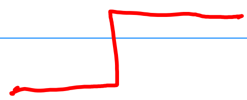
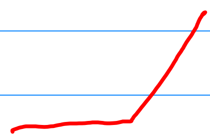
Transient Responses (H.1)

20150603

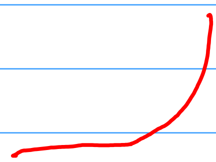
step fn $u(t) \Longleftrightarrow \frac{1}{s}$



ramp fn $t\,u(t) \Longleftrightarrow \frac{1}{s^2}$



parabola fn $t^2 u(t) \Longleftrightarrow \frac{2}{s^3}$



1st Order Circuit

$$\boxed{a_1 \frac{dx}{dt} + a_0 x(t) = b_0 f(t)}$$

$$\boxed{\frac{a_1}{a_0} \frac{dx}{dt} + 1 \cdot x(t) = \frac{b_0}{a_0} f(t)}$$

$$\boxed{\tau \frac{dx}{dt} + 1 \cdot x(t) = K_s f(t)}$$

$$\boxed{\tau \frac{dx}{dt} + 1\, x(t) = K_s\, f(t)}$$

time constant τ dc gain K_s

associated homogeneous eq

$$\tau \frac{dx}{dt} + 1\, x(t) = 0 \Rightarrow x_h(t)$$ homogeneous solution

$x_n(t)$ natural response

$\tau m + 1 = 0$

$m = -1/\tau$

$e^{-\frac{t}{\tau}} \rightarrow 0$ as $t \to \infty$

$$\boxed{\alpha\, e^{-\frac{t}{\tau}}}$$

$$\tau \frac{dx}{dt} + 1\, x(t) = K_s\, f(t) \Rightarrow x_p(t)$$ particular solution

$x_f(t)$ forced response

similar form as $f(t)$

natural response

$$\tau \frac{dx}{dt} + 1\, x(t) = 0$$

$\tau m + 1 = 0$ auxiliary equation

$$m = -\frac{1}{\tau}$$

$$x_h(t) = c e^{-\frac{1}{\tau} t}$$

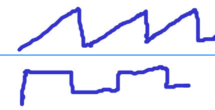
$$x_n(t) = \alpha e^{-t/\tau}$$

$$x_n(\infty) = \alpha \frac{1}{e^{\infty/\tau}} = 0$$

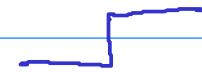
* at the steady state the natural response '$\boxed{\text{vanishes}}$'.

* at the steady state only part of $x_p(t)$ particular solution is left.

forced response



$$\tau \frac{dx}{dt} + 1\,x(t) = \boxed{K_s\, f(t)}$$ time varying $f(t)$



$$\tau \frac{dx}{dt} + 1\,x(t) = \boxed{K_s\, F}$$ constant $F \rightarrow$ dc

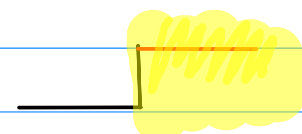
assume $x_p(t) = A$ ← constant

substitute

$$\tau \cdot 0 + 1\,A = K_s\, F$$

$\therefore x_p(t) = K_s\, F$: particular solution

$$\boxed{\tau \frac{dx}{dt} + 1\,x(t) = K_s\, F}$$



$(t \geq 0)$

this eq holds only when $t \geq 0$

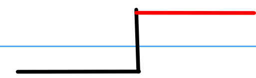
$x_f(t) = K_s F \quad (t \geq 0)$

$x_f(\infty) = K_s F$ — steady state response

constant input F

* particular solution
= forced response
= steady state response

Complete Response



dc input
constant, not varying

$$\tau \frac{dx}{dt} + 1\, x(t) = K_s F \qquad (t \geq 0)$$

this eq holds only when $t \geq 0$

$$x(t) = x_n(t) + x_f(t) = \alpha e^{-t/\tau} + K_s F$$

$$= \alpha e^{-t/\tau} + x(\infty)$$

$t=0$ coefficient α

$$x(0) = x_n(0) + x_f(0) = \alpha e^{-0/\tau} + K_s F$$

$$= \alpha e^{-0/\tau} + x(\infty)$$

$$x(0) = \alpha + x(\infty)$$

$$\alpha = x(0) - x(\infty)$$

$$\boxed{x(t) = \left[x(0) - x(\infty)\right] e^{-t/\tau} + x(\infty)} \qquad t \geq 0$$

2nd Order Circuits

$$a_2 \boxed{\frac{d^2x(t)}{dt}} + a_1 \boxed{\frac{dx(t)}{dt}} + a_0 \boxed{x(t)} = b_0 f(t)$$

$$\boxed{a_2 m^2 + a_1 m + a_0 = 0}$$ auxilary eq

$$a_2 \boxed{\frac{d^2x(t)}{dt}} + a_1 \boxed{\frac{dx(t)}{dt}} + a_0 \boxed{x(t)} = 0 \quad *$$

Homogeneous Solution ⇒ natural response

1. $\boxed{D > 0}$ 2 distinct real roots m_1, m_2

$$\boxed{x_h(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}}$$

2. $\boxed{D = 0}$ repeated real root m_1

$$\boxed{x_h(t) = C_1 e^{m_1 t} + C_2 t \cdot e^{m_2 t}}$$

3. $\boxed{D < 0}$ 2 complex conjugate root $m_1 = \alpha + j\beta$, $m_2 = \alpha - j\beta$

$$\boxed{\begin{aligned} x_h(t) &= C_1 e^{(\alpha + j\beta)t} + C_2 e^{(\alpha - j\beta)t} \\ &= e^{\alpha t}\left(C_3 \cos(\beta t) + C_4 \sin(\beta t)\right) \end{aligned}}$$

$$a_2 \boxed{\frac{d^2 x(t)}{dt}} + a_1 \boxed{\frac{dx(t)}{dt}} + a_0 \boxed{x(t)} = b_0 f(t)$$

Particular solution $\Rightarrow$ forced response

$x_p(t)$ — similar form of $f(t)$

$A \leftarrow C$

$At + B \leftarrow t+1$

$Ae^{2t} \leftarrow e^{2t}$

$A\cos(kt) + B\sin(kt) \leftarrow \sin(kt)$

Laplace Transform

$$\text{(I)}\quad \boxed{a_2 \frac{d^2x(t)}{dt} + a_1 \frac{dx(t)}{dt} + a_0\, x(t) = b_0\, f(t)}$$

$$\text{(II)}\quad \boxed{\frac{1}{\omega_n^2} \frac{d^2x(t)}{dt} + 2\frac{\zeta}{\omega_n} \frac{dx(t)}{dt} + 1\, x(t) = K f(t)}$$

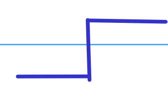
$$\text{(III)}\quad \boxed{1 \frac{d^2x(t)}{dt} + 2\zeta\omega_n \frac{dx(t)}{dt} + \omega_n^2\, x(t) = \omega_n^2 K f(t)}$$

$$s^2 X(s) + 2\zeta\omega_n s X(s) + \omega_n^2 X(s) = \omega_n^2 K F(s)$$

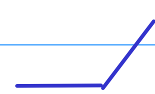
$$\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right) X(s) = \omega_n^2 K F(s)$$

$$X(s) = \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} K F(s)$$

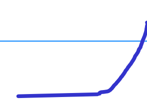
$$X(s) = \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} \frac{K}{s}$$



$$X(s) = \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} \frac{K}{s^2}$$



$$X(s) = \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} \frac{2K}{s^3}$$



Natural Response $\leftarrow$ $x_h(t)$

homogeneous eq.

(I) $$a_2 \frac{d^2x(t)}{dt} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = b_0 f(t)$$

(II) $$\frac{1}{\omega_n^2} \frac{d^2x(t)}{dt} + 2\frac{\zeta}{\omega_n} \frac{dx(t)}{dt} + 1\, x(t) = K f(t)$$

(III) $$1\, \frac{d^2x(t)}{dt} + 2\zeta\omega_n \frac{dx(t)}{dt} + \omega_n^2 x(t) = \omega_n^2 K f(t)$$

(I) $$a_2 \frac{d^2x(t)}{dt} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = 0$$

(II) $$\frac{1}{\omega_n^2} \frac{d^2x(t)}{dt} + 2\frac{\zeta}{\omega_n} \frac{dx(t)}{dt} + 1\, x(t) = 0$$

(III) $$1\, \frac{d^2x(t)}{dt} + 2\zeta\omega_n \frac{dx(t)}{dt} + \omega_n^2 x(t) = 0$$

associated homogeneous equation

$D > 0$ 2 distinct real roots m_1, m_2

$x_n(t) \Leftarrow$ $$\boxed{x_h(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}}$$

$D = 0$ repeated real root m_1

$x_n(t) \Leftarrow$ $$\boxed{x_h(t) = C_1 e^{m_1 t} + C_2 t \cdot e^{m_2 t}}$$

$D < 0$ 2 complex conjugate root $m_1 = \alpha + j\beta$, $m_2 = \alpha - j\beta$

$x_n(t) \Leftarrow$ $$\boxed{\begin{aligned} x_h(t) &= C_1 e^{(\alpha + j\beta)t} + C_2 e^{(\alpha - j\beta)t} \\ &= e^{\alpha t}\left(C_3 \cos(\beta t) + C_4 \sin(\beta t)\right) \end{aligned}}$$

* coefficients C_1, C_2 determined from initial conditions

$x(0)$; $\dot{x}(0)$ $\rightarrow$ $x(t) = x_n(t) + x_f(t)$

$$1\,\frac{d^2x(t)}{dt} + 2\zeta\omega_n \frac{dx(t)}{dt} + \omega_n^2\, x(t) = \omega_n^2\, K f(t)$$

$$1\,\frac{d^2x(t)}{dt} + 2\zeta\omega_n \frac{dx(t)}{dt} + \omega_n^2\, x(t) = 0.$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

$$\boxed{m^2 + 2\zeta\omega_n m + \omega_n^2 = 0}$$

$$m = \frac{-\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}}{a} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\;\omega_n$$

$\omega_n > 0$

$$D = (\zeta\omega_n)^2 - \omega_n^2 = (\zeta^2 - 1)\,\omega_n^2$$

$D > 0$ $\quad \zeta^2 > 1$ $\quad$ Overdamping

$D = 0$ $\quad \zeta^2 = 1$ $\quad$ Critical damping

$D < 0$ $\quad \zeta^2 < 1$ $\quad$ Underdamping

* consider $\underline{\zeta > 0}$

Over damping $\quad \zeta > 1$

$$m_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$$
$$m_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$$

$$\begin{aligned} x_n(t) &= \alpha_1 e^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t} + \alpha_2 e^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t} \\ &= \alpha_1 e^{-(\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t} + \alpha_2 e^{-(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t} \\ &= \alpha_1 e^{-t/\tau_1} + \alpha_2 e^{-t/\tau_2} \end{aligned}$$

Critical damping $\quad \zeta = 1$

$$m_1 = -\zeta\omega_n = -\omega_n$$
$$m_2 = -\zeta\omega_n = -\omega_n$$

$$\begin{aligned} x_n(t) &= \alpha_1 e^{(-\zeta\omega_n)t} + \alpha_2 t \cdot e^{(-\zeta\omega_n)t} \\ &= \alpha_1 e^{-(\omega_n)t} + \alpha_2 t \cdot e^{-(\omega_n)t} \\ &= \alpha_1 e^{-t/\tau_1} + \alpha_2 t \cdot e^{-t/\tau_2} \end{aligned}$$

Under damping $\quad \zeta < 1$

$$m_1 = -\zeta\omega_n + j\sqrt{1-\zeta^2}\,\omega_n$$
$$m_2 = -\zeta\omega_n - j\sqrt{1-\zeta^2}\,\omega_n$$

$$\begin{aligned} x_n(t) &= \alpha_1 e^{(-\zeta\omega_n + j\sqrt{1-\zeta^2}\,\omega_n)t} + \alpha_2 e^{(-\zeta\omega_n - j\sqrt{1-\zeta^2}\,\omega_n)t} \\ &= e^{-(\zeta\omega_n)t}\left(\alpha_3 \cos(\sqrt{1-\zeta^2}\,\omega_n t) + \alpha_4 \sin(\sqrt{1-\zeta^2}\,\omega_n t)\right) \\ &= A \cdot e^{-t/\tau} \cos(\sqrt{1-\zeta^2}\,\omega_n t + \theta) \end{aligned}$$

* Time Response

2nd Order System

$$\frac{-b \pm \sqrt{b^2-4a}}{2a} \qquad \frac{-b \pm \sqrt{b^2-ac}}{a}$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s-s_1)(s-s_2)}$$

$$= \frac{A}{(s-s_1)} + \frac{B}{(s-s_2)}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

$$= -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

$\omega_n > 0$

$D = \zeta^2 - 1 > 0$ 두 실근 (Overdamping)

$$s_1 = -\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n$$

$$s_2 = -\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n$$

$D = \zeta^2 - 1 = 0$ 중근 (Critically Damping)

$$s_1 = s_2 = -\zeta\omega_n + \sqrt{\overbrace{\zeta^2-1}^{0}}\,\omega_n = -\zeta\omega_n$$

$D = \zeta^2 - 1 < 0$ (Underdamping)

$$s_1 = -\zeta\omega_n + j\sqrt{1-\zeta^2}\,\omega_n = \sigma + j\omega$$

$$s_2 = -\zeta\omega_n - j\sqrt{1-\zeta^2}\,\omega_n = \sigma - j\omega$$

$D = \zeta^2 - 1 > 0$ 두 실근 (Overdamping)

$$G(s) = \frac{\omega_n^2}{(s - s_1)(s - s_2)}$$

$$\boxed{\begin{aligned} s_1 &= -\zeta \omega_n + \sqrt{\zeta^2 - 1}\,\omega_n \\ s_2 &= -\zeta \omega_n - \sqrt{\zeta^2 - 1}\,\omega_n \end{aligned}}$$

$D = \zeta^2 - 1 = 0$ 중근 (Critically Damping)

$$G(s) = \frac{\omega_n^2}{(s - s_1)^2}$$

$$\boxed{s_1 = s_2 = -\zeta \omega_n}$$

$D = \zeta^2 - 1 < 0$ (Underdamping)

$$G(s) = \frac{\omega_n^2}{(s - s_1)(s - s_2)}$$

$$\boxed{\begin{aligned} s_1 &= -\zeta \omega_n + j\sqrt{1 - \zeta^2}\,\omega_n \\ s_2 &= -\zeta \omega_n - j\sqrt{1 - \zeta^2}\,\omega_n \end{aligned}}$$

$$(s - s_1)(s - s_2) \Leftarrow s^2 - (s_1 + s_2)s + s_1 s_2$$

Verification

$(a+b)(a-b) = a^2 - b^2$
$(a+jb)(a-jb) = a^2 - (jb)^2$
$= a^2 + b^2$

$D = \zeta^2 - 1 > 0$ 두 실근 (Overdamping)

$S_1 + S_2 = -2\zeta\omega_n$

$S_1 \cdot S_2 = (-\zeta\omega_n)^2 - \left(\sqrt{\zeta^2 - 1}\,\omega_n\right)^2$

$= \zeta^2\omega_n^2 - (\zeta^2 - 1)\omega_n^2$

$= \omega_n^2$

$$\boxed{\begin{aligned} S_1 &= -\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n \\ S_2 &= -\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n \end{aligned}}$$

$D = \zeta^2 - 1 = 0$ 중근 (Critically Damping)

$S_1 + S_2 = 2S_1 = -2\zeta\omega_n$

$S_1 \cdot S_2 = S_1^2 = \zeta^2\omega_n^2 = \omega_n^2$

$\boxed{\zeta^2 = 1}$

$$\boxed{S_1 = S_2 = -\zeta\omega_n}$$

$D = \zeta^2 - 1 < 0$ (Underdamping)

$S_1 + S_2 = -2\zeta\omega_n$

$S_1 \cdot S_2 = (-\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2$

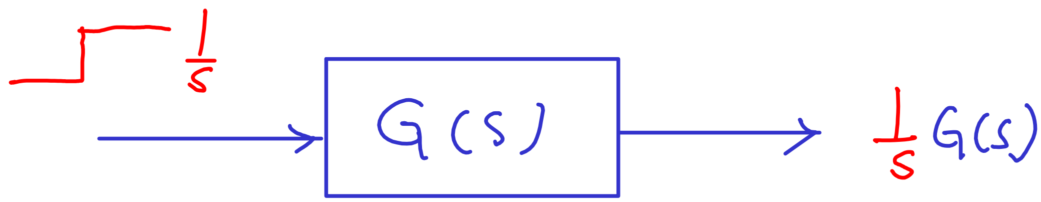
$= \zeta^2\omega_n^2 + (1-\zeta^2)\omega_n^2$

$= \omega_n^2$

$$\boxed{\begin{aligned} S_1 &= -\zeta\omega_n + j\sqrt{1-\zeta^2}\,\omega_n \\ S_2 &= -\zeta\omega_n - j\sqrt{1-\zeta^2}\,\omega_n \end{aligned}}$$

$$\boxed{\begin{aligned} (s-S_1)(s-S_2) &= s^2 - (S_1+S_2)s + S_1S_2 \\ &= s^2 + 2\zeta\omega_n s + \omega_n^2 \end{aligned}}$$

Step response $= \mathcal{L}^{-1}\left\{ \frac{1}{s} G(s) \right\}$



$D = \zeta^2 - 1 > 0$ 두 실근 (Overdamping)

$$\frac{1}{s}G(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{k_0}{s} + \frac{k_1}{(s-s_1)} + \frac{k_2}{(s-s_2)}$$

$D = \zeta^2 - 1 = 0$ 중근 (Critically Damping)

$$\frac{1}{s}G(s) = \frac{\omega_n^2}{s(s-s_1)^2} = \frac{k_0}{s} + \frac{k_1}{(s-s_1)} + \frac{k_2}{(s-s_1)^2}$$

$D = \zeta^2 - 1 < 0$ (Underdamping)

$$\frac{1}{s}G(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{k_0}{s} + \frac{k_1}{s-s_1} + \frac{k_2}{s-s_2}$$

or $$= \frac{k_0}{s} + \frac{k_1 s + k_2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$\frac{k}{s^2+k^2} \Longleftrightarrow \sin(kt)$

$\frac{s}{s^2+k^2} \Longleftrightarrow \cos(kt)$

Use $\Rightarrow$ $$\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{1}{(s^2+2\zeta\omega_n s+\zeta^2\omega_n^2) - \zeta^2\omega_n^2 + \omega_n^2}$$

$$= \frac{1}{(s+\zeta\omega_n)^2 + (1-\zeta^2)\omega_n^2}$$

Underdamping $\zeta^2 < 1$ ①

$$\boxed{\frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{k_0}{s} + \frac{k_1 s + k_2}{(s^2+2\zeta\omega_n s+\omega_n^2)}}$$

$$k_0 s^2 + 2k_0\zeta\omega_n s + k_0\omega_n^2 + k_1 s^2 + k_2 s = \omega_n^2$$

$$\underbrace{(k_0+k_1)}_{=0}s^2 + \underbrace{(2k_0\zeta\omega_n + k_2)}_{=0}s + \underbrace{k_0\omega_n^2}_{=1} = \omega_n^2$$

$$k_1 = -1 \qquad -2\zeta\omega_n = k_2$$

$$\boxed{\frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{1}{s} - \frac{s}{(s^2+2\zeta\omega_n s+\omega_n^2)} - \frac{2\zeta\omega_n}{(s^2+2\zeta\omega_n s+\omega_n^2)}}$$

$$\frac{1}{(s^2+2\zeta\omega_n s+\omega_n^2)} = \frac{1}{(s^2+2\zeta\omega_n s+\zeta^2\omega_n^2) - \zeta^2\omega_n^2 + \omega_n^2}$$

$$= \frac{1}{(s+\zeta\omega_n)^2 + (1-\zeta^2)\omega_n^2}$$

$$= \frac{1}{(s+\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2}$$

$$\boxed{= \frac{1}{s} - \left\{ \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} + \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} \right\}}$$

Underdamping $\zeta^2 < 1$ ②

$$\boxed{= \frac{1}{s} - \left\{ \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} + \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} \right\}}$$

$$\boxed{\begin{aligned} \frac{s}{s^2+k^2} &\Longleftrightarrow \cos(kt) \\ \frac{(s+a)}{(s+a)^2+k^2} &\Longleftrightarrow e^{-at}\cos(kt) \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{k}{s^2+k^2} &\Longleftrightarrow \sin(kt) \\ \frac{k}{(s+a)^2+k^2} &\Longleftrightarrow e^{-at}\sin(kt) \end{aligned}}$$

$$\left\{ \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} + \frac{\zeta\left(\sqrt{1-\zeta^2}\,\omega_n\right)\frac{1}{\sqrt{1-\zeta^2}}}{(s+\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} \right\}$$

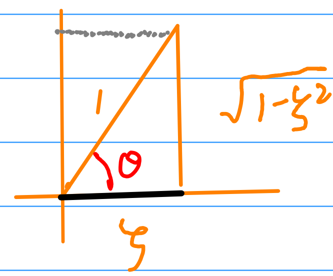
$$= e^{-\zeta\omega_n t}\left\{ \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right) \right\}$$

$$\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left\{ \sqrt{1-\zeta^2}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + \zeta \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right) \right\}$$

$$\boxed{\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left\{\left(\sqrt{1-\zeta^2}\right)\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right)+\left(\zeta\right)\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}}$$

① underdamping $\zeta^2<1$

$\cos\theta=\zeta$

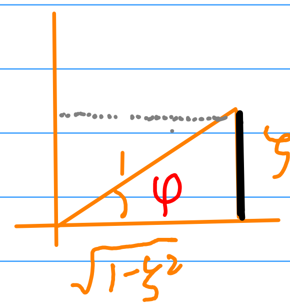


$$\boxed{\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left\{\sin\theta\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right)+\cos\theta\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}}$$

$\sin(\alpha+\beta)$

② underdamping $\zeta^2<1$

$\sin\varphi=\zeta$



$$\boxed{\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left\{\cos\varphi\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right)+\sin\varphi\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}}$$

$\cos(\alpha-\beta)$

$$\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \left\{ \sin\theta \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + \cos\theta \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right) \right\}$$

① $$= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \theta\right) \qquad \cos\theta = \zeta$$

② $$= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \varphi\right) \qquad \sin\varphi = \zeta$$

$$\theta + \varphi = \frac{\pi}{2}$$

Underdamping $\zeta^2 < 1$ ③

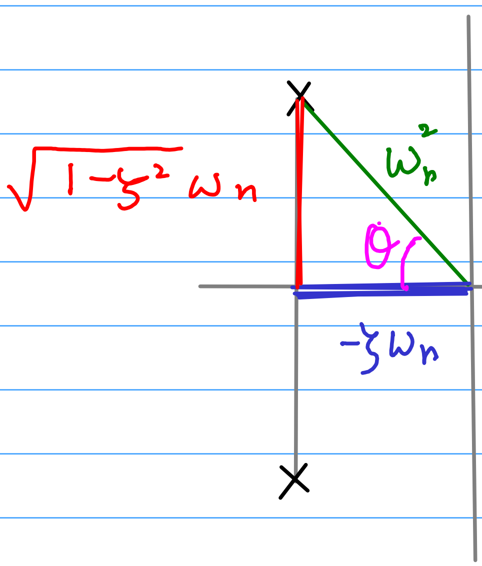
$$\frac{1}{s}G(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$= \frac{1}{s} - \left\{\frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2-(\sqrt{1-\zeta^2}\,\omega_n)^2} + \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2-(\sqrt{1-\zeta^2}\,\omega_n)^2}\right\}$$

$$\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left\{\sqrt{1-\zeta^2}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + \zeta\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

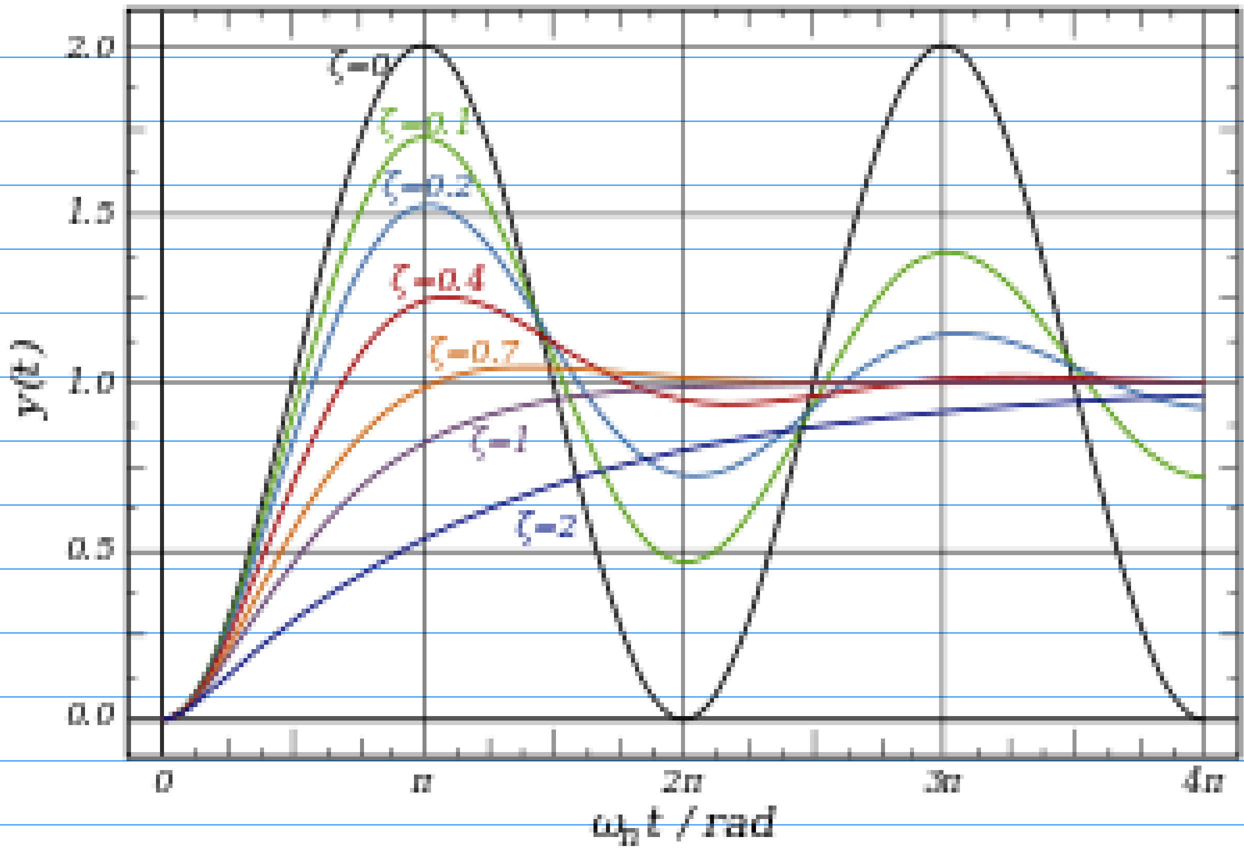
$$= \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+\theta\right) \qquad \cos\theta = \zeta$$

$$= \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t+\theta\right) \qquad \sin\varphi = \zeta$$



$$(-\zeta\omega_n)^2 + (\sqrt{1-\zeta^2}\,\omega_n)^2$$
$$= \zeta^2\omega_n^2 + (1-\zeta^2)\omega_n^2 = \omega_n^2$$

$$s_1 = -\zeta\omega_n + j\sqrt{1-\zeta^2}\,\omega_n$$
$$s_2 = -\zeta\omega_n - j\sqrt{1-\zeta^2}\,\omega_n$$



$$= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \theta\right) \qquad \cos\theta = \zeta$$

$$= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\sqrt{1-\zeta^2}\,\omega_n t + \theta\right) \qquad \sin\varphi = \zeta$$

$X\cos(\omega t) + Y\sin(\omega t)$

$\sqrt{X^2+Y^2}\cos(\omega t - \theta)$

$$
\begin{aligned}
&X\cos(\omega t) + Y\sin(\omega t) \\
&= \sqrt{X^2+Y^2}\left[\tfrac{X}{\sqrt{X^2+Y^2}}\cos(\omega t) + \tfrac{Y}{\sqrt{X^2+Y^2}}\sin(\omega t)\right] \\
&= \sqrt{X^2+Y^2}\left[\cos(\theta)\cos(\omega t) + \sin(\theta)\sin(\omega t)\right] \\
&= \sqrt{X^2+Y^2}\cos(\theta - \omega t) \\
&= \sqrt{X^2+Y^2}\cos(\omega t - \theta)
\end{aligned}
$$

$$
\boxed{\begin{aligned}
&X\cos(\omega t) + Y\sin(\omega t) \\
&= \sqrt{X^2+Y^2}\cos(\omega t - \theta)
\end{aligned}}
$$

$$\cos(\theta) = \frac{X}{\sqrt{X^2+Y^2}}$$

$$\sin(\theta) = \frac{Y}{\sqrt{X^2+Y^2}}$$

$X\cos(\omega t) - Y\sin(\omega t)$

$\sqrt{X^2+Y^2}\cos(\omega t + \theta)$