CLTI Impulse Response (4A)

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Impulse Response and Differential Equations

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_1 \frac{d\,y(t)}{dt} + a_0\, y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{d\,x(t)}{dt} + b_0\, x(t)$$

y(t) = h(t) : impulse response

when x(t) = δ(t) : forcing function

interval of existence / uniqueness

forcing function

	t < 0	t = 0	t > 0
x(t) =	0	δ(t)	0

	causal system	particular solution	homogeneous solution

impulse response

	t < 0	t = 0	t > 0
y(t) =	0	Aδ(t)	$y_h(t)$

only when N=M

Solutions of Differential Equations

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_1 \frac{d\,y(t)}{dt} + a_0\, y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1}\frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{d\,x(t)}{dt} + b_0\, x(t)$$

(t > 0)

Homogeneous solution

$$a_N \frac{d^N h(t)}{dt^N} + a_{N-1}\frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_1 \frac{d\,h(t)}{dt} + a_0\, h(t) = 0$$

$h(t) = y_h(t)u(t)$

$y_h(t)$ **homogeneous solution**
characteristic modes only

linear combination of all the derivatives of $h(t)$
must add to zero for any time $t \neq 0$

(t = 0)

Particular solution

$$a_N \frac{d^N h(t)}{dt^N} + a_{N-1}\frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_1 \frac{d\,h(t)}{dt} + a_0\, h(t) = b_M \frac{d^M \delta(t)}{dt^M} + b_{M-1}\frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_1 \frac{d\,\delta(t)}{dt} + b_0\, \delta(t)$$

$y_p(t)$ **particular solution**

linear combination of the **forcing function** and **all its unique derivatives**

linear combination of an **impulse** and **its unique derivatives** (the doublet, the triplet) : all these exist at time $t = 0$

Particular Solutions

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_1 \frac{d\,y(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{d\,x(t)}{dt} + b_0 x(t)$$

Method of Undetermined coefficients

Particular solution

$$a_N \frac{d^N h(t)}{dt^N} + a_{N-1} \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_1 \frac{d\,h(t)}{dt} + a_0 h(t) = \boxed{b_M \frac{d^M \delta(t)}{dt^M} + b_{M-1} \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_1 \frac{d\,\delta(t)}{dt} + b_0 \delta(t)}$$

h(t) is differentiated up to n times ⇐ All these derivatives must match ⇒ δ(t) is differentiated up to m times

Finding particular solution

We can determine whether h(t) can contain an impulse or its unique derivatives

$$\int_{0^-}^{0^+} h(t)\,dt = 0$$

when h(t) does not have an impulse or its derivatives

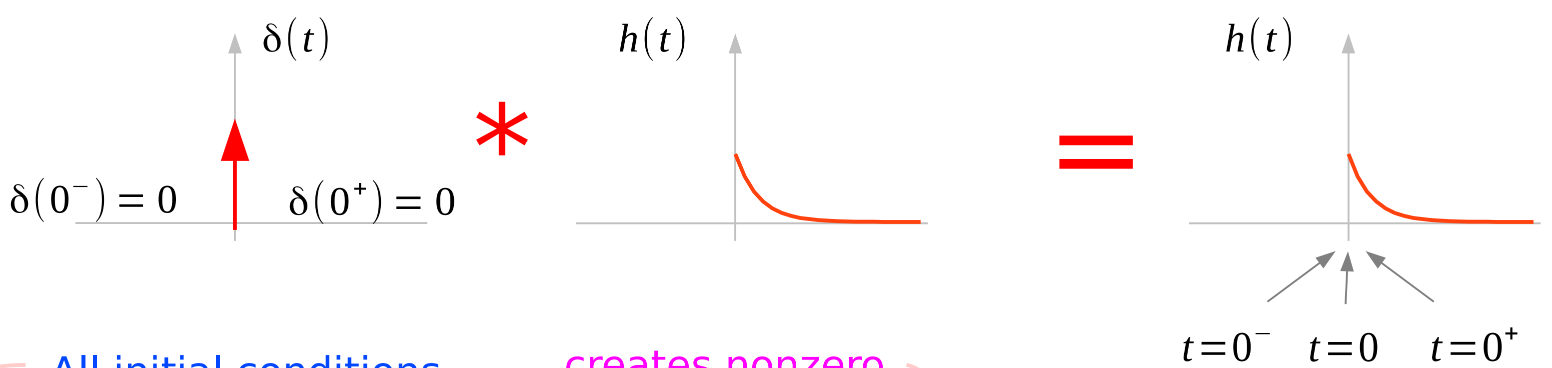
$$\int_{0^-}^{0^+} h(t)\,dt \neq 0$$

Otherwise

Impulse Response h(t)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,\delta(t)}{dt} + b_N \delta(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot h(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot \delta(t)$$



All initial conditions are zero at $t = 0^-$

creates nonzero initial condition at $t = 0^+$

all zero	$y(0^-) = y^{(1)}(0^-) = \cdots = y^{(N-1)}(0^-) = 0$
non-zero	$y(0^+),\ y^{(1)}(0^+),\ \cdots,\ y^{(N-1)}(0^+)$

$t \geq 0^+$ $(t \neq 0)$	h(t) = characteristic mode terms only
$t = 0$	h(t) can have at most an impulse $A_0 \delta(t)$

$$h(t) = b_0 \delta(t) + \textit{char mode terms} \quad t \geq 0$$

h(t) can have at most a δ(t)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^N \delta(t)}{dt^N} + b_1 \frac{d^{N-1}\delta(t)}{dt^{N-1}} + \cdots + b_{N-1}\frac{d\delta(t)}{dt} + b_N \delta(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) h(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\delta(t)$$

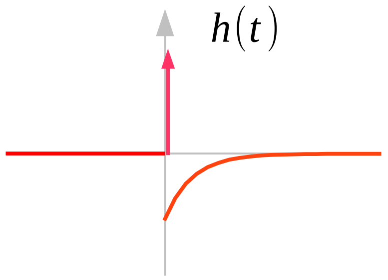
$M = N$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) h(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\delta(t)$$

If $\delta^{(1)}(t)$ is included in $h(t)$, then the highest order term

$D^N \delta^{(1)}(t) \rightarrow \delta^{(N+1)}(t)$ $\qquad$ $b_0 D^N \delta(t) \rightarrow \delta^{(N)}(t)$ $\qquad$ Contradiction!!!

h(t) cannot contain $\delta^{(i)}(t)$ at all ➡ *h(t)* can contain at most δ(t) $\quad M \le N$



$$h(t) = b_0 \delta(t) + \textit{char mode terms} \quad t \ge 0$$

Finding Impulse Response from Diff Equations

- Impulse Matching Method
- Simplified Impulse Matching Method
- Green's Function

- Impulse Matching

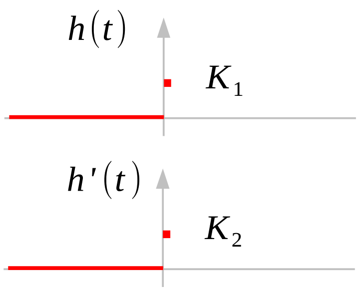
Impulse Matching (N>M) (1)

$$(D^2 + 5D + 6)\,y(t) = (D + 1)\,x(t)$$

$$h''(t) + 5h'(t) + 6h(t) = \delta'(t) + \delta(t)$$

$$(m^2 + 5m + 6) = 0 \qquad m = -2,\ -3$$

$$h(t) = (c_1 e^{-2t} + c_2 e^{-3t})\,u(t)$$



$$h(0^+) = K_1 \Rightarrow h'(0) = K_1\delta(t)$$

$$h'(0^+) = K_2 \Rightarrow h''(0) = K_2\delta(t) + K_1\delta'(t)$$

$$h''(0) + 5h'(0) + 6h(0)$$

$$= [K_2\delta(t) + K_1\delta'(t)] + 5[K_1\delta(t)] + 6[const]$$

$$= K_1\delta'(t) + (5K_1 + K_2)\delta(t) + 6[const]$$

$$= \delta'(t) + \delta(t)$$

$$K_1 = 1 \Rightarrow h(0^+) = 1$$

$$K_2 = -4 \Rightarrow h'(0^+) = -4$$

Impulse Matching (N>M) (2)

$$(D^2 + 5D + 6)\,y(t) = (D+1)\,x(t)$$

$$h''(t) + 5h'(t) + 6h(t) = \delta'(t) + \delta(t)$$

$$(m^2 + 5m + 6) = 0 \qquad m = -2,\ -3$$

$$h(t) = (\boxed{c_1} e^{-2t} + \boxed{c_2} e^{-3t})\,u(t)$$

$$h(t) = (c_1 e^{-2t} + c_2 e^{-3t})\,u(t)$$

$$h'(t) = (-2c_1 e^{-2t} - 3c_2 e^{-3t})\,u(t) + (c_1 e^{-2t} + c_2 e^{-3t})\,\delta(t)$$

$$= (-2c_1 e^{-2t} - 3c_2 e^{-3t})\,u(t) + (c_1 + c_2)\,\delta(t)$$

$$h(0^+) = (c_1 + c_2)$$

$$h'(0^+) = (-2c_1 - 3c_2)$$

$$h(0^+) = 1$$

$$h'(0^+) = -4$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

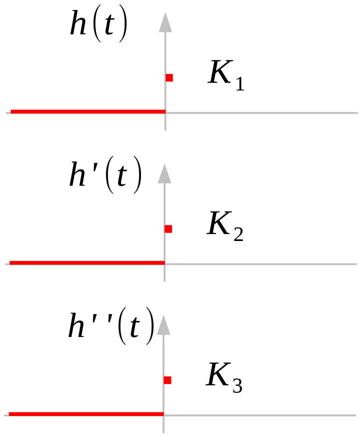
$$\boxed{c_1} = \frac{\begin{vmatrix} 1 & 1 \\ -4 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix}} = \frac{+1}{-1} = -1 \qquad \boxed{c_2} = \frac{\begin{vmatrix} 1 & 1 \\ -2 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix}} = \frac{-2}{-1} = +2$$

$$h(t) = (\boxed{-} e^{-2t} + \boxed{2} e^{-3t})\,u(t)$$

Impulse Matching (N>M) (3)

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot y(t) = \left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) \cdot x(t)$$

$$\left(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N\right) = 0 \quad \Rightarrow \quad h(t) = \left(\sum_{k=1}^{N} c_k e^{\lambda_k t}\right) u(t)$$



$$h(0^+) = K_1 \qquad h(0) = K_1 \delta(t)$$

$$h'(0^+) = K_2 \qquad h'(0) = K_2 \delta(t) + K_1 \delta'(t)$$

$$h''(0^+) = K_3 \qquad h''(0) = K_3 \delta''(t) + K_2 \delta'(t) + K_1 \delta''(t)$$

$$h(t) = \left(\sum_{k=1}^{N} c_k e^{\lambda_k t}\right) u(t)$$

Impulse matching

$$h(0^+) = K_1$$
$$h'(0^+) = K_2$$
$$h''(0^+) = K_3$$

$\Rightarrow$

$$h(0)$$
$$h'(0)$$
$$h''(0)$$

$\Rightarrow$

Find coeff's

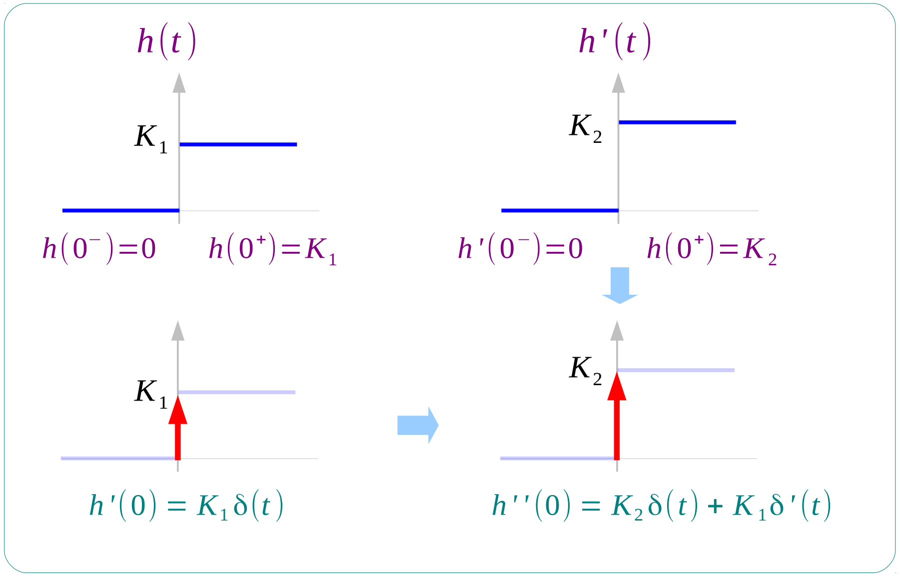
$$c_1$$
$$c_2$$
$$c_3$$

Impulse Matching (N=M) (1)

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_2 y(t) = b_0 \frac{d^2 x(t)}{dt^2} + b_1 \frac{dx(t)}{dt} + b_2 x(t)$$

$$\begin{cases} h''(t) + a_1 h'(t) + a_2 h(t) = b_0 \delta''(t) + b_1 \delta'(t) + b_2 \delta(t) \\ h(0^+) = K_1 \\ h'(0^+) = K_2 \end{cases}$$

IVP (Initial Value Problem)



$$m^2 + a_1 m + a_2 = 0 \quad \Rightarrow \quad m = m_1,\ m_2$$

$$h(t) = b_0\delta(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x}) u(t)$$

$$h'(t) = b_0\delta'(t) + (c_1 m_1 e^{m_1 x} + c_2 m_2 e^{m_2 x}) u(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x}) \delta(t)$$

$$h''(t) = b_0\delta''(t) + (c_1 m_1^2 e^{m_1 x} + c_2 m_2^2 e^{m_2 x}) u(t) + (c_1 m_1 e^{m_1 x} + c_2 m_2 e^{m_2 x})(\delta(t) + \delta(t)) + (c_1 e^{m_1 x} + c_2 e^{m_2 x}) \delta'(t)$$

Impulse Matching (N=M) (2)

we can determine $K_{1,}\ K_2$

the initially at rest conditions $h(0^-) = h'(0^-) = 0$

and the property of step and delta $\frac{d}{dt}u(t) = \delta(t)$

$$h(t) = b_0\delta(t) + \left(c_1 e^{m_1 x} + c_2 e^{m_2 x}\right)u(t)$$

don't have to consider the term $b_0\delta(t)$

$$h''(0) + a_1 h'(0) + a_2 h(0) = b_0\delta''(t) + b_1\delta'(t) + b_2\delta(t)$$

thus h(0) has a finite jump

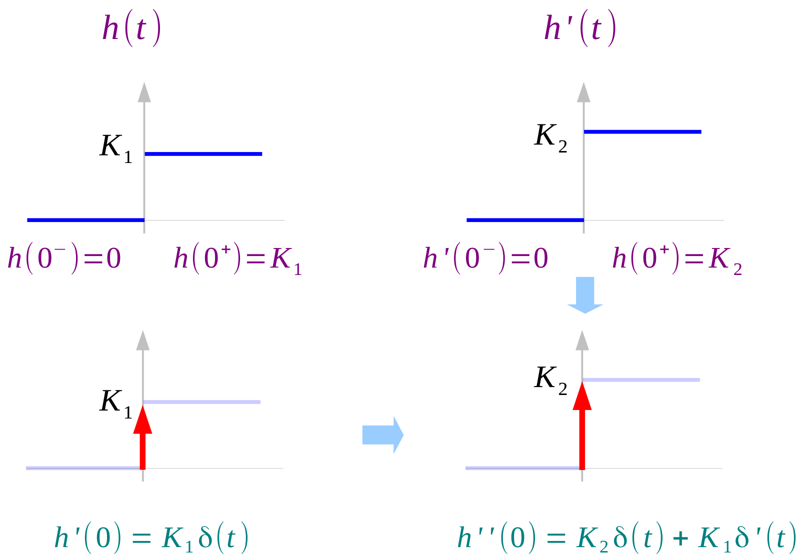
$$h''(0) + a_1 h'(0) = b_1\delta'(t) + b_2\delta(t)$$

$$K_2\delta(t) + K_1\delta'(t) + a_1 K_1\delta(t) = b_1\delta'(t) + b_2\delta(t)$$

$$K_1\delta'(t) + (a_1 K_1 + K_2)\delta(t) = b_1\delta'(t) + b_2\delta(t)$$

$$\begin{cases} K_1 = b_1 \\ a_1 K_1 + K_2 = b_2 \end{cases}$$

K_1
K_2



Impulse Matching (N=M) (3)

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_2 y(t) = b_0 \frac{d^2 x(t)}{dt^2} + b_1 \frac{dx(t)}{dt} + b_2 x(t)$$

$$\begin{cases} h''(t) + a_1 h'(t) + a_2 h(t) = b_0 \delta''(t) + b_1 \delta'(t) + b_2 \delta(t) \\ h(0^+) = K_1 \\ h'(0^+) = K_2 \end{cases}$$

IVP (Initial Value Problem)

$$m^2 + a_1 m + a_2 = 0 \quad \Rightarrow \quad m = m_1,\ m_2$$

$$h(t) = b_0 \delta(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x}) u(t)$$

$$h(t) = b_0 \delta(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x}) u(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x}) \delta(t)$$

$$h(0^+) = (c_1 + c_2) \qquad h'(0^+) = (c_1 m_1 + c_2 m_2)$$

$$h(0^+) = K_1 \Rightarrow c_1 \qquad h'(0^+) = K_2 \Rightarrow c_2$$

$$h(t) = b_0 \delta(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x}) u(t)$$

Impulse Matching Example

$$\frac{d^2 y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + 1x(t)$$

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$

$$(\lambda^2 + 5\lambda + 6) = (\lambda+2)(\lambda+3) = 0$$

$$h(t) = (c_1 e^{-2t} + c_2 e^{-3t}) \cdot u(t)$$

$$\dot{h}(t) = (-2c_1 e^{-2t} - 3c_2 e^{-3t}) \cdot u(t) + (c_1 e^{-2t} + c_2 e^{-3t}) \cdot \delta(t)$$

$$\begin{cases} h(0^+) = (c_1 e^0 + c_2 e^0) \cdot u(t) \\ \dot{h}(0^+) = (-2c_1 e^0 - 3c_2 e^0) \cdot u(t) \end{cases}$$

$$\begin{cases} h(0^+) = c_1 + c_2 = +1 = K_1 \\ \dot{h}(0^+) = -2c_1 - 3c_2 = -4 = K_2 \end{cases}$$

$$c_1 = -1$$
$$c_2 = +2$$

$$h(t) = (-e^{-2t} + 2e^{-3t}) \cdot u(t)$$

$$\ddot{h}(t) + 5\dot{h}(t) + 6h(t) = \dot{\delta}(t) + 1\delta(t)$$

$$\begin{cases} h(0^-) = 0 \\ \dot{h}(0^-) = 0 \end{cases} \qquad \begin{cases} h(0^+) = K_1 \\ \dot{h}(0^+) = K_2 \end{cases}$$

$$\begin{cases} \dot{h}(0) = K_1 \delta(t) \\ \ddot{h}(0) = K_1 \dot{\delta}(t) + K_2 \delta(t) \end{cases}$$

$$\ddot{h}(0) + 5\dot{h}(0) + 6h(0) = \dot{\delta}(t) + 1\delta(t)$$

$$(K_1 \dot{\delta}(t) + K_2 \delta(t)) + 5K_1 \delta(t) + 6h(0)$$

$$K_1 \dot{\delta}(t) + (5K_1 + K_2)\delta(t) + \cdots = \dot{\delta}(t) + 1\delta(t)$$

$$K_1 = 1$$
$$5K_1 + K_2 = 1$$

$$\begin{cases} K_1 = 1 = h(0^+) \\ K_2 = -4 = \dot{h}(0^+) \end{cases}$$

- Simplified Impulse Matching

Simplified Impulse Matching (1)

$$\frac{d^2 y(t)}{dt^2} + \mathbf{a_1} \frac{d\,y(t)}{dt} + \mathbf{a_2}\, y(t) = \mathbf{b_0} \frac{d^2 x(t)}{dt^2} + \mathbf{b_1} \frac{d\,x(t)}{dt} + \mathbf{b_2}\, x(t)$$

$$\begin{cases} h''(t) + \mathbf{a_1} h'(t) + \mathbf{a_2} h(t) = \boxed{\mathbf{b_0} \delta''(t)} + \boxed{\mathbf{b_1} \delta'(t)} + \boxed{\mathbf{b_2} \delta(t)} \\ h(0^+) = K_1 \\ h'(0^+) = K_2 \end{cases}$$

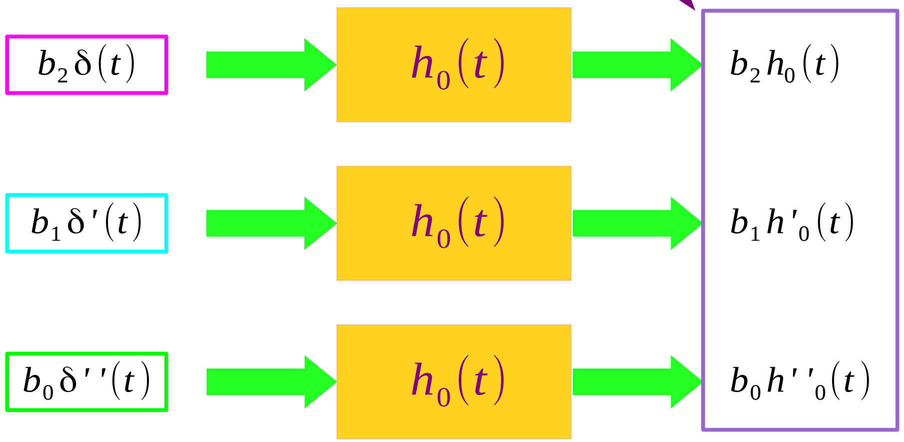
IVP (Initial Value Problem)

$$\begin{cases} h_0''(t) + \mathbf{a_1} h_0'(t) + \mathbf{a_2} h_0(t) = \delta(t) \\ h(0^+) = 0 \\ h'(0^+) = 1 \quad \rightarrow \quad h''(0^+) = \delta(t) \end{cases}$$

Simpler IVP

$$P(D) = (b_0 D^2 + b_1 D + b_2)$$

$$h(t) = b_0 \delta(t) + [P(D)\, h_0(t)]$$



Simplified Impulse Matching (2)

$$\begin{cases} h_0''(t) + \mathbf{a_1} h_0'(t) + \mathbf{a_2} h_0(t) = \delta(t) \\ h_0(0^+) = 0 \\ h_0'(0^+) = 1 \end{cases}$$

Simpler IVP

$y_n(t)$ Only a linear combination of characteristic modes Excluding u(t)

$$m^2 + a_1 m + a_2 = 0 \quad \Rightarrow \quad m = m_1,\ m_2$$

$$h_0(t) = y_n(t)\, u(t) \qquad \text{causality}$$

$$h_0(t) = (c_1 e^{m_1 x} + c_2 e^{m_2 x})\, u(t)$$

$$h_0'(t) = (c_1 m_1 e^{m_1 x} + c_2 m_2 e^{m_2 x})\, u(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x})\, \delta(t)$$

$$y_n(t) = (c_1 e^{m_1 x} + c_2 e^{m_2 x})$$

$$y_n'(t) = (c_1 m_1 e^{m_1 x} + c_2 m_2 e^{m_2 x})$$

$$\begin{cases} h_0(0^+) = 0 = c_1 + c_2 \\ h_0'(0^+) = 1 = c_1 m_1 + c_2 m_2 \end{cases} \qquad c_1, \quad c_2$$

$$\begin{cases} y_n(0^+) = 0 = c_1 + c_2 \\ y_n'(0^+) = 1 = c_1 m_1 + c_2 m_2 \end{cases}$$

$$(b_0 D^2 + b_1 D + b_2)\, [(\mathbf{c_1} e^{m_1 x} + \mathbf{c_2} e^{m_2 x})\, u(t)]$$

$$[(b_0 D^2 + b_1 D + b_2)\, (\mathbf{c_1} e^{m_1 x} + \mathbf{c_2} e^{m_2 x})]\, u(t)$$

$$h(t) = b_0 \delta(t) + [P(D) h_0(t)]$$

$$= b_0 \delta(t) + [P(D) [y_n(t)\, u(t)]]$$

$$\Longleftrightarrow$$

$$h(t) = b_0 \delta(t) + [P(D) y_n(t)]\, u(t)$$

Simplified Impulse Matching Example

$$\frac{d^2 y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + 1x(t)$$

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$

$$(\lambda^2 + 5\lambda + 6) = (\lambda+2)(\lambda+3) = 0$$

$$\begin{cases} y_n(t) = (c_1 e^{-2t} + c_2 e^{-3t}) \\ \dot{y}_n(t) = (-2c_1 e^{-2t} - 3c_2 e^{-3t}) \end{cases} \quad \begin{cases} y_n(0) = 0 \\ \dot{y}_n(0) = 1 \end{cases}$$

$$\begin{cases} y_n(0) = (c_1 e^0 + c_2 e^0) = 0 \\ \dot{y}_n(0) = (-2c_1 e^0 - 3c_2 e^0) = 1 \end{cases}$$

$$\begin{cases} y_n(0) = c_1 + c_2 = 0 \\ \dot{y}_n(0) = -2c_1 - 3c_2 = 1 \end{cases}$$

$$c_1 = +1$$
$$c_2 = -1$$

$$y_n(t) = (+e^{-2t} - e^{-3t})$$

$$h(t) = b_0\delta(t) + [P(D)y_n(t)]u(t)$$

$$(D+1)y_n(t) = (-2e^{-2t} + 3e^{-3t}) + (e^{-2t} - e^{-3t})$$
$$= (-e^{-2t} + 2e^{-3t})$$

$$h(t) = (-e^{-2t} + 2e^{-3t})\cdot u(t)$$

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11/18/14

- Integrating both sides successively

First Order ODE (n > m) – Impulse Response

$$y'(t) + a\,y(t) = x(t)$$

$$h'(t) + a\,h(t) = 0$$ Homogeneous equation

$$h'(t) + a\,h(t) = \delta(t)$$ Particular equation

$$\int_{0^-}^{0^+} h'(t)dt + a\int_{0^-}^{0^+} h(t)dt = \int_{0^-}^{0^+} \delta(t)dt$$

$$\underbrace{h(0^+) - h(0^-)}_{\to 0} + a\int_{0^-}^{0^+} h(t)dt = \underbrace{\int_{0^-}^{0^+} \delta(t)dt}_{\to 1}$$

$h(t) = 0$	$(t < 0)$	causal system
$h(t) = K e^{-at}$	$(t > 0)$	homogeneous solution
?	$(t = 0)$	particular solution

$$\int_{0^-}^{0^+} h(t)dt = 0$$ when h(t) does not have an impulse or its derivatives

$$\int_{0^-}^{0^+} h(t)dt \neq 0$$ Otherwise

if $\delta(t)$ is included in $h(t)$

$\delta'(t)$ must present in $h'(t)$

But RHS has no doublet or triplet

➡ no $\delta(t)$ included in $h(t)$

➡ $h(0^+) = 1$ $\quad h(0^+) = K e^{-a0^+} = K = 1$

$$h(t) = e^{-at} \quad (t > 0)$$

$$h(t) = e^{-at} u(t)$$

First Order ODE (n > m) - Verification

$$y'(t) + a\,y(t) = x(t)$$

$$h'(t) + a\,h(t) = \delta(t)$$

$$\begin{cases} h'(t) + a\,h(t) = 0 & (t < 0) \\ h'(t) + a\,h(t) = 0 & (t > 0) \\ h'(t) + a\,h(t) = \delta(t) & (t = 0) \end{cases}$$

$h(t) = e^{-at}\,u(t)$ satisfies

$h'(t) + a\,h(t) = \delta(t)$ for all t

$$h(t) = e^{-at}\,u(t)$$

$$h'(t) = -a\,e^{-at}\,u(t) + a\,e^{-at}\,\delta(t)$$

$$h'(t) + a\,h(t)$$

$$= -a\,e^{-at}\,u(t) + e^{-at}\,\delta(t) + a\,e^{-at}\,u(t)$$

$$= e^{-at}\,\delta(t) = e^{0}\,\delta(t)$$

$$= \delta(t)$$

$h(t)$ has exactly the right size discontinuity at time t=0

$$h(t) = 0 \quad (t < 0)$$

$$h(0^{+}) = 1 \quad (t > 0)$$

First Order ODE (n = m) – Impulse Response

$$h'(t) + a\,h(t) \;=\; x'(t)$$

$$u_1(t) \;=\; \frac{d}{dt}\delta(t)$$

$$h'(t) + a\,h(t) \;=\; 0$$

Homogeneous equation

$$h'(t) + a\,h(t) \;=\; \delta'(t) \;=\; u_1(t)$$

Particular equation

$h(t) = 0$	$(t < 0)$	causal system
$h(t) = K\,e^{-at}$	$(t > 0)$	homogeneous solution
?	$(t = 0)$	particular solution

The highest derivative is the same for the excitation and response (n = m)

➡ $\delta(t)$ included in $h(t)$

$$h(t) \;=\; Ke^{-at}u(t) + K_\delta\,\delta(t)$$

yh yp

Determining Coefficients

$$y'(t) + a\,y(t) = x'(t)$$

$$h'(t) + a\,h(t) = \delta'(t) = u_1(t)$$

$$\int_{0^-}^{0^+} h'(t)\,dt + a\int_{0^-}^{0^+} h(t)\,dt = \int_{0^-}^{0^+} \delta'(t)\,dt$$

$$\int_{-\infty}^{t} h'(t)\,dt + a\int_{-\infty}^{t} h(t)\,dt = \int_{-\infty}^{t} \delta'(t)\,dt$$

$$h(t) + a\int_{-\infty}^{t} h(t)\,dt = \delta(t)$$

$$\int_{0^-}^{0^+} h(t)\,dt + a\int_{0^-}^{0^+}\int_{-\infty}^{t} h(\lambda)\,d\lambda\,dt = \int_{0^-}^{0^+} \delta(t)\,dt$$

$$\underline{h(0^+)} - \underset{\to 0}{h(0^-)} + \underline{a\int_{0^-}^{0^+} h(t)\,dt} = \underset{\to 0}{\delta(0^+)} - \underset{\to 0}{\delta(0^-)}$$

$$\int_{0^-}^{0^+} h(t)\,dt + a\left\{\int_{0^-}^{0^+} dt\right\}\left\{\int_{-\infty}^{t} h(\lambda)\,d\lambda\right\} = \int_{0^-}^{0^+} \delta(t)\,dt$$

$$K \qquad + a K_\delta \quad = 0$$

$$K_\delta = u(0^+) - u(0^-) = 1 - 0$$

$$K = -a K_\delta = -a$$

$$h(t) = \underbrace{K e^{-at}u(t)}_{y_h} + \underbrace{K_\delta \delta(t)}_{y_p}$$

$$h(t) = \delta(t) - a e^{-at} u(t)$$

First Order ODE (n = m) - Verification

$$h'(t) + a\,h(t) \;=\; x'(t)$$

$$h'(t) + a\,h(t) \;=\; \delta'(t) \;=\; u_1(t)$$

$$\begin{cases} h'(t) + a\,h(t) = 0 & (t < 0) \\ h'(t) + a\,h(t) = 0 & (t > 0) \\ h'(t) + a\,h(t) = \delta'(t) & (t = 0) \end{cases}$$

$$h(t) \;=\; \delta(t) - a e^{-at} u(t)$$ satisfies

$$h'(t) + a\,h(t) = \delta'(t)$$ for all t

$$h'(t) \;=\; \delta'(t) + a^2 e^{-at} u(t) - a e^{-at} \delta(t)$$

$$\;=\; \delta'(t) + a^2 e^{-at} u(t) - a\,\delta(t)$$

$$a\,h(t) \;=\; a\,\delta(t) - a^2 e^{-at} u(t)$$

$$h'(t) + a\,h(t)$$

$$= \delta'(t) + a^2 e^{-at} u(t) - a\,\delta(t)$$

$$+ \; a\,\delta(t) - a^2 e^{-at} u(t)$$

$$= \delta'(t)$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] B.P. Lathi, Linear Systems and Signals (2nd Ed)

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