 3^2 + 4}} = \frac{1}{20 + 19.925} = 0.25$$

$T = 3$

$n = 32 \; bits$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Combining two cases (odd & even **r**) (3)

$r=2k$ $\quad X = 4 - T^2$

$$mT-D \le T + \frac{-8(T/n)+4}{\sqrt{4(T/n)+8(T/n^2)} + \sqrt{4(T/n)+4/n^2}} = T + \frac{-8(T/n)+4}{\sqrt{4(T/n)} + \sqrt{4(T/n)}} \approx T + \frac{-8(T/n)+4}{4\sqrt{(T/n)}}$$

$r=2k+1$ $\quad X = 4$

$$\approx T + -2\sqrt{T/n} + \frac{1}{\sqrt{(T/n)}}$$

$$mT-D \le T + \frac{(T-2)^2/n}{\sqrt{4(T/n)+4(T/n^2)} + \sqrt{4(T/n)+(T/n)^2+4/n^2}} = T + \frac{(T-2)^2/n}{\sqrt{4(T/n)} + \sqrt{4(T/n)}} \approx T + \frac{(T-2)^2/n}{4\sqrt{(T/n)}}$$

$T = 3$

$n = 32\ bits$

$2 \le T \le 7$

$n = 32\ bits$

$$mT-D < T + 1$$

$$\frac{2}{32} \le \frac{T}{32} \le \frac{7}{32}$$

$$0.0625 \le \frac{T}{32} \le 0.21875$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers

Delay model

For n sufficiently large,

We have $mT - D < T + 1$

Since $mT - D$ is an integer

$$mT - D \leq T$$

Oklobdzija: High-Speed VLSI arithmetic units : adders and multipliers