Complex Integration (2B)

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1/17/14

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Complex Integration

Indefinite Integration of Analytic Functions

analytic $f(z)$
in a simply connected domain **D**

➡ exists an indefinite integral of $f(z)$

➡ **analytic** $F(z)$

such that $F'(z) = f(z)$

in **D**, and all paths in **D**

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

Integration by the use of the Path

analyticity is not required

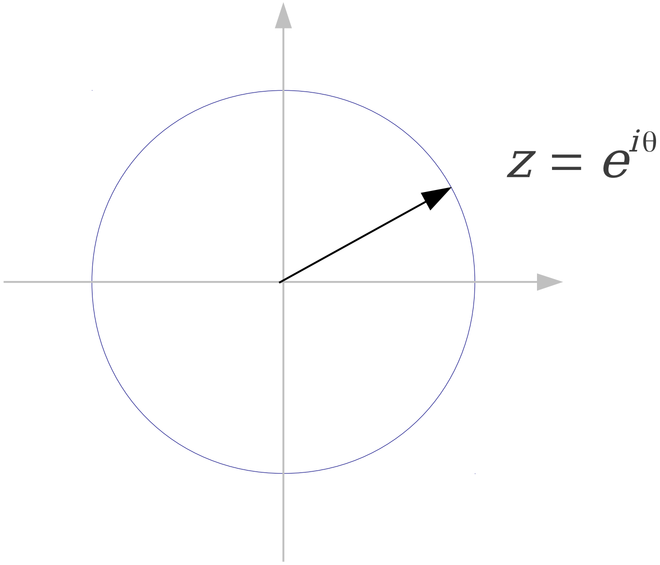
a piecewise smooth path **C** represented by

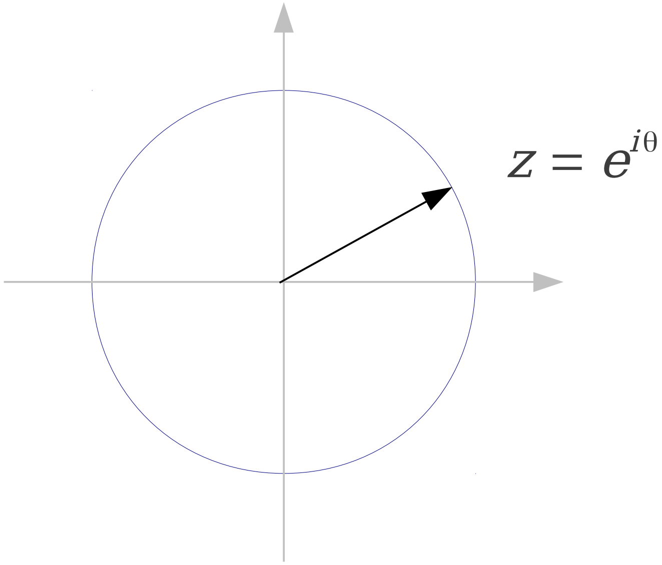
$z(t) = x(t) + iy(t) \quad a \le t \le b$

continuous on **C** $f(z)$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

Unit Circular Contour





$$z(t) = x(t) + iy(t) \qquad a \le t \le b$$

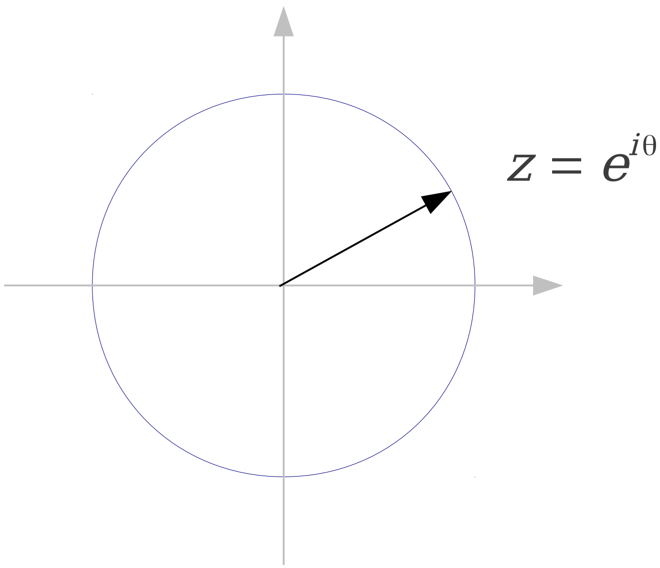
$$\int_C f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt$$

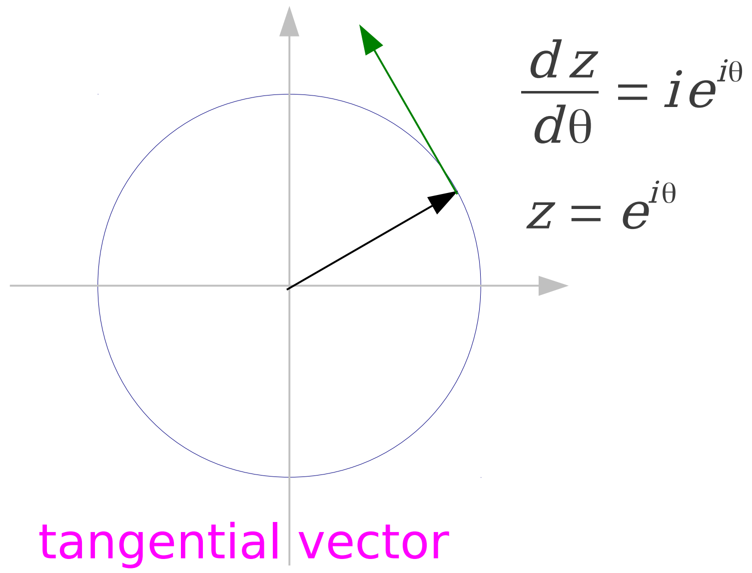
$$z(r, \theta) = r(\cos\theta + i\sin\theta) = r e^{i\theta}$$

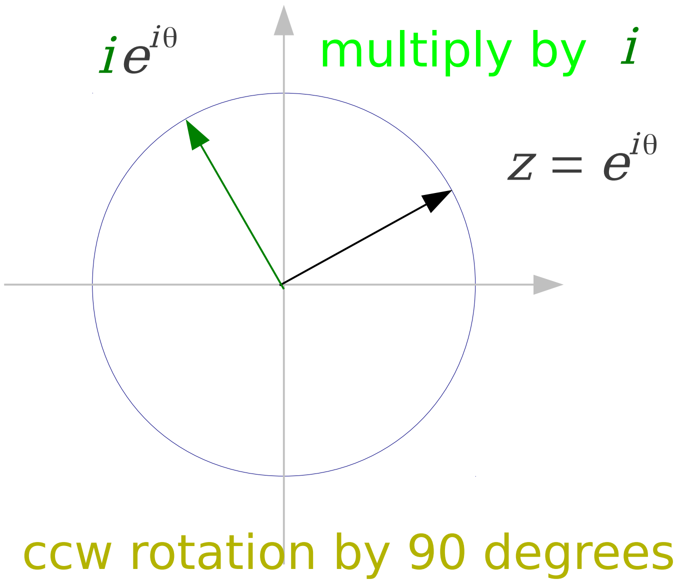
along the circle r is fixed

➡ $$\frac{dz}{d\theta} = i e^{i\theta}$$

Unit Circular Contour







No radial axis change

leading phase by 90 degrees

$$dz = i e^{i\theta} d\theta$$

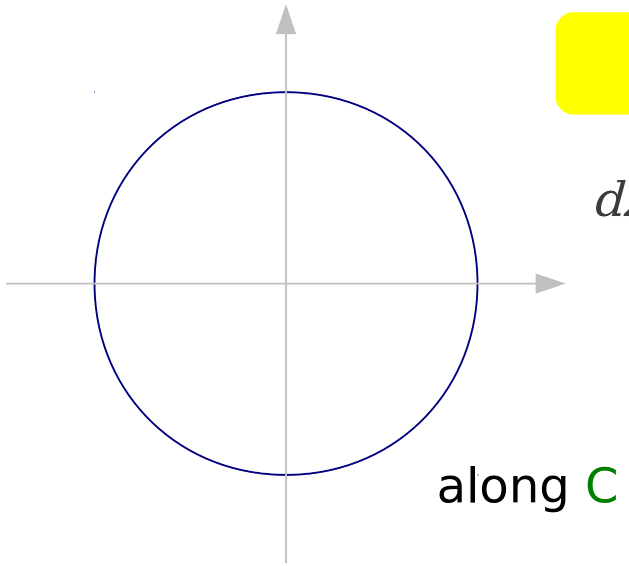
Contour Integration

$$\oint_C z\,dz = \int_0^{2\pi} e^{i\theta}\, i e^{i\theta}\, d\theta$$

$$= \left[\frac{1}{2} e^{i2\theta}\right]_0^{2\pi} = 0$$

$$\oint_C z^2\,dz = \int_0^{2\pi} e^{i2\theta}\, i e^{i\theta}\, d\theta$$

$$= \left[\frac{1}{3} e^{i3\theta}\right]_0^{2\pi} = 0$$



$$z = e^{i\theta}$$

$$dz = i e^{i\theta}\, d\theta$$

$$\oint_C \frac{1}{z}\,dz = \int_0^{2\pi} e^{-i\theta}\, i e^{i\theta}\, d\theta$$

$$= \left[i\right]_0^{2\pi} = 2\pi i$$

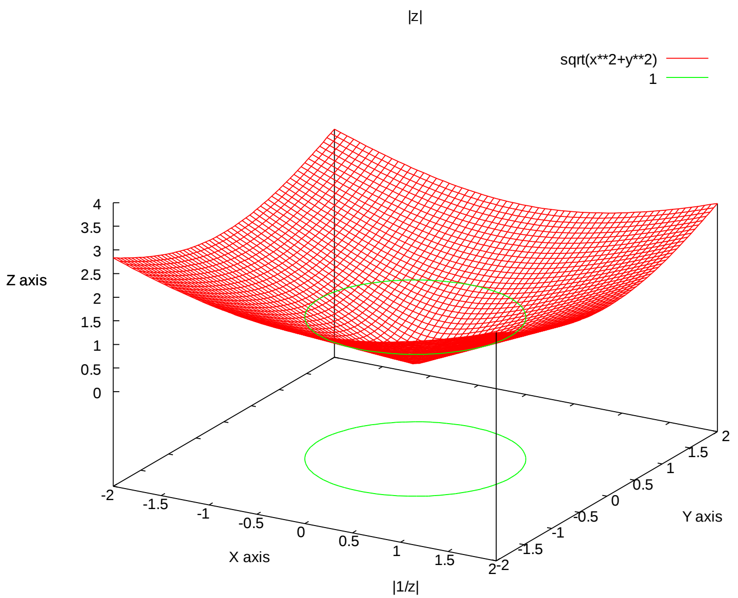
$$\oint_C z^2\,dz = \int_0^{2\pi} e^{-i2\theta}\, i e^{i\theta}\, d\theta$$

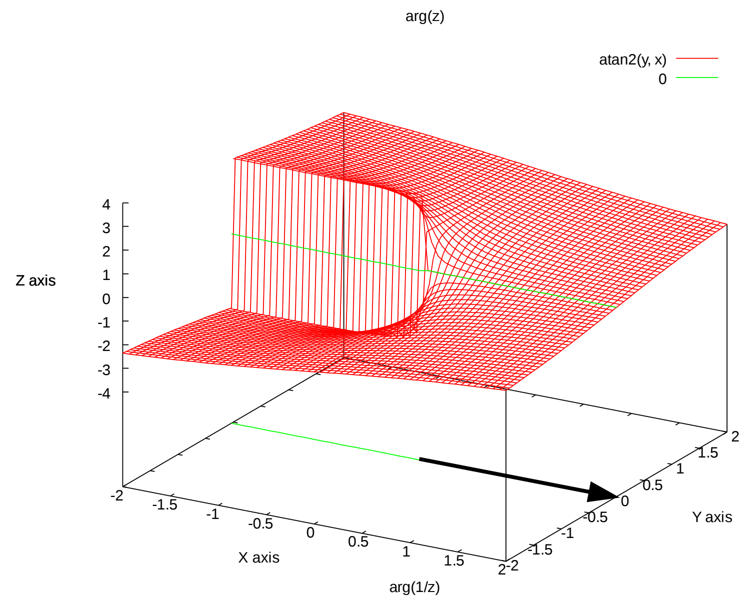
$$= \left[-e^{-i\theta}\right]_0^{2\pi} = 0$$

$$\oint_C dz = \int_0^{2\pi} i e^{i\theta}\, d\theta$$

$$= \left[e^{i\theta}\right]_0^{2\pi} = 0$$

Functions z, 1/z on the unit circle (1)

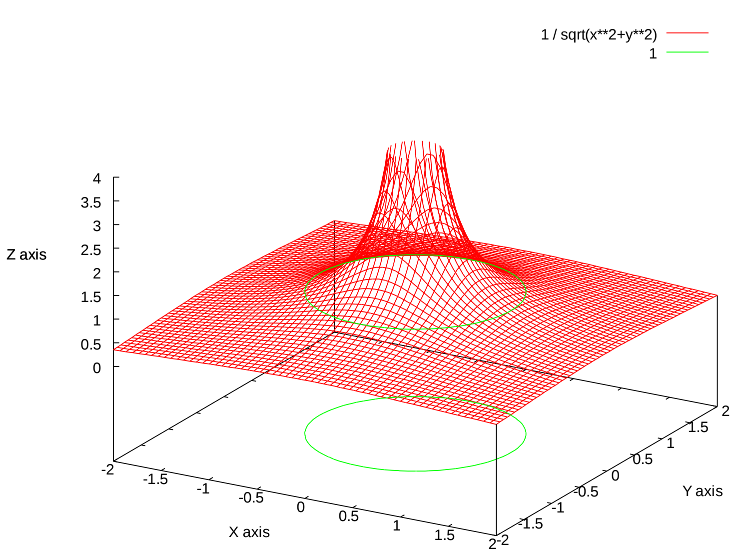


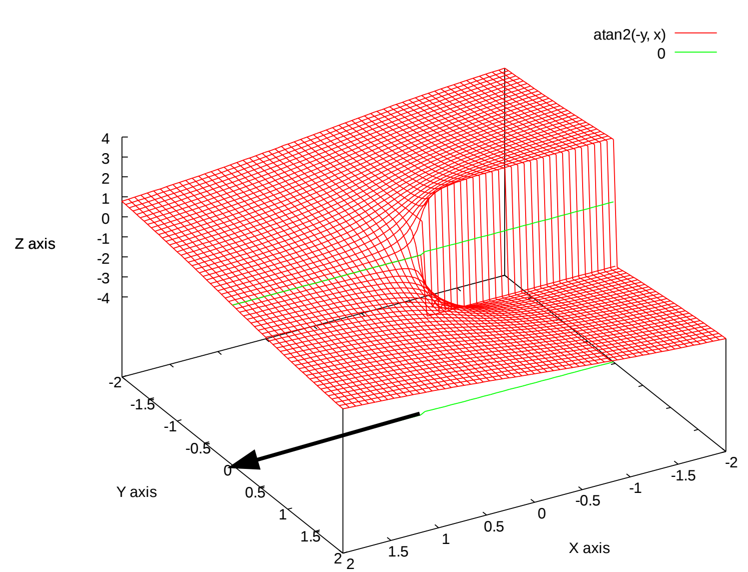


$$\oint_C z\,dz$$

$$= \int_0^{2\pi} e^{i\theta}\, i e^{i\theta}\, d\theta$$

$$= \left[\frac{1}{2} e^{i2\theta}\right]_0^{2\pi} = 0$$





$$\oint_C \frac{1}{z}\,dz$$

$$= \int_0^{2\pi} e^{-i\theta}\, i e^{i\theta}\, d\theta$$

$$= [i]_0^{2\pi} = 2\pi i$$

splot code for f(z)=z

```
# Plot f(z) = z
# Base on 3D gnuplot demo - contour plot
# Licensing: This code is distributed under the GNU LGPL
license.
# Modified: 2012.12.17
# Author: Young W. Lim

# set terminal pngcairo  transparent enhanced font "arial,10"
fontscale 0.8 size 400, 250
# set output 'contours.1.png'
set view 60, 30, 0.85, 1.1
set samples 60, 60
set isosamples 61, 61
#set contour base
#set contour surface
set contour both
set cntrparam levels discrete 1, 4

set title "|z|"
set xlabel "X axis"
set ylabel "Y axis"
set zlabel "Z axis"
set zlabel  offset character 1, 0, 0 font "" textcolor lt -1 norotate
.emf'
replot
set term wxt
```

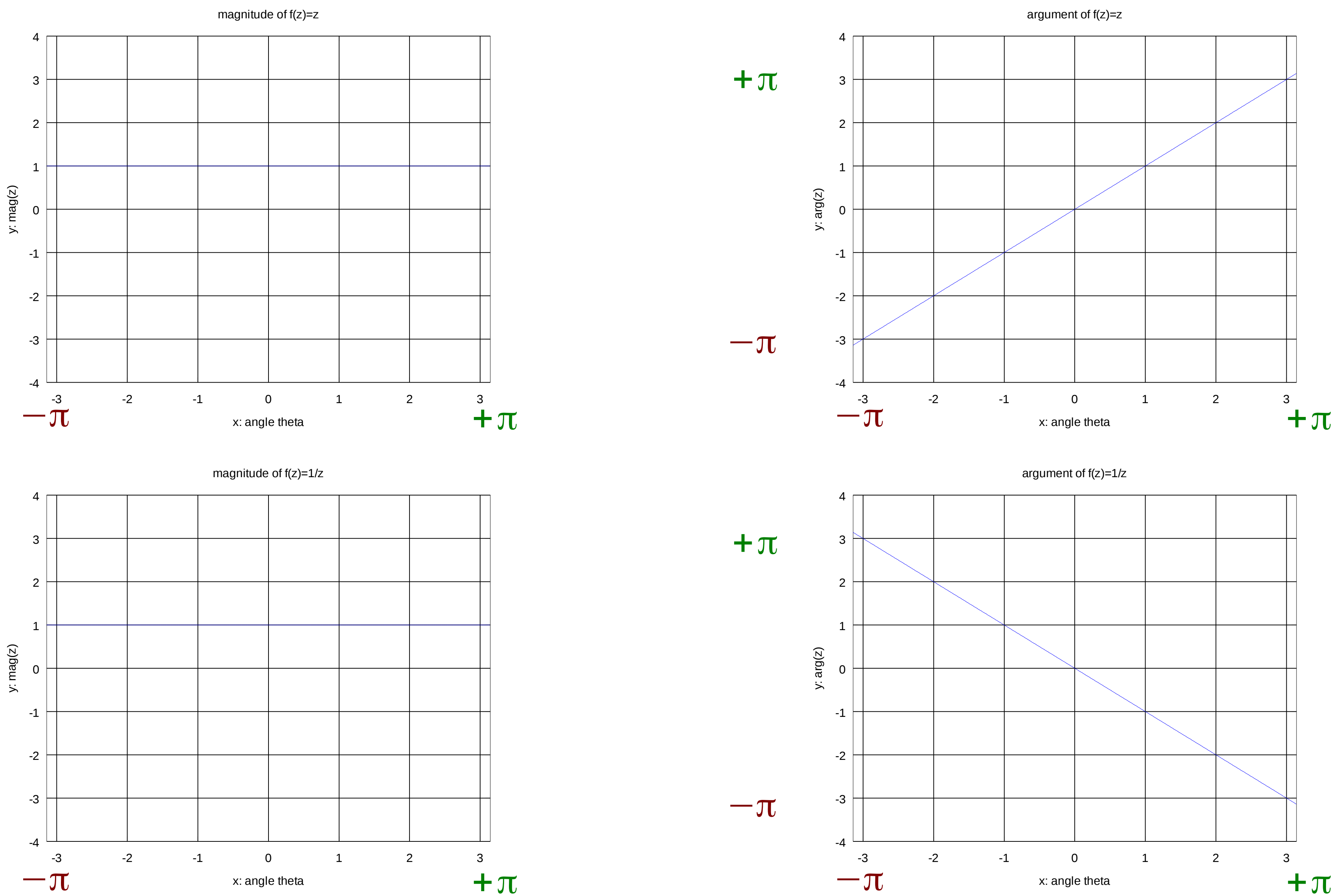
```
set xrange [-2: 2]
set yrange [-2: 2]
set zrange [0: 4]
splot sqrt(x**2+y**2)

set term emf
set output 'splot_z.mag.emf'
replot
set term wxt

pause -1

set cntrparam levels discrete 0
set zrange [-4: 4]
set title "arg(z)"

splot atan2(y, x)

set term emf
set output 'splot_z.arg
```

splot code for f(z)=1/z

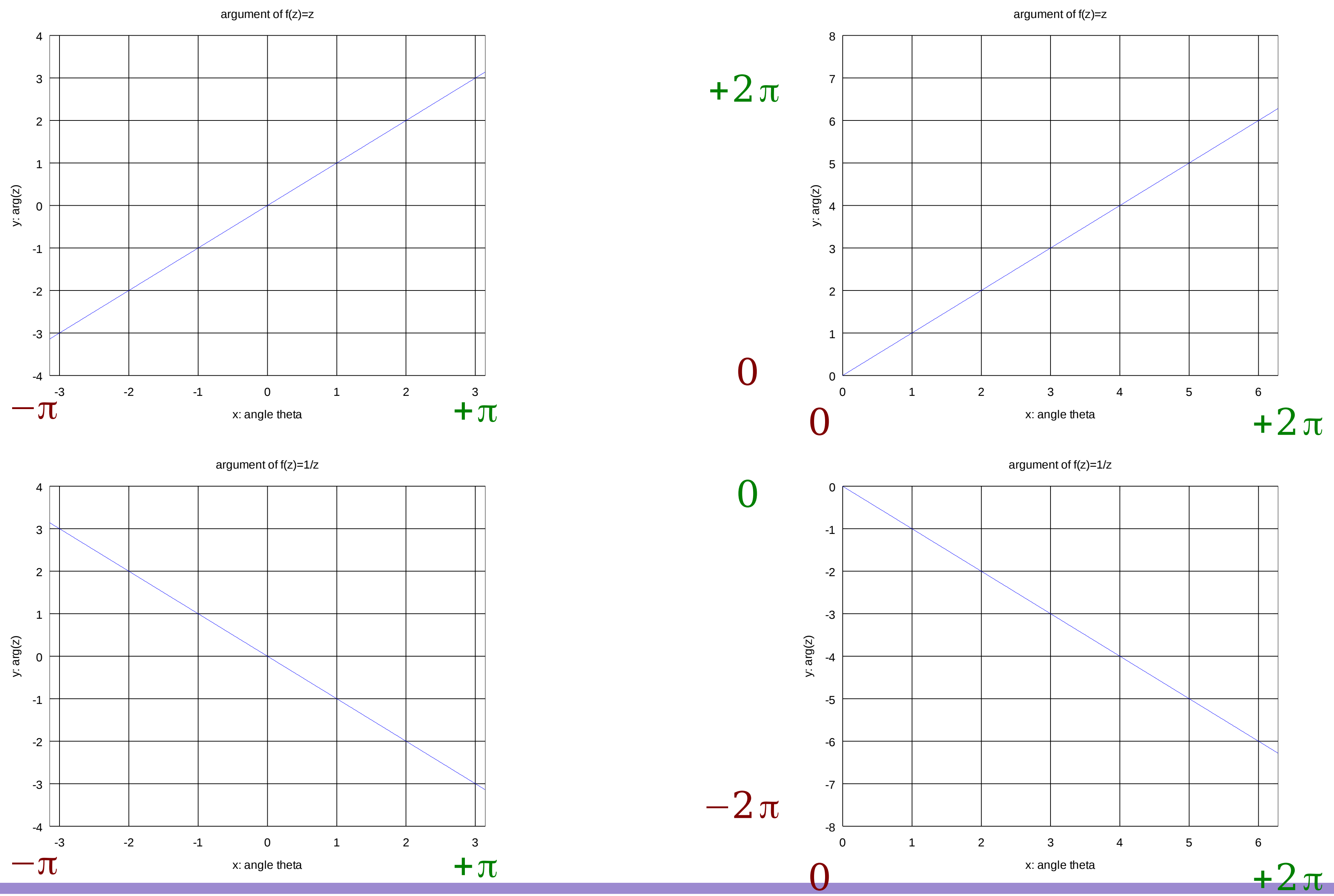
```
# Plot f(z) = z
# Base on 3D gnuplot demo - contour plot
# Licensing: This code is distributed under the GNU LGPL
license.
# Modified: 2012.12.17
# Author: Young W. Lim

# set terminal pngcairo  transparent enhanced font "arial,10"
fontscale 0.8 size 400, 250
# set output 'contours.1.png'
set view 60, 30, 0.85, 1.1
set samples 60, 60
set isosamples 61, 61
#set contour base
#set contour surface
set contour both
set cntrparam levels discrete 1, 4

set title "|1/z|"
set xlabel "X axis"
set ylabel "Y axis"
set zlabel "Z axis"
set zlabel  offset character 1, 0, 0 font "" textcolor lt -1 norotate
```

```
set xrange [-2: 2]
set yrange [-2: 2]
set zrange [0: 4]
splot 1 / sqrt(x**2+y**2)

set term emf
set output 'splot_1_z.mag.emf'
replot
set term wxt

pause -1

set view 47, 150, 0.85, 1.1
set cntrparam levels discrete 0
set zrange [-4: 4]
set title "arg(1/z)"
splot atan2(-y, x)


set term emf
set output 'splot_1_z.arg.emf'
replot
set term wxt
```

Plot around the unit circle (1)



Plot around the unit circle (2)

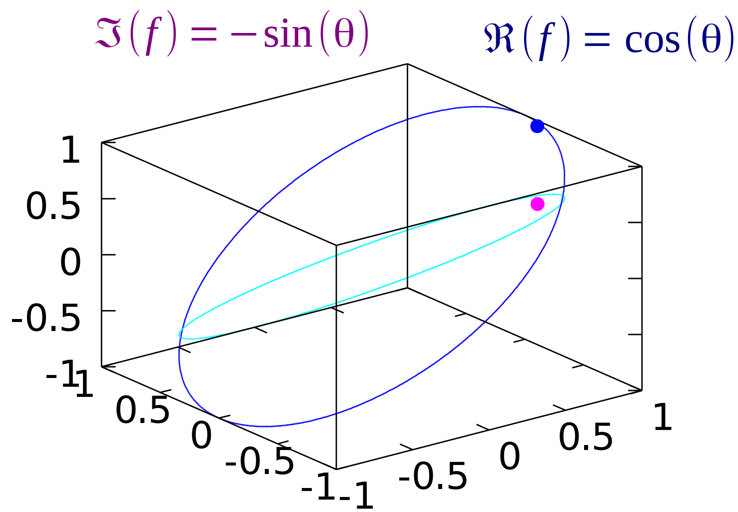


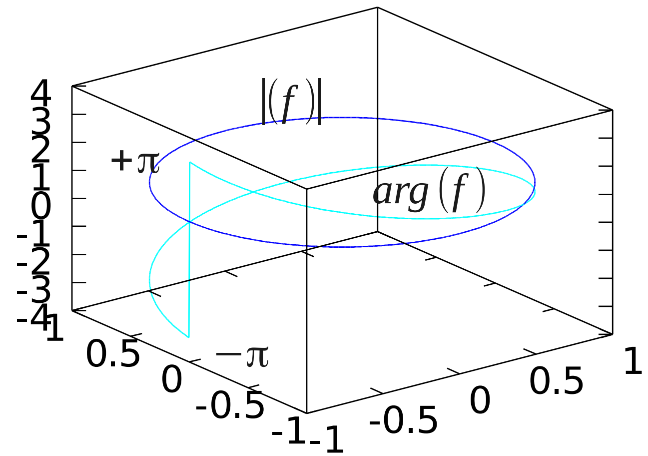
plot unit circle code

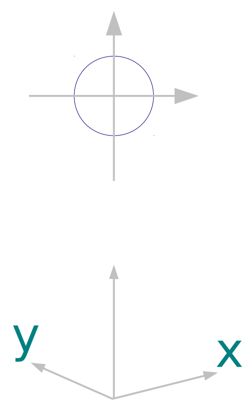
```
%-----------------------------------------------------------------------
% Plot f(z) = z on the unit circle
% Licensing: This code is distributed under the GNU LGPL license.
% Modified: 2012.12.17
% Author: Young W. Lim
%-----------------------------------------------------------------------
t = -pi : 0.01 : pi;
z = e.^(j*t);

plot(t, abs(z))
title("magnitude of f(z)=z");
xlabel("x: angle theta");
ylabel("y: mag(z)");
grid on
axis([-pi pi -4 +4]);
print -demf uc_z.mag.emf
pause
plot(t, arg(z))
title("argument of f(z)=z");
xlabel("x: angle theta");
ylabel("y: arg(z)");
grid on
axis([-pi pi -4 +4]);
print -demf uc_z.arg.emf
```

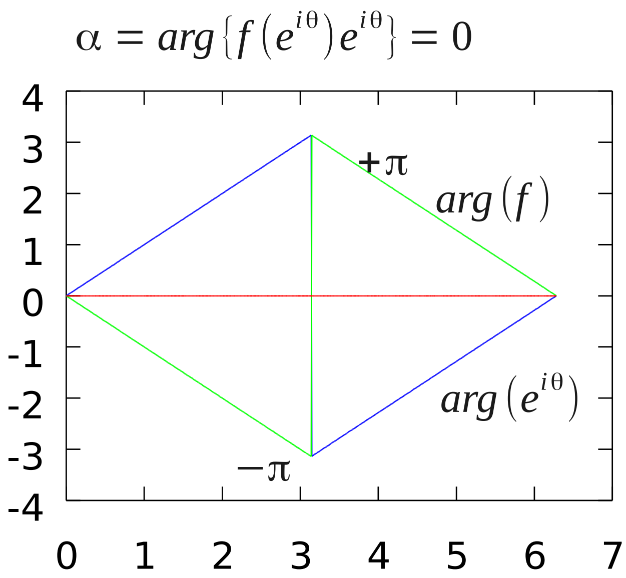
```
t = -pi : 0.01 : pi;
z = e.^(-j*t);

plot(t, abs(z))
title("magnitude of f(z)=1/z");
xlabel("x: angle theta");
ylabel("y: mag(z)");
grid on
axis([-pi pi -4 +4]);
print -demf uc_1_z.mag.emf
pause
plot(t, arg(z))
title("argument of f(z)=1/z");
xlabel("x: angle theta");
ylabel("y: arg(z)");
grid on
axis([-pi pi -4 +4]);
print -demf uc_1_z.arg.emf
```

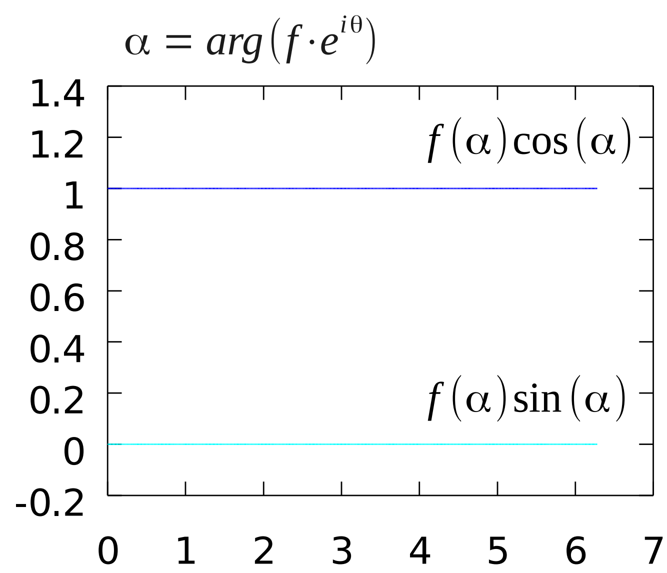
Contour Integration of f(z)=1/z











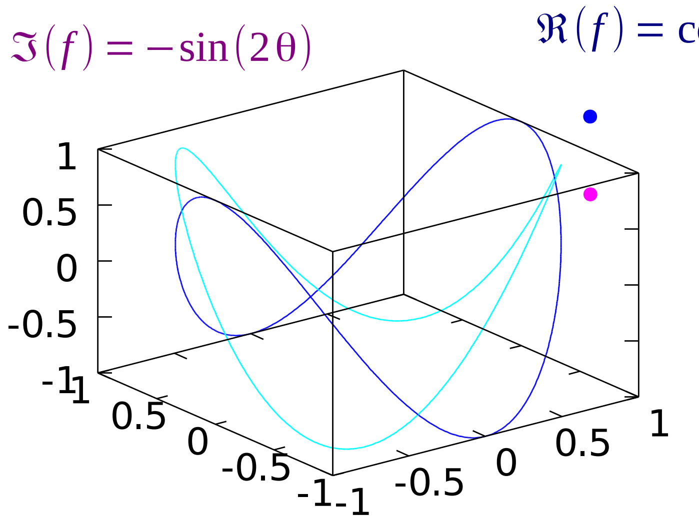
$$\int_{-\pi}^{+\pi} f(r,\theta)\, i e^{i\theta}\, d\theta$$

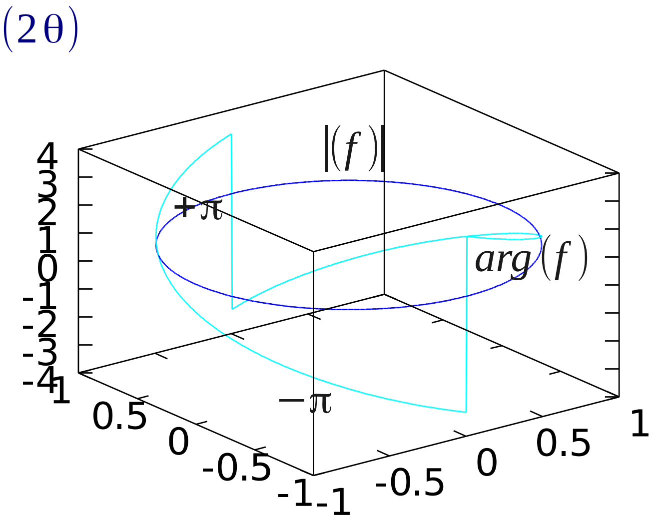
$$= \int_{-\pi}^{+\pi} e^{-i\theta}\, i e^{i\theta}\, d\theta$$

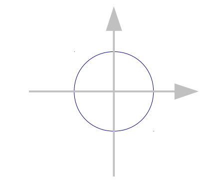
$$= \int_{-\pi}^{+\pi} i\, d\theta$$

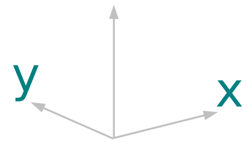
$$= 2\pi i$$

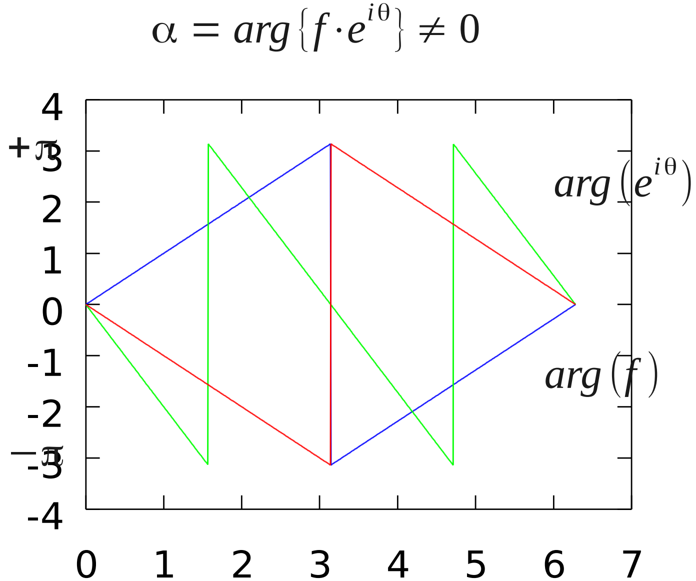
Contour Integration of f(z)=1/z²

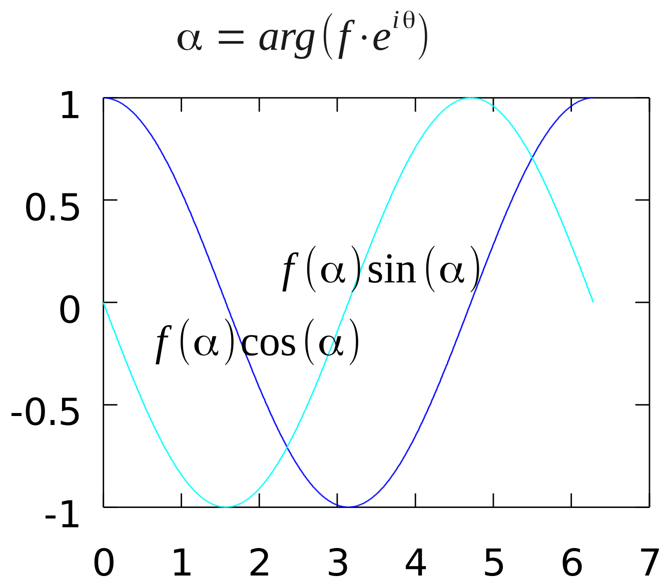












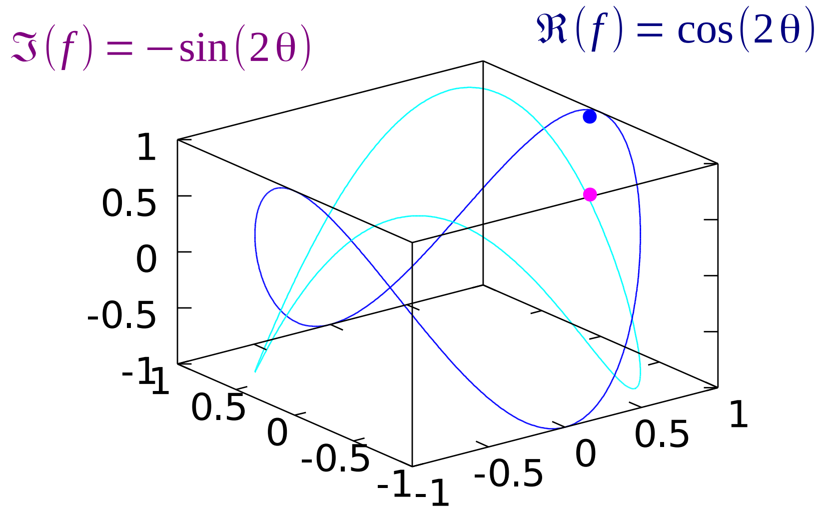
$$\int_{-\pi}^{+\pi} f(r,\theta)\, i e^{i\theta}\, d\theta$$

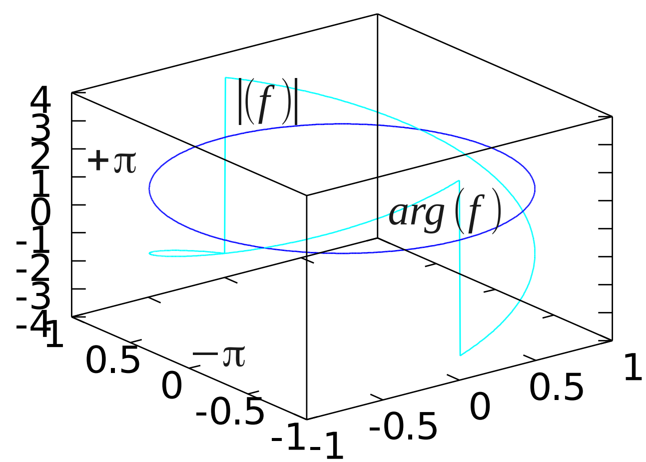
$$= \int_{-\pi}^{+\pi} e^{-i2\theta} i e^{i\theta}\, d\theta$$

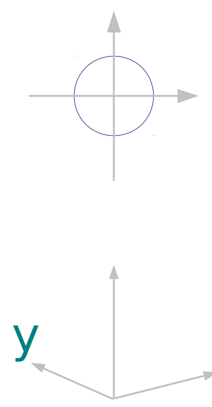
$$= \int_{-\pi}^{+\pi} i\, e^{-i\theta}\, d\theta$$

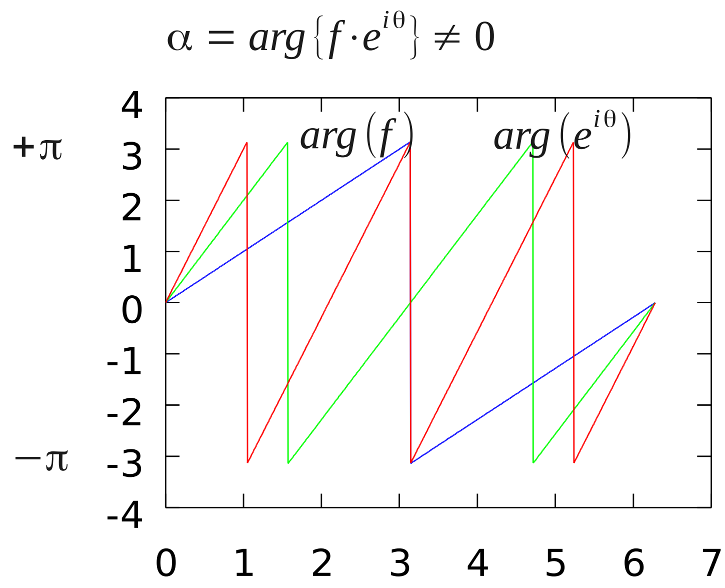
$$= 0$$

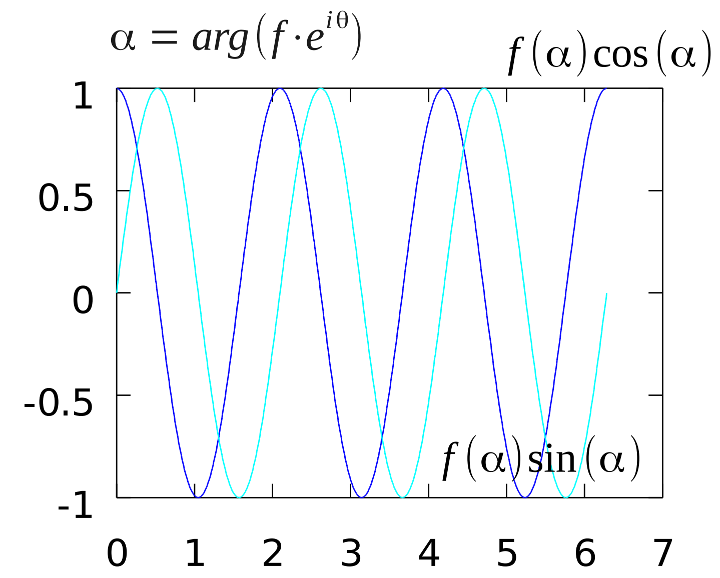
Contour Integration of f(z)=z²











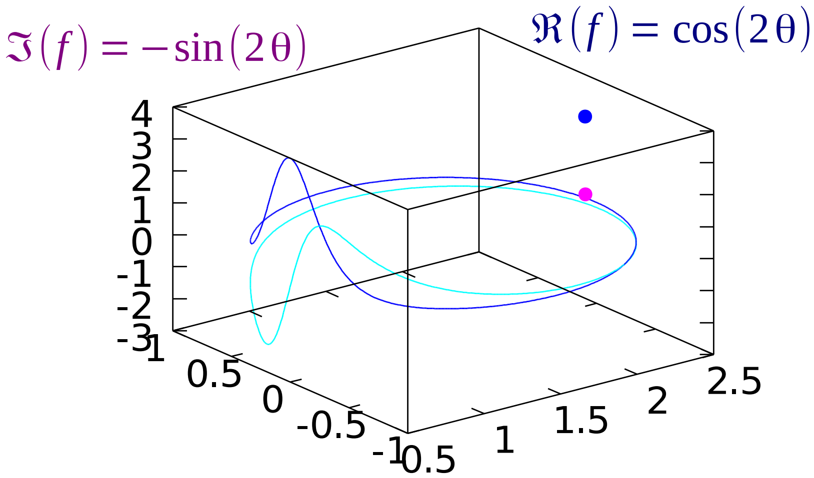
$$\int_{-\pi}^{+\pi} f(r,\theta) i e^{i\theta} d\theta$$

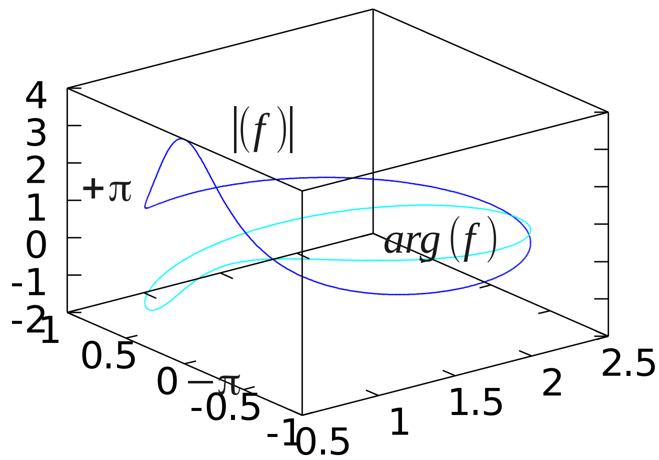
$$= \int_{-\pi}^{+\pi} e^{i2\theta} i e^{i\theta} d\theta$$

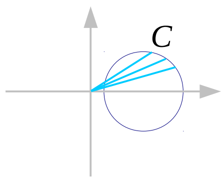
$$= \int_{-\pi}^{+\pi} i e^{i3\theta} d\theta$$

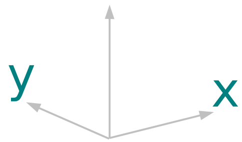
$$= 0$$

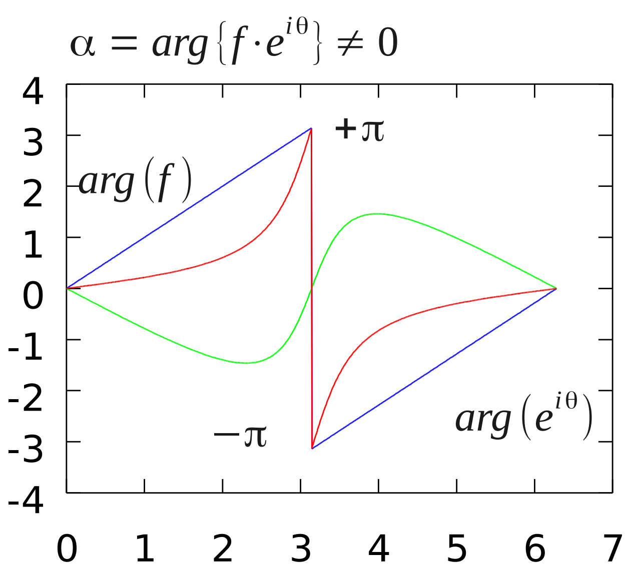
Contour Integration of f(z)=1/z along unit circle at (1.5, 0)

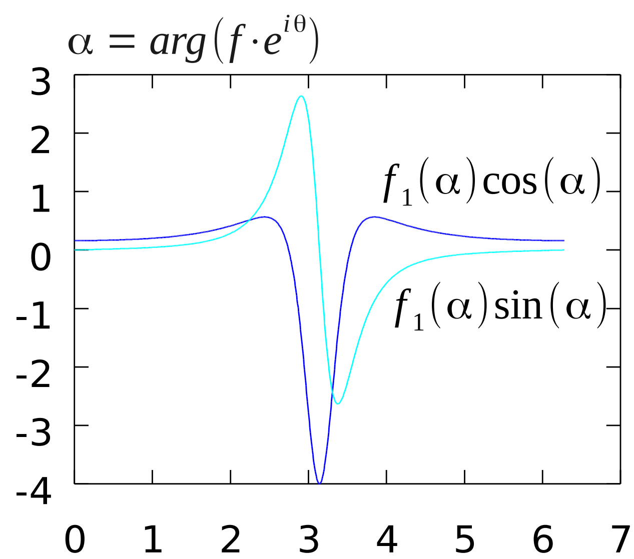












$$\int_C f(r,\theta)\, i e^{i\theta}\, d\theta$$

$$\int_{-\pi}^{+\pi} f(r+1.5,\theta)\, i e^{i\theta}\, d\theta$$

$$\int_{-\pi}^{+\pi} f_1(\theta)\, i e^{i\theta}\, d\theta$$

$$= 0$$

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1/17/14

Plotting Code for Contour Integrals

```
%-----------------------------------------------------------------------------
% Plot the integration of f(z) = 1/z along the unit circle
% Licensing: This code is distributed under the GNU LGPL license.
% Modified: 2014.01.15
% Author: Young W. Lim
%-----------------------------------------------------------------------------

clf; clear *;

t=linspace(0, 2*pi, 1000);
a = 1.5
xt = a+cos(t);
yt = sin(t);
zt = (xt + i*yt);
zt = zt .* zt;
ft =  1./zt;  % ft = 1./zt, ft = zt;

theta = rem(arg((xt-a+i*yt).*(ft)), 2*pi);

xx = abs(ft).*cos(theta);
yy = abs(ft).*sin(theta);

subplot(2,2, 1)
plot3(xt, yt, real(ft), "b", xt, yt, imag(ft), "c")

subplot(2,2, 2)
plot3(xt, yt, abs(ft), "b", xt, yt, arg(ft), "c")

subplot(2,2, 3)
plot(t, arg(xt-a+i*yt), "b", t, arg(ft), "g", t, theta, "r")

subplot(2,2, 4)
plot(t, xx, "b", t, yy, "c")

sum(xx(1:length(xx)-1))
sum(yy(1:length(xx)-1))
```

Cauchy's Integral Formula

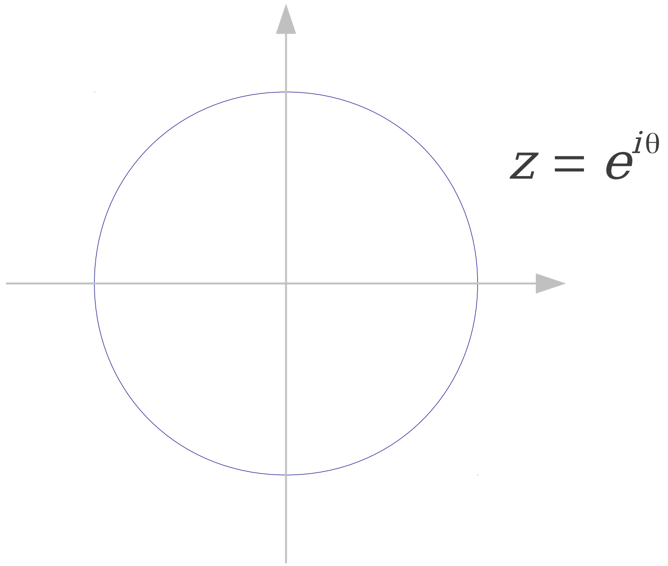
$f(z)$: **analytic** on and inside simple close curve C

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$$

the value of $f(z)$

at a point $z = a$ inside C

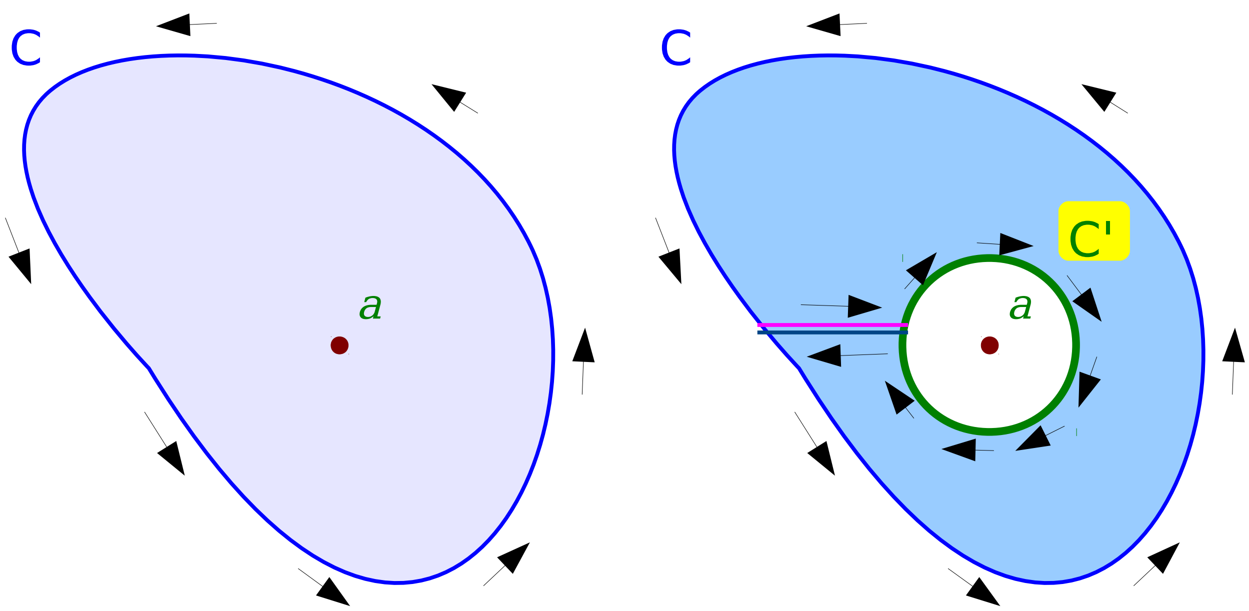
$$f(z) = \frac{1}{2\pi i}\oint \frac{f(w)}{w-z}\,dw$$



Cauchy's Integral Formula

$f(z)$: **analytic** on and inside simple close curve C

$$\Rightarrow \quad f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$$



$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \oint_{ccw\ C'} \frac{f(z)\,dz}{z-a}$$

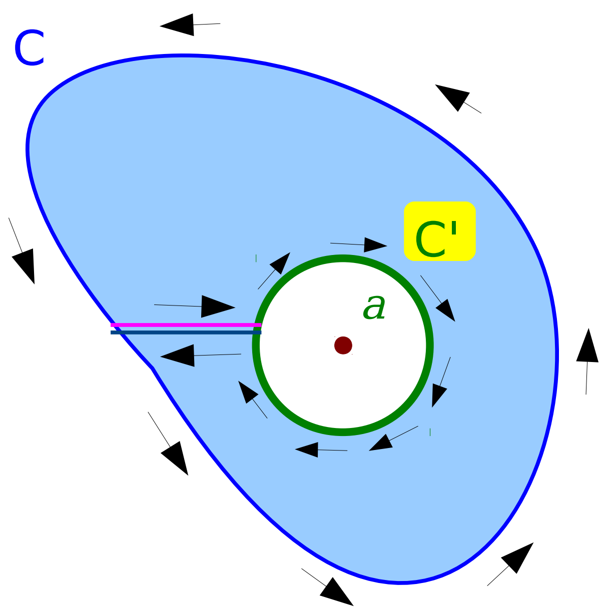
$$\oint_C f(z)\,dz = 0$$

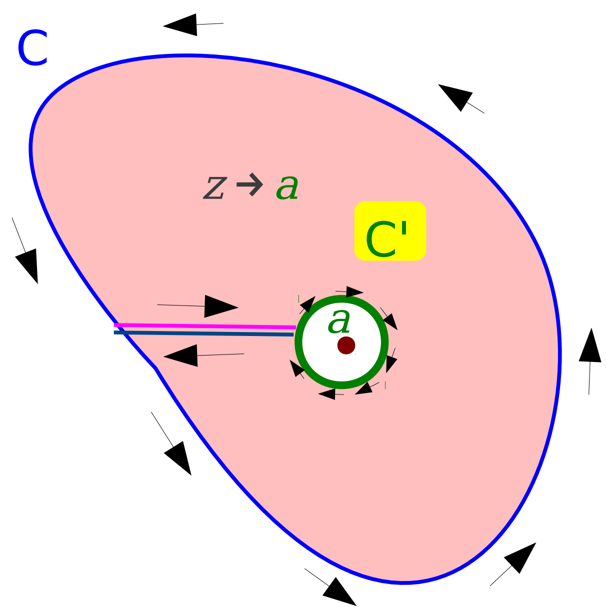
$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} + \oint_{cw\ C'} \frac{f(z)\,dz}{z-a} = 0$$

Cauchy's Integral Formula

$f(z)$: **analytic** on and inside simple close curve C

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$$





$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a} = \oint_{ccw\ C'} \frac{f(z)\,dz}{z-a}$$

$$= 2\pi i\ f(a)$$

along C' $\quad z - a = \rho e^{i\theta}$

$$z = a - \rho e^{i\theta}$$

$$dz = i\rho e^{i\theta}\,d\theta$$

$$\frac{dz}{z-a} = \frac{i\rho e^{i\theta}\,d\theta}{\rho e^{i\theta}}$$

$$\oint_{ccw\ C} \frac{f(z)\,dz}{z-a}$$

$$= \int_0^{2\pi} f(z)\,i\,d\theta$$

$$= 2\pi i\ f(a)$$

as $\quad z \to a \ \Rightarrow\ \rho \to 0$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”