Stationarity

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

First Order Stationary
N Gaussian random variables

Definition

if the first order density function does not change with a shift in time origin

$$f_X(x_1; t_1) = f_X(x_1; t_1 + \Delta)$$

must be true for any timet_1 and any real number Δ if $X(t)$ is to be a first-order stationary

Consequences of stationarity

N Gaussian random variables

Definition

$f_X(x, t_1)$is independent of t_1
the process mean value is a constant

$$m_X(t) = \overline{X} = constant$$

the process mean value
N Gaussian random variables

Definition

$$m_X(t) = \overline{X} = constant$$

$$m_X(t_1) = \int_{-\infty}^{\infty} x f_X(x; t_1) dx$$

$$m_X(t_2) = \int_{-\infty}^{\infty} x f_X(x; t_2) dx$$

$$m_X(t_1) = m_X(t_1 + \Delta)$$

Second-Order Stationary Process

N Gaussian random variables

Definition

if the second order density function does not change with a shift in time origin

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

must be true for any timet_1,t_2 and any real number Δ if $X(t)$ is to be a second-order stationary
Auto-correlation function

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] = R_{XX}(\tau)$$

N^{th}-order Stationary Processes

N Gaussian random variables

Definition

if the second order density function does not change with a shift in time origin

$$f_X(x_1, \cdots, x_N; t_1, \cdots, t_N) = f_X(x_1, \cdots, x_N; t_1 + \Delta, \cdots, t_N + \Delta)$$

must be true for any time$t_1, ..., t_N$ and any real number Δ if $X(t)$ is to be a second-order stationary

Wide Sense Stationary Process

N Gaussian random variables

Definition

$$m_X(t) = \overline{X} = constant$$

$$E\left[X(t)X(t+\tau)\right] = R_{XX}(\tau)$$

The properties of autocorrelation functions (1)

N Gaussian random variables

Definition

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$

$$R_{XX}(-\tau) = R_{XX}(\tau)$$

$$R_{XX}(0) = E\left[X^2(t)\right]$$

$$P\left[|X(t+\tau) - X(t)| > \varepsilon\right] = \frac{2}{\varepsilon^2}\left(R_{XX}(0) - R_{XX}(\tau)\right)$$

The properties of autocorrelation functions (2)

N Gaussian random variables

Definition

if $X(t) = \overline{X} + N(t)$ where $N(t)$ is WSS, is zero-mean, and has autocorrelation function $R_{NN}(\tau) \to 0$ as $|\tau| \to \infty$, then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \overline{X}^2$$

if $X(t)$ is mean square periodic, i.e, there exists a $T \neq 0$ such that $E\left[(X(t+T) - X(t))^2\right] = 0$ for all t, then $R_{XX}(t)$ will have a periodic component with the same period

$R_{XX}(\tau)$ cannot have an arbitrary shape