Laurent Series and z-Transform - Geometric Series Applications A

20200113 Mon

Combinations of a and z -- common ratio

$$
\begin{array}{ccc}
a^{n} & \qquad & a^{-n} \\
\boxed{a\ z} & & \boxed{a^{\boxed{-1}}\ z} \\
\updownarrow \boxed{-1} \ \updownarrow \boxed{-1} & & \updownarrow \boxed{-1} \ \updownarrow \boxed{-1} \\
\boxed{a^{\boxed{-1}}\ z^{\boxed{-1}}} & & \boxed{a\ z^{\boxed{-1}}} \\
a^{n} & & a^{-n}
\end{array}
$$

the same formula,
different representations

$$a\,z \qquad \frac{1}{1-az} = \frac{a^{-1}z^{-1}}{a^{-1}z^{-1}-1}$$

$$a^{\boxed{-1}}z^{\boxed{-1}} \qquad \frac{1}{1-a^{-1}z^{-1}} = \frac{az}{az-1}$$

$$a^{\boxed{-1}}\,z \qquad \frac{1}{1-a^{-1}z} = \frac{az^{-1}}{az^{-1}-1}$$

$$a\,z^{\boxed{-1}} \qquad \frac{1}{1-az^{-1}} = \frac{a^{-1}z}{a^{-1}z-1}$$

the same formula
with different ROCs

different Geometric Series

$$a\,z$$

$$\frac{1}{1-az} \quad |z|<a^{-1} \qquad\qquad -\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1}$$

$$a^0z^0 + a^1z^1 + a^2z^2 + \cdots \qquad\qquad a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots$$

$$a^{\boxed{-1}}z^{\boxed{-1}}$$

$$\frac{1}{1-a^{-1}z^{-1}} \quad |z|>a^{-1} \qquad\qquad -\frac{az}{1-az} \quad |z|<a^{-1}$$

$$a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots \qquad\qquad a^1z^1 + a^2z^2 + a^3z^3 + \cdots$$

$$a^{\boxed{-1}}\,z$$

$$\frac{1}{1-a^{-1}z} \quad |z|<a \qquad\qquad -\frac{az^{-1}}{1-az^{-1}} \quad |z|>a$$

$$a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots \qquad\qquad a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$$

$$a\,z^{\boxed{-1}}$$

$$\frac{1}{1-az^{-1}} \quad |z|>a \qquad\qquad -\frac{a^{-1}z}{1-a^{-1}z} \quad |z|<a$$

$$a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots \qquad\qquad a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \cdots$$

geometric series
starting with
a unit term

geometric series
starting with
a non-unit term

the same formula with different ROCs

different Geometric Series -- Shifted Relationship

$a \; z$	$\frac{1}{1-az}$ $\quad |z|<a^{-1}$ $a^0z^0+a^1z^1+a^2z^2+\cdots$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z|>a^{-1}$ $a^{-1}z^{-1}+a^{-2}z^{-2}+a^{-3}z^{-3}+\cdots$	left shifted, inverted
$a^{\boxed{-1}} \; z^{\boxed{-1}}$	$\frac{1}{1-a^{-1}z^{-1}}$ $\quad |z|>a^{-1}$ $a^0z^0+a^{-1}z^{-1}+a^{-2}z^{-2}+\cdots$	$-\frac{az}{1-az}$ $\quad |z|<a^{-1}$ $a^1z^1+a^2z^2+a^3z^3+\cdots$	right shifted, inverted
$a^{\boxed{-1}} \; z$	$\frac{1}{1-a^{-1}z}$ $\quad |z|<a$ $a^0z^0+a^{-1}z^1+a^{-2}z^2+\cdots$	$-\frac{az^{-1}}{1-az^{-1}}$ $\quad |z|>a$ $a^1z^{-1}+a^2z^{-2}+a^3z^{-3}+\cdots$	left shifted, inverted
$a \; z^{\boxed{-1}}$	$\frac{1}{1-az^{-1}}$ $\quad |z|>a$ $a^0z^0+a^1z^{-1}+a^2z^{-2}+\cdots$	$-\frac{a^{-1}z}{1-a^{-1}z}$ $\quad |z|<a$ $a^{-1}z^1+a^{-2}z^2+a^{-3}z^3+\cdots$	right shifted, inverted
	geometric series starting with a unit term	geometric series starting with a non-unit term	

the same formula with different ROCs

different Geometric Series -- Complementary Relation

* inverted relation is ignored

$a z$	causal: $\dfrac{1}{1-az}$ $\quad|z|<a^{-1}$	anti-causal: $\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad|z|>a^{-1}$ — left shifted,
	$a^0z^0 + a^1z^1 + a^2z^2 + \cdots$	$a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots$
$a^{-1}z^{-1}$	anti-causal: $\dfrac{1}{1-a^{-1}z^{-1}}$ $\quad|z|>a^{-1}$	causal: $\dfrac{az}{1-az}$ $\quad|z|<a^{-1}$ — right shifted,
	$a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots$	$a^1z^1 + a^2z^2 + a^3z^3 + \cdots$
$a^{-1}z$	causal: $\dfrac{1}{1-a^{-1}z}$ $\quad|z|<a$	anti-causal: $\dfrac{az^{-1}}{1-az^{-1}}$ $\quad|z|>a$ — left shifted,
	$a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots$	$a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$
$a z^{-1}$	anti-causal: $\dfrac{1}{1-az^{-1}}$ $\quad|z|>a$	causal: $-\dfrac{a^{-1}z}{1-a^{-1}z}$ $\quad|z|<a$ — right shifted,
	$a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots$	$a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \cdots$
	geometric series starting with a unit term	geometric series starting with a common ratio term

a^n $\quad n = 0, 1, 2, \cdots$

right shifted, inverted $\quad n = 1, 2, 3, \cdots$

$\boxed{a\,z}$ $\quad |z| < a^{-1}$

$\updownarrow {-1} \quad \updownarrow {-1}$

$\boxed{a^{-1}\,z^{-1}}$ $\quad |z| > a^{-1}$

a^n $\quad n = 0, -1, -2, \cdots$

left shifted, inverted $\quad n = -1, -2, -3, \cdots$

a^{-n} $\quad n = 0, 1, 2, \cdots$

$n = 1, 2, 3, \cdots$

$\boxed{a^{-1}\,z}$ $\quad |z| < a$

$\updownarrow {-1} \quad \updownarrow {-1}$

$\boxed{a\,z^{-1}}$ $\quad |z| > a$

a^{-n} $\quad n = 0, -1, -2, \cdots$

$n = -1, -2, -3, \cdots$

All combinations of sequences

$a\,z$

$$\frac{1}{1-az} \quad |z| < a^{-1}$$

$a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots$

starting with a common ratio term

$$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

$a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots$

left shifted, inverted

$a^{\boxed{-1}}\,z^{\boxed{-1}}$

$$\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

$a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \dots$

starting with a common ratio term

$$-\frac{az}{1-az} \quad |z| < a^{-1}$$

$a^1 z^1 + a^2 z^2 + a^3 z^3 + \dots$

right shifted, inverted

$a^{\boxed{-1}}\,z$

$$\frac{1}{1-a^{-1}z} \quad |z| < a$$

$a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \dots$

starting with a common ratio term

$$-\frac{az^{-1}}{1-az^{-1}} \quad |z| > a$$

$a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \dots$

left shifted, inverted

$a\,z^{\boxed{-1}}$

$$\frac{1}{1-az^{-1}} \quad |z| > a$$

$a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots$

starting with a common ratio term

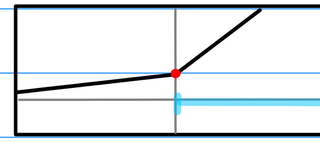
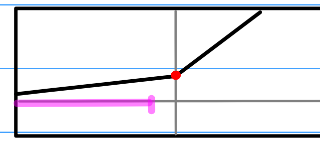
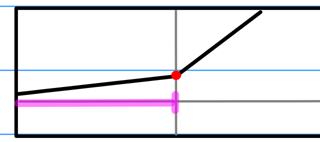
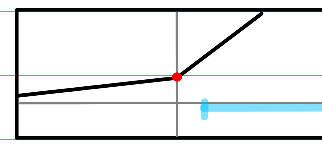
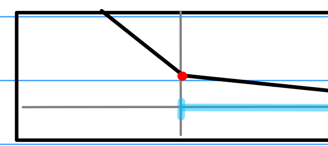
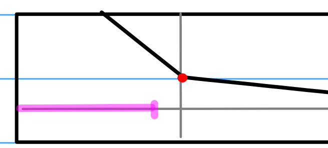
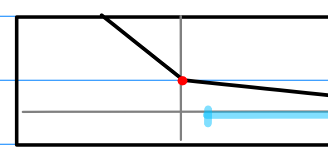
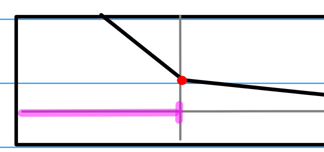
$$-\frac{a^{-1}z}{1-a^{-1}z} \quad |z| < a$$

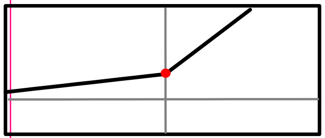
$a^{-1} z^1 + a^{-2} z^2 + a^{-3} z^3 + \dots$

right shifted, inverted

Shifted, Inverted Relation
Complementary, Inverted Relation

* inverted relation is ignored

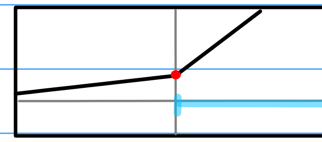
$a\,z$	$\frac{1}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^0z^0 + a^1z^1 + a^2z^2 + \cdots$ a^n	
starting with a common ratio term	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots$ a^n	
$a^{\boxed{-1}}z^{\boxed{-1}}$	$\frac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots$ a^n	
starting with a common ratio term	$\frac{az}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^1z^1 + a^2z^2 + a^3z^3 + \cdots$ a^n	
$a^{\boxed{-1}}z$	$\frac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$	$a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots$ a^{-n}	
starting with a common ratio term	$\frac{az^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$ a^{-n}	
$a\,z^{\boxed{-1}}$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > a$	$a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots$ a^{-n}	
starting with a common ratio term	$\frac{a^{-1}z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \cdots$ a^{-n}	

a^n 

Geometric Series Combinations (1)

$\boxed{az}$

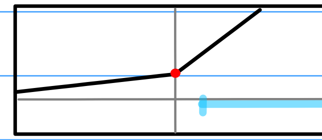
$\dfrac{1}{1-az}$ $|z| < a^{-1}$



a^n

a^n $n = 0, 1, 2, \cdots$

right shifted $\dfrac{az}{1-az}$ $|z| < a^{-1}$

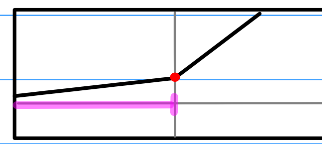


a^n

a^n $n = 1, 2, 3, \cdots$

$\boxed{a^{\boxed{-1}} z^{\boxed{-1}}}$

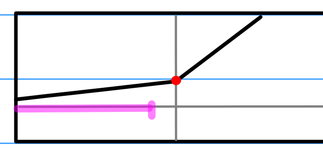
$\dfrac{1}{1-a^{-1}z^{-1}}$ $|z| > a^{-1}$



a^n

a^n $n = 0, -1, -2, \cdots$

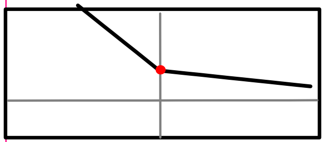
left shifted $\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $|z| > a^{-1}$



a^n

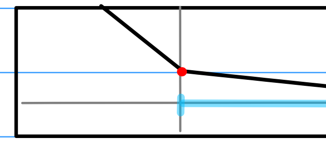
a^n $n = -1, -2, -3, \cdots$

* inverted relation is ignored

a^{-n}  Geometric Series Combinations (2)

$\boxed{a^{\boxed{-1}} z}$

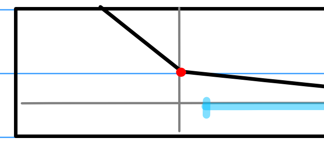
$\boxed{\dfrac{1}{1-a^{-1}z} \quad |z|<a}$



a^{-n}

$a^{-n} \quad n = 0, 1, 2, \cdots$

right shifted $\boxed{\dfrac{a^{-1}z}{1-a^{-1}z} \quad |z|<a}$

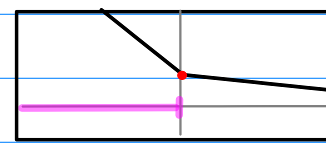


a^{-n}

$a^{-n} \quad n = 1, 2, 3, \cdots$

$\boxed{a\, z^{\boxed{-1}}}$

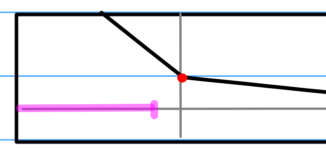
$\boxed{\dfrac{1}{1-a z^{-1}} \quad |z|>a}$



a^{-n}

$a^{-n} \quad n = 0, -1, -2, \cdots$

left shifted $\boxed{\dfrac{a z^{-1}}{1-a z^{-1}} \quad |z|>a}$

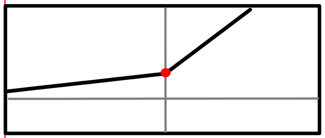


a^{-n}

$a^{-n} \quad n = -1, -2, -3, \cdots$

* inverted relation is ignored

a^n



*a SHL.Seq

/a SHR.Seq

Shifting a sequence

SHL.Seq

$$\boxed{\frac{1}{1-az} \quad |z|<a^{-1}} \xrightarrow{*a} \boxed{\frac{a}{1-az} \quad |z|<a^{-1}}$$

a^n $n = 0, 1, 2, \cdots$ $\qquad$ a^{n+1} $n = 0, 1, 2, \cdots$

SHR.Seq

$$\boxed{\frac{az}{1-az} \quad |z|<a^{-1}} \xrightarrow{/a} \boxed{\frac{z}{1-az} \quad |z|<a^{-1}}$$

a^n $n = 1, 2, 3, \cdots$ $\qquad$ a^{n-1} $n = 1, 2, 3, \cdots$

SHR.Seq

$$\boxed{\frac{1}{1-a^{-1}z^{-1}} \quad |z|>a^{-1}} \xrightarrow{/a} \boxed{\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1}}$$

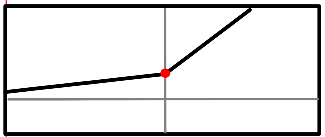
a^n $n = 0, -1, -2, \cdots$ $\qquad$ a^{n-1} $n = 0, -1, -2, \cdots$

SHL.Seq

$$\boxed{\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1}} \xrightarrow{*a} \boxed{\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1}}$$

a^n $n = -1, -2, -3, \cdots$ $\qquad$ a^{n+1} $n = -1, -2, -3, \cdots$

a^n



/z SHL.Seq, SHL.Rng

*z SHR.Seq, SHR.Rng

Shifting a sequence

SHL.Seq, SHL.Rng

$\frac{1}{1-az}$ $|z|<a^{-1}$ | $\frac{a}{1-az}$ $|z|<a^{-1}$

/z

a^n $n = 0, 1, 2, \cdots$ | a^{n+1} $n = 0, 1, 2, \cdots$

*z

SHR.Seq, SHR.Rng

$\frac{az}{1-az}$ $|z|<a^{-1}$ | $\frac{z}{1-az}$ $|z|<a^{-1}$

a^n $n = 1, 2, 3, \cdots$ | a^{n-1} $n = 1, 2, 3, \cdots$

SHR.Seq, SHR.Rng

$\frac{1}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ | $\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$

*z

a^n $n = 0, -1, -2, \cdots$ | a^{n-1} $n = 0, -1, -2, \cdots$

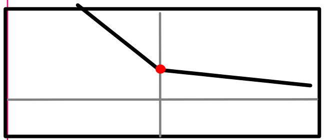
/z

SHL.Seq, SHL.Rng

$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ | $\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$

a^n $n = -1, -2, -3, \cdots$ | a^{n+1} $n = -1, -2, -3, \cdots$

* inverted relation is ignored

a^{-n} 

/a SHL.Seq

*a SHR.Seq

Shifting a sequence

SHL.Seq

$$\boxed{\frac{1}{1-a^{-1}z} \quad |z|<a} \xrightarrow{/a} \boxed{\frac{a^{-1}}{1-a^{-1}z} \quad |z|<a}$$

$a^{-n} \quad n = 0, 1, 2, \cdots$ $\qquad$ $a^{-n-1} \quad n = 0, 1, 2, \cdots$

SHR.Seq

$$\boxed{\frac{a^{-1}z}{1-a^{-1}z} \quad |z|<a} \xrightarrow{*a} \boxed{\frac{z}{1-a^{-1}z} \quad |z|<a}$$

$a^{-n} \quad n = 1, 2, 3, \cdots$ $\qquad$ $a^{-n+1} \quad n = 1, 2, 3, \cdots$

SHR.Seq

$$\boxed{\frac{1}{1-az^{-1}} \quad |z|>a} \xrightarrow{*a} \boxed{\frac{a}{1-az^{-1}} \quad |z|>a}$$

$a^{-n} \quad n = 0, -1, -2, \cdots$ $\qquad$ $a^{-n+1} \quad n = 0, -1, -2, \cdots$

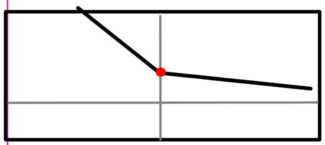
SHL.Seq

$$\boxed{\frac{az^{-1}}{1-az^{-1}} \quad |z|>a} \xrightarrow{/a} \boxed{\frac{z^{-1}}{1-az^{-1}} \quad |z|>a}$$

$a^{-n} \quad n = -1, -2, -3, \cdots$ $\qquad$ $a^{-n-1} \quad n = -1, -2, -3, \cdots$

Shifting a sequence

a^{-n}



/z SHL.Seq, SHL.Rng

*z SHR.Seq, SHR.Rng

$$\frac{1}{1-a^{-1}z} \quad |z|<a$$

$a^{-n} \quad n = 0, 1, 2, \cdots$

/z

SHL.Seq, SHL.Rng

$$\frac{a^{-1}}{1-a^{-1}z} \quad |z|<a$$

$a^{-n-1} \quad n = 0, 1, 2, \cdots$

*z

$$\frac{a^{-1}z}{1-a^{-1}z} \quad |z|<a$$

$a^{-n} \quad n = 1, 2, 3, \cdots$

SHR.Seq, SHR_Rng

$$\frac{z}{1-a^{-1}z} \quad |z|<a$$

$a^{-n+1} \quad n = 1, 2, 3, \cdots$

$$\frac{1}{1-az^{-1}} \quad |z|>a$$

$a^{-n} \quad n = 0, -1, -2, \cdots$

*z

SHR.Seq, SHR.Rng

$$\frac{a}{1-az^{-1}} \quad |z|>a$$

$a^{-n+1} \quad n = 0, -1, -2, \cdots$

/z

$$\frac{az^{-1}}{1-az^{-1}} \quad |z|>a$$

$a^{-n} \quad n = -1, -2, -3, \cdots$

SHL.Seq, SHL.Rng

$$\frac{z^{-1}}{1-az^{-1}} \quad |z|>a$$

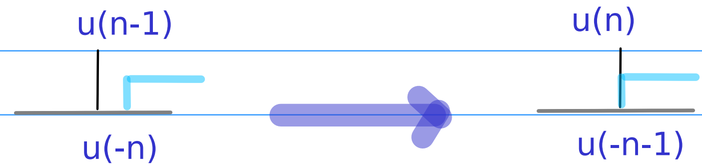
$a^{-n-1} \quad n = -1, -2, -3, \cdots$

* inverted relation is ignored

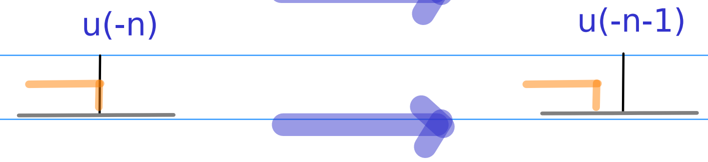
SHL.Seq	Shift Right(Sequence Function)
SHR.Seq	Shift Right(Sequence Function)
SHL.ROC	Shift Right(Region of Convergence)
SHR.ROC	Shift Right(Region of Convergence)

Shifting of a Range

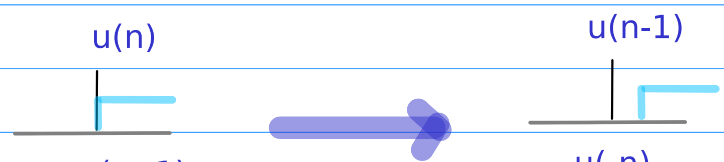
SHL.Rng



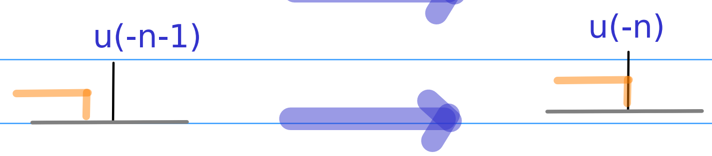
SHL.Rng



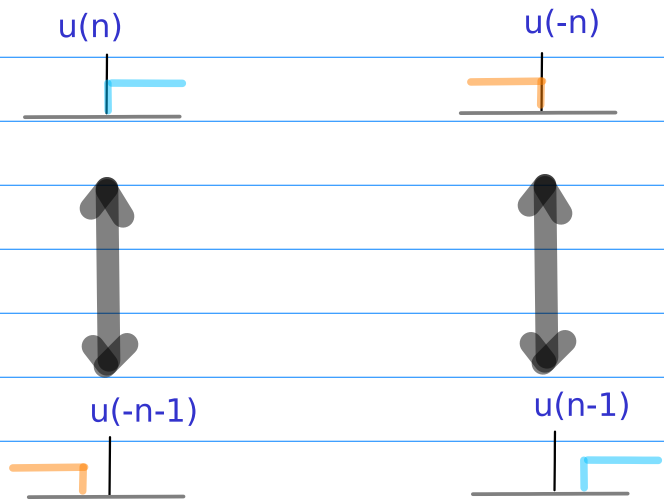
SHR.Rng



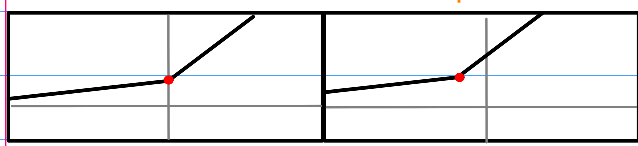
SHR.Rng



Complement

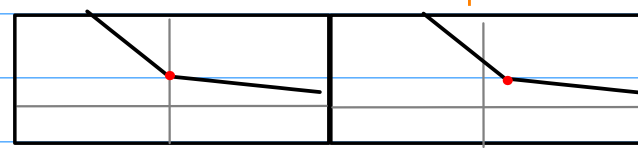


SHL.Seq



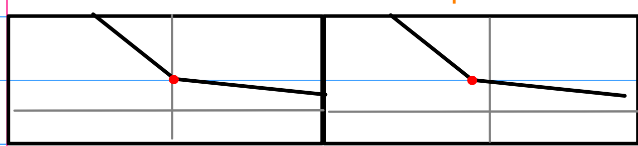
a^n a^{n+1}

SHR.Seq



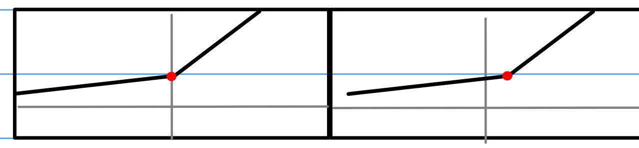
a^{-n} a^{-n+1}

SHL.Seq



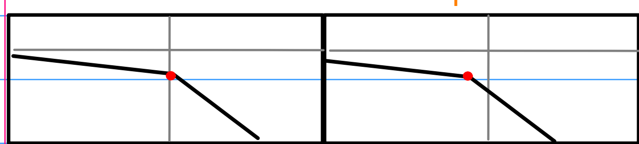
a^{-n} a^{-n-1}

SHR.Seq



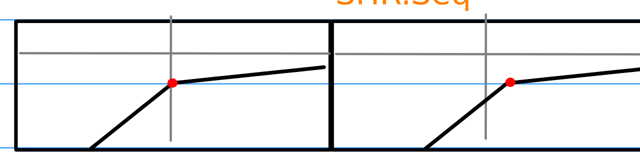
a^n a^{n-1}

SHL.Seq



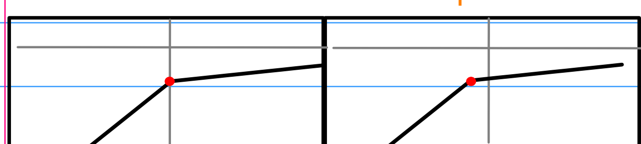
$-a^n$ $-a^{n+1}$

SHR.Seq



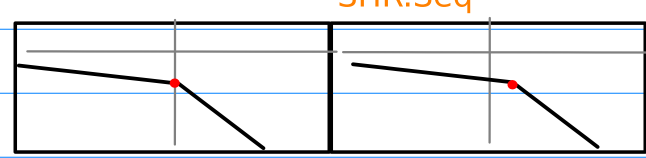
$-a^{-n}$ $-a^{-n+1}$

SHL.Seq



$-a^{-n}$ $-a^{-n-1}$

SHR.Seq



$-a^n$ $-a^{n-1}$

Left Shifted Sequence

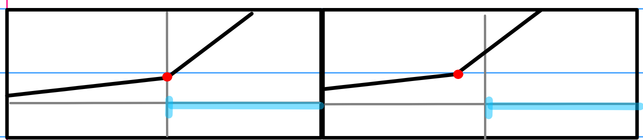
Right Shifted Sequence

1-A $\boxed{\text{Left}}$ Shfted Sequence (a, a^{-1}) $\boxed{\text{Causal}}$

$$a^{n} \cdot u(n) \longrightarrow a^{n+1} \cdot u(n)$$
$$a^{-n} \cdot u(n) \longrightarrow a^{-n-1} \cdot u(n)$$

SHL.Seq

a

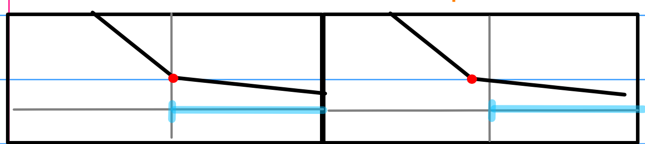


a^{n} a^{n+1}

$\dfrac{1}{1-az}$ $|z|<a^{-1}$ $\dfrac{a}{1-az}$ $|z|<a^{-1}$

SHL.Seq

a^{-1}



a^{-n} a^{-n-1}

$\dfrac{1}{1-a^{-1}z}$ $|z|<a$ $\dfrac{a^{-1}}{1-a^{-1}z}$ $|z|<a$

$$\frac{1}{1-az} = a^{0}z^{0} + a^{1}z^{1} + a^{2}z^{2} + \cdots \Rightarrow \underbrace{(a^{0}, a^{1}, a^{2}, \cdots)}_{} \quad (n \geq 0)$$

0 1 2

$|z|<a^{-1}$

$$\frac{a}{1-az} = a^{1}z^{0} + a^{2}z^{1} + a^{3}z^{2} + \cdots \Rightarrow (a^{1}, a^{2}, a^{3}, \cdots) \quad (n \geq 0)$$

0 1 2

$|z|<a^{-1}$

$$\frac{1}{1-a^{-1}z} = a^{0}z^{0} + a^{-1}z^{1} + a^{-2}z^{2} + \cdots \Rightarrow (a^{0}, a^{-1}, a^{-2}, \cdots) \quad (n \geq 0)$$

0 1 2

$|z|<a^{-1}$

$$\frac{a^{-1}}{1-a^{-1}z} = a^{-1}z^{0} + a^{-2}z^{1} + a^{-3}z^{2} + \cdots \Rightarrow (a^{-1}, a^{-2}, a^{-3}, \cdots) \quad (n \geq 0)$$

0 1 2

$|z|<a^{-1}$

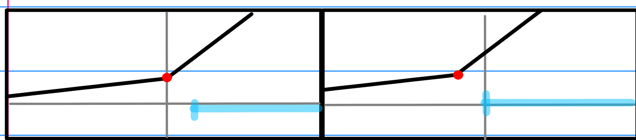
1-B $\boxed{\text{Left}}$ Shfted Sequence (z^{-1}, z^{-1}) $\boxed{\text{Causal}}$

$$a^{n} \cdot u(n-1) \longrightarrow a^{n+1} \cdot u(n)$$

$$a^{-n} \cdot u(n-1) \longrightarrow a^{-n-1} \cdot u(n)$$

SHL.Seq, SHL.Rng

z^{-1}



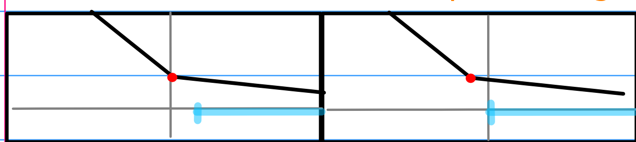
a^{n} $\quad$ a^{n+1}

$\dfrac{az}{1-az}$ $\quad$ $|z| < a^{-1}$

$\dfrac{a}{1-az}$ $\quad$ $|z| < a^{-1}$

SHL.Seq, SHL.Rng

z^{-1}



a^{-n} $\quad$ a^{-n-1}

$\dfrac{a^{-1}z}{1-a^{-1}z}$ $\quad$ $|z| < a$

$\dfrac{a^{-1}}{1-a^{-1}z}$ $\quad$ $|z| < a$

$$\frac{az}{1-az} = a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots \Rightarrow (\overset{0}{0}, \overset{1}{a^{1}}, \overset{2}{a^{2}}, \cdots) \quad (n \geq 1)$$

$|z| < a^{-1}$

$$\frac{a}{1-az} = a^{1}z^{0} + a^{2}z^{1} + a^{3}z^{2} + \cdots \Rightarrow (\underset{0}{a^{1}}, \underset{1}{a^{2}}, \underset{2}{a^{3}}, \cdots) \quad (n \geq 0)$$

$|z| < a^{-1}$

$$\frac{a^{-1}z}{1-a^{-1}z} = a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots \Rightarrow (\overset{0}{0}, \overset{1}{a^{-1}}, \overset{2}{a^{-2}} \cdots) \quad (n \geq 1)$$

$|z| < a^{-1}$

$$\frac{a^{-1}}{1-a^{-1}z} = a^{-1}z^{0} + a^{-2}z^{1} + a^{-3}z^{2} + \cdots \Rightarrow (\underset{0}{a^{-1}}, \underset{1}{a^{-2}}, \underset{2}{a^{-3}}, \cdots) \quad (n \geq 0)$$

$|z| < a^{-1}$

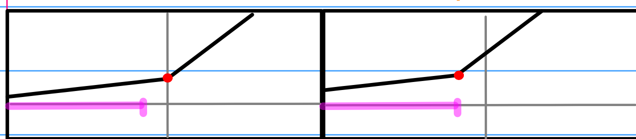
1-C $\boxed{\text{Left}}$ Shfted Sequence (a, a^{-1}) $\boxed{\text{Anti-Causal}}$

$$a^{n} \cdot u(-n-1) \longrightarrow a^{n+1} \cdot u(-n-1)$$

$$a^{-n} \cdot u(-n-1) \longrightarrow a^{-n-1} \cdot u(-n-1)$$

SHL.Seq

a



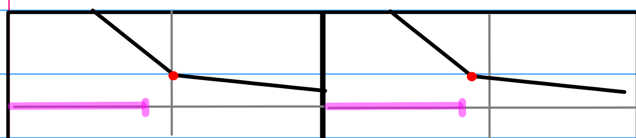
a^{n} $\quad$ a^{n+1}

$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad$ $\frac{z^{-1}}{1-a^{-1}z^{-1}}$

$|z|>a^{-1}$ $\quad$ $|z|>a^{-1}$

SHL.Seq

a^{-1}



a^{-n} $\quad$ a^{-n-1}

$\frac{az^{-1}}{1-az^{-1}}$ $\quad$ $\frac{z^{-1}}{1-az^{-1}}$

$|z|>a$ $\quad$ $|z|>a$

$$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} = \cdots + a^{-3}z^{-3} + a^{-2}z^{-2} + a^{-1}z^{-1} \Rightarrow (\ldots, \overset{-3}{a^{-3}}, \overset{-2}{a^{-2}}, \overset{-1}{a^{-1}}) \quad (n<0)$$

$|z|>a^{-1}$

$$\frac{z^{-1}}{1-a^{-1}z^{-1}} = \cdots + a^{-2}z^{-3} + a^{-1}z^{-2} + a^{0}z^{-1} \Rightarrow (\ldots, \underset{-3}{a^{-2}}, \underset{-2}{a^{-1}}, \underset{-1}{a^{0}}) \quad (n<0)$$

$|z|>a^{-1}$

$$\frac{az^{-1}}{1-az^{-1}} = \cdots + a^{3}z^{-3} + a^{2}z^{-2} + a^{1}z^{-1} \Rightarrow (\ldots, \overset{-3}{a^{3}}, \overset{-2}{a^{2}}, \overset{-1}{a^{1}}) \quad (n<0)$$

$|z|>a$

$$\frac{z^{-1}}{1-az^{-1}} = \cdots + a^{2}z^{-3} + a^{1}z^{-2} + a^{0}z^{-1} \Rightarrow (\ldots, \underset{-3}{a^{2}}, \underset{-2}{a^{1}}, \underset{-1}{a^{0}}) \quad (n<0)$$

$|z|>a$

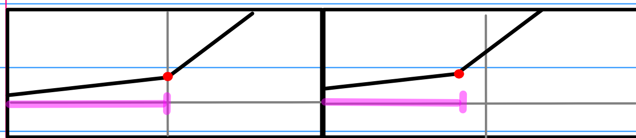
1-D $\boxed{\text{Left}}$ Shfted Sequence (z^{-1}, z^{-1}) $\boxed{\text{Anti-Causal}}$

$$a^{n} \cdot u(-n) \longrightarrow a^{n+1} \cdot u(-n-1)$$

$$a^{-n} \cdot u(-n) \longrightarrow a^{-n-1} \cdot u(-n-1)$$

z^{-1}

SHL.Seq, SHL.Rng



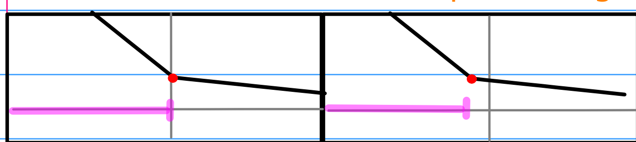
a^{n} $\qquad$ a^{n+1}

$\dfrac{1}{1-a^{-1}z^{-1}}$ $\quad |z| > a^{-1}$

$\dfrac{z^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z| > a^{-1}$

z^{-1}

SHL.Seq, SHL.Rng



a^{-n} $\qquad$ a^{-n-1}

$\dfrac{1}{1-az^{-1}}$ $\quad |z| > a^{-1}$

$\dfrac{z^{-1}}{1-az^{-1}}$ $\quad |z| > a^{-1}$

$$\frac{1}{1-a^{-1}z^{-1}} = \cdots + a^{-2}z^{-2} + a^{-1}z^{-1} + a^{0}z^{0} \Rightarrow (\cdots, \overset{-2}{a^{-2}}, \overset{-1}{a^{-1}}, \overset{0}{a^{0}}) \quad (n<1)$$

$|z| > a^{-1}$

$$\frac{z^{-1}}{1-a^{-1}z^{-1}} = \cdots + a^{-2}z^{-3} + a^{-1}z^{-2} + a^{0}z^{-1} \Rightarrow (\cdots, \underset{-2}{a^{-1}}, \underset{-1}{a^{0}}, \underset{0}{0}) \quad (n<0)$$

$|z| > a^{-1}$

$$\frac{1}{1-az^{-1}} = \cdots + a^{2}z^{-2} + a^{1}z^{-1} + a^{0}z^{0} \Rightarrow (\cdots, \overset{-2}{a^{2}}, \overset{-1}{a^{1}}, \overset{0}{a^{0}}) \quad (n<1)$$

$|z| > a^{-1}$

$$\frac{z^{-1}}{1-az^{-1}} = \cdots + a^{2}z^{-3} + a^{1}z^{-2} + a^{0}z^{-1} \Rightarrow (\cdots, \underset{-2}{a^{1}}, \underset{-1}{a^{0}}, \underset{0}{0}) \quad (n<0)$$

$|z| > a^{-1}$

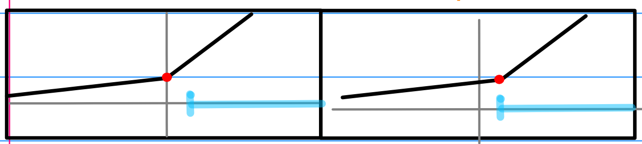
2-A $\boxed{\text{Right}}$ Shfted Sequence (a^{-1}, a) $\boxed{\text{Causal}}$

$$a^{n} \cdot u(n-1) \longrightarrow a^{n-1} \cdot u(n-1)$$

$$a^{-n} \cdot u(n-1) \longrightarrow a^{-n+1} \cdot u(n-1)$$

SHR.Seq

a^{-1}



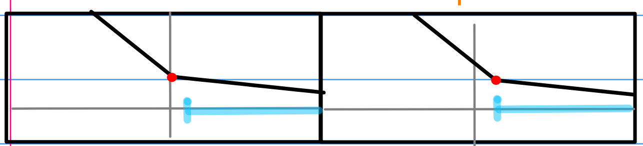
a^{n} a^{n-1}

$$\frac{az}{1-az}$$ $|z| < a^{-1}$

$$\frac{z}{1-az}$$ $|z| < a^{-1}$

SHR.Seq

a



a^{-n} a^{-n+1}

$$\frac{a^{-1}z}{1-a^{-1}z}$$ $|z| < a$

$$\frac{z}{1-a^{-1}z}$$ $|z| < a$

$$\frac{az}{1-az} = a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots \Rightarrow \overset{\;1\qquad 2\qquad 3}{(a^{1}, a^{2}, a^{3}, \cdots)} \quad (n \geq 1)$$

$|z| < a^{-1}$

$$\frac{z}{1-az} = a^{0}z^{1} + a^{1}z^{2} + a^{2}z^{3} + \cdots \Rightarrow \underset{\;1\qquad 2\qquad 3}{(a^{0}, a^{1}, a^{2}, \cdots)} \quad (n \geq 1)$$

$|z| < a^{-1}$

$$\frac{a^{-1}z}{1-a^{-1}z} = a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots \Rightarrow \overset{\;1\qquad 2\qquad 3}{(a^{-1}, a^{-2}, a^{-3}, \cdots)} \quad (n \geq 1)$$

$|z| < a^{-1}$

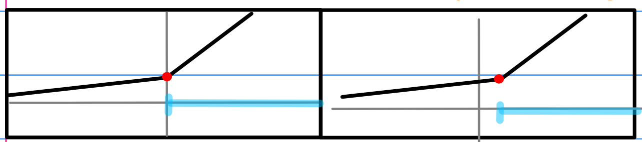
$$\frac{z}{1-a^{-1}z} = a^{0}z^{1} + a^{-1}z^{2} + a^{-2}z^{3} + \cdots \Rightarrow \underset{\;1\qquad 2\qquad 3}{(a^{0}, a^{-1}, a^{-2}, \cdots)} \quad (n \geq 1)$$

$|z| < a^{-1}$

2-B $\boxed{\text{Right}}$ Shfted Sequence (z, z) $\boxed{\text{Causal}}$

$$a^{n} \cdot u(n) \longrightarrow a^{n-1} \cdot u(n-1)$$
$$a^{-n} \cdot u(n) \longrightarrow a^{-n+1} \cdot u(n-1)$$

SHR.Seq, SHR.Rng

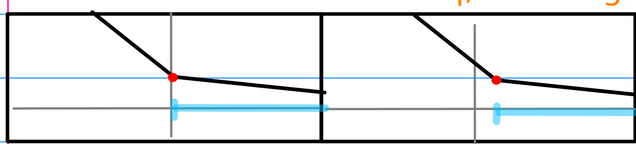
z 

a^{n} a^{n-1}

$\frac{1}{1-az}$ $\frac{z}{1-az}$

$|z|<a^{-1}$ $|z|<a^{-1}$

SHR.Seq, SHR.Rng

z 

a^{-n} a^{-n+1}

$\frac{1}{1-a^{-1}z}$ $\frac{z}{1-a^{-1}z}$

$|z|<a$ $|z|<a$

$$\frac{1}{1-az} = a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots \Rightarrow \underset{\text{0 1 2}}{(a^0, a^1, a^2, \cdots)} \quad (n \geq 0)$$

$|z|<a^{-1}$

$$\frac{z}{1-az} = a^0 z^1 + a^1 z^2 + a^2 z^3 + \cdots \Rightarrow \underset{\text{0 1 2}}{(0, a^0, a^1, \cdots)} \quad (n \geq 1)$$

$|z|<a^{-1}$

$$\frac{1}{1-a^{-1}z} = a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \cdots \Rightarrow \underset{\text{0 1 2}}{(a^0, a^{-1}, a^{-2}, \cdots)} \quad (n \geq 0)$$

$|z|<a^{-1}$

$$\frac{z}{1-a^{-1}z} = a^0 z^1 + a^{-1} z^2 + a^{-2} z^3 + \cdots \Rightarrow \underset{\text{0 1 2}}{(0, a^0, a^{-1}, \cdots)} \quad (n \geq 1)$$

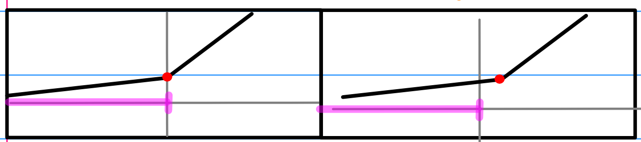
$|z|<a^{-1}$

2-c $\boxed{\text{Right}}$ Shfted Sequence (a^{-1}, a) Anti-Causal

$$a^{n} \cdot u(-n) \longrightarrow a^{n-1} \cdot u(-n)$$
$$a^{-n} \cdot u(-n) \longrightarrow a^{-n+1} \cdot u(-n)$$

a^{-1}

SHR.Seq



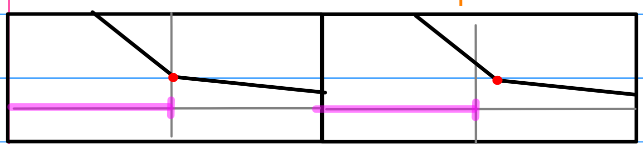
a^{n} $\quad a^{n-1}$

$$\frac{1}{1-a^{-1}z^{-1}} \qquad \frac{a^{-1}}{1-a^{-1}z^{-1}}$$

$|z| > a^{-1}$ $\quad |z| > a^{-1}$

a

SHR.Seq



a^{-n} $\quad a^{-n+1}$

$$\frac{1}{1-a z^{-1}} \qquad \frac{a}{1-a z^{-1}}$$

$|z| > a$ $\quad |z| > a$

$$\frac{1}{1-a^{-1}z^{-1}} = \cdots + a^{-2}z^{-2} + a^{-1}z^{-1} + a^{0}z^{0} \Rightarrow (\ldots, \underset{-2}{a^{-2}}, \underset{-1}{a^{-1}}, \underset{0}{a^{0}}) \quad (n<1)$$

$|z| > a^{-1}$

$$\frac{a^{-1}}{1-a^{-1}z^{-1}} = \cdots + a^{-3}z^{-2} + a^{-2}z^{-1} + a^{-1}z^{0} \Rightarrow (\ldots, \underset{-2}{a^{-3}}, \underset{-1}{a^{-2}}, \underset{0}{a^{-1}}) \quad (n<1)$$

$|z| > a^{-1}$

$$\frac{1}{1-a z^{-1}} = \cdots + a^{2}z^{-2} + a^{1}z^{-1} + a^{0}z^{0} \Rightarrow (\ldots, \underset{-2}{a^{2}}, \underset{-1}{a^{1}}, \underset{0}{a^{0}}) \quad (n<1)$$

$|z| > a$

$$\frac{a}{1-a z^{-1}} = \cdots + a^{3}z^{-2} + a^{2}z^{-1} + a^{1}z^{0} \Rightarrow (\ldots, \underset{-2}{a^{3}}, \underset{-1}{a^{2}}, \underset{0}{a^{1}}) \quad (n<1)$$

$|z| > a$

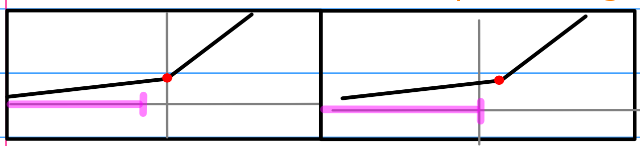
2-D $\boxed{\text{Right}}$ Shfted Sequence (z, z) $\boxed{\text{Anti-Causal}}$

$$a^{n} \cdot u(-n-1) \longrightarrow a^{n-1} \cdot u(-n)$$

$$a^{-n} \cdot u(-n-1) \longrightarrow a^{-n+1} \cdot u(-n)$$

SHR.Seq, SHR.Rng

z

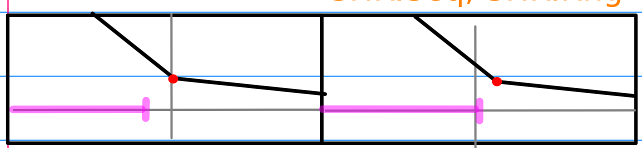


$\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z| > a^{-1}$

$\dfrac{a^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z| > a^{-1}$

SHR.Seq, SHR.Rng

z



$\dfrac{az^{-1}}{1-az^{-1}}$ $\quad |z| > a$

$\dfrac{a}{1-az^{-1}}$ $\quad |z| > a$

$$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} = \cdots + a^{-3}z^{-3} + a^{-2}z^{-2} + a^{-1}z^{-1} \Rightarrow (\ldots, \overset{-3}{a^{-2}}, \overset{-2}{a^{-1}}, \overset{-1}{0}) \quad (n<0)$$

$|z| > a^{-1}$

$$\frac{a^{-1}}{1-a^{-1}z^{-1}} = \cdots + a^{-3}z^{-2} + a^{-2}z^{-1} + a^{-1}z^{0} \Rightarrow (\ldots, \underset{-3}{a^{-3}}, \underset{-2}{a^{-2}}, \underset{-1}{a^{-1}}) \quad (n<1)$$

$|z| > a^{-1}$

$$\frac{az^{-1}}{1-az^{-1}} = \cdots + a^{3}z^{-3} + a^{2}z^{-2} + a^{1}z^{-1} \Rightarrow (\ldots, \overset{-2}{a^{2}}, \overset{-1}{a^{1}}, \overset{0}{0}) \quad (n<0)$$

$|z| > a$

$$\frac{a}{1-az^{-1}} = \cdots + a^{3}z^{-3} + a^{2}z^{-2} + a^{1}z^{-1} \Rightarrow (\ldots, \underset{-2}{a^{3}}, \underset{-1}{a^{2}}, \underset{0}{a^{1}}) \quad (n<1)$$

$|z| > a$

Original Sequence		Shifted Sequence	Original Sequence		Shifted Sequence
$a^{n} \cdot u(n)$	$\longrightarrow$	$a^{n+1} \cdot u(n)$	$a^{n} \cdot u(-n-1)$	$\longrightarrow$	$a^{n+1} \cdot u(-n-1)$
$a^{-n} \cdot u(n)$	$\longrightarrow$	$a^{-n-1} \cdot u(n)$	$a^{-n} \cdot u(-n-1)$	$\longrightarrow$	$a^{-n-1} \cdot u(-n-1)$
$a^{n} \cdot u(n-1)$	$\longrightarrow$	$a^{n+1} \cdot u(n)$	$a^{n} \cdot u(-n)$	$\longrightarrow$	$a^{n+1} \cdot u(-n-1)$
$a^{-n} \cdot u(n-1)$	$\longrightarrow$	$a^{-n-1} \cdot u(n)$	$a^{-n} \cdot u(-n)$	$\longrightarrow$	$a^{-n-1} \cdot u(-n-1)$
$a^{n} \cdot u(n-1)$	$\longrightarrow$	$a^{n-1} \cdot u(n-1)$	$a^{n} \cdot u(-n)$	$\longrightarrow$	$a^{n-1} \cdot u(-n)$
$a^{-n} \cdot u(n-1)$	$\longrightarrow$	$a^{-n+1} \cdot u(n-1)$	$a^{-n} \cdot u(-n)$	$\longrightarrow$	$a^{-n+1} \cdot u(-n)$
$a^{n} \cdot u(n)$	$\longrightarrow$	$a^{n-1} \cdot u(n-1)$	$a^{n} \cdot u(-n-1)$	$\longrightarrow$	$a^{n-1} \cdot u(-n)$
$a^{-n} \cdot u(n)$	$\longrightarrow$	$a^{-n+1} \cdot u(n-1)$	$a^{-n} \cdot u(-n-1)$	$\longrightarrow$	$a^{-n+1} \cdot u(-n)$

Complementary Ranges

$u(n)$ $u(-n-1)$

$u(n-1)$ $u(-n)$

Original Sequence	Shifted Sequence	Original Sequence	Shifted Sequence
$a^{n} \cdot u(n) \longrightarrow$	$a^{n+1} \cdot u(n)$	$a^{n} \cdot u(-n-1) \longrightarrow$	$a^{n+1} \cdot u(-n-1)$
$a^{-n} \cdot u(n) \longrightarrow$	$a^{-n-1} \cdot u(n)$	$a^{-n} \cdot u(-n-1) \longrightarrow$	$a^{-n-1} \cdot u(-n-1)$
$\ll(a^0, a^1, a^2, \cdots)$	$(a^1, a^2, a^3, \cdots)$	$(\cdots, a^{-3}, a^{-2}, a^{-1})$	$(\cdots, a^{-2}, a^{-1}, a^0)\ll$
$\ll(a^0, a^{-1}, a^{-2}, \cdots)$	$(a^{-1}, a^{-2}, a^{-3}, \cdots)$	$(\cdots, a^{3}, a^{2}, a^{1})$	$(\cdots, a^{2}, a^{1}, a^0)\ll$
shift out			shift in
$a^{n} \cdot u(n-1) \longrightarrow$	$a^{n+1} \cdot u(n)$	$a^{n} \cdot u(-n) \longrightarrow$	$a^{n+1} \cdot u(-n-1)$
$a^{-n} \cdot u(n-1) \longrightarrow$	$a^{-n-1} \cdot u(n)$	$a^{-n} \cdot u(-n) \longrightarrow$	$a^{-n-1} \cdot u(-n-1)$
$\ll(0, a^1, a^2, \cdots)$	$(a^1, a^2, a^3, \cdots)$	$(\cdots, a^{-2}, a^{-1}, a^0)$	$(\cdots, a^{-1}, a^0, 0)\ll$
$\ll(0, a^{-1}, a^{-2} \cdots)$	$(a^{-1}, a^{-2}, a^{-3}, \cdots)$	$(\cdots, a^{2}, a^{1}, a^0)$	$(\cdots, a^{1}, a^0, 0)\ll$
shift out			shift in
$a^{n} \cdot u(n-1) \longrightarrow$	$a^{n-1} \cdot u(n-1)$	$a^{n} \cdot u(-n) \longrightarrow$	$a^{n-1} \cdot u(-n)$
$a^{-n} \cdot u(n-1) \longrightarrow$	$a^{-n+1} \cdot u(n-1)$	$a^{-n} \cdot u(-n) \longrightarrow$	$a^{-n+1} \cdot u(-n)$
$(a^1, a^2, a^3, \cdots)$	$\gg(a^0, a^1, a^2, \cdots)$	$(\cdots, a^{-2}, a^{-1}, a^0)\gg$	$(\cdots, a^{-3}, a^{-2}, a^{-1})$
$(a^{-1}, a^{-2}, a^{-3}, \cdots)$	$\gg(a^0, a^{-1}, a^{-2}, \cdots)$	$(\cdots, a^{2}, a^{1}, a^0)\gg$	$(\cdots, a^{3}, a^{2}, a^{1})$
	shift in	shift out	
$a^{n} \cdot u(n) \longrightarrow$	$a^{n-1} \cdot u(n-1)$	$a^{n} \cdot u(-n-1) \longrightarrow$	$a^{n-1} \cdot u(-n)$
$a^{-n} \cdot u(n) \longrightarrow$	$a^{-n+1} \cdot u(n-1)$	$a^{-n} \cdot u(-n-1) \longrightarrow$	$a^{-n+1} \cdot u(-n)$
$(a^0, a^1, a^2, \cdots)$	$\gg(0, a^0, a^1, \cdots)$	$(\cdots, a^{-2}, a^{-1}, 0)\gg$	$(\cdots, a^{-3}, a^{-2}, a^{-1})$
$(a^0, a^{-1}, a^{-2} \cdots)$	$\gg(0, a^0, a^{-1}, \cdots)$	$(\cdots, a^{2}, a^{1}, 0)\gg$	$(\cdots, a^{3}, a^{2}, a^{1})$
	shift in	shift out	

Complementary and Symmetric Relations

a^n	$\longrightarrow$	a^{n+1}
a^{-n}	$\longrightarrow$	a^{-n-1}

$u(n)$	$\longrightarrow$	$u(n)$
$u(-n-1)$	$\longrightarrow$	$u(-n-1)$
$u(n-1)$	$\longrightarrow$	$u(n)$
$u(-n)$	$\longrightarrow$	$u(-n-1)$

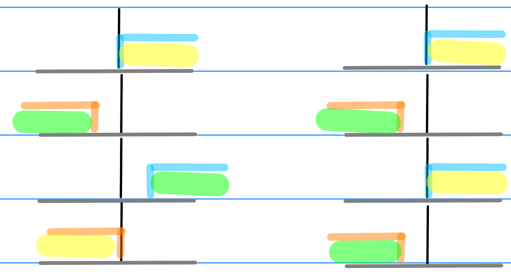
a^n	$\longrightarrow$	a^{n-1}
a^{-n}	$\longrightarrow$	a^{-n+1}

$u(n-1)$	$\longrightarrow$	$u(n-1)$
$u(-n)$	$\longrightarrow$	$u(-n)$
$u(n)$	$\longrightarrow$	$u(n-1)$
$u(-n-1)$	$\longrightarrow$	$u(-n)$

$u(n)$ complementary $u(-n-1)$ symmetric $u(n-1)$

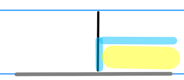
$u(-n)$ complementary $u(n-1)$ symmetric $u(-n-1)$

$u(n)$	$\longrightarrow$	$u(n)$	no shift
$u(-n-1)$	$\longrightarrow$	$u(-n-1)$	no shift
$u(n-1)$	$\longrightarrow$	$u(n)$	left shift
$u(-n)$	$\longrightarrow$	$u(-n-1)$	left shift

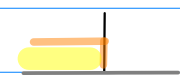


$u(n-1)$	$\longrightarrow$	$u(n-1)$	no shift
$u(-n)$	$\longrightarrow$	$u(-n)$	no shift
$u(n)$	$\longrightarrow$	$u(n-1)$	right shift
$u(-n-1)$	$\longrightarrow$	$u(-n)$	right shift

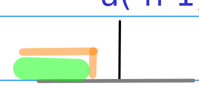
u(n)



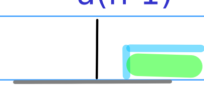
u(-n)



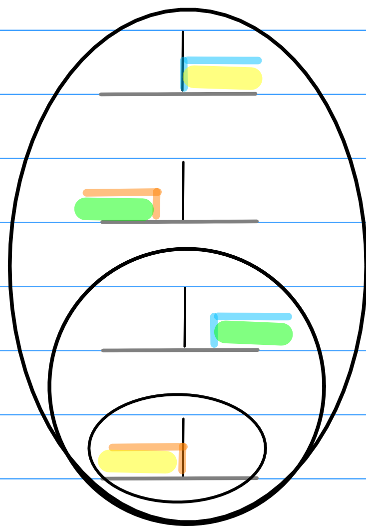
u(-n-1)

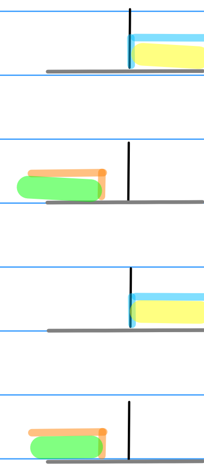


u(n-1)

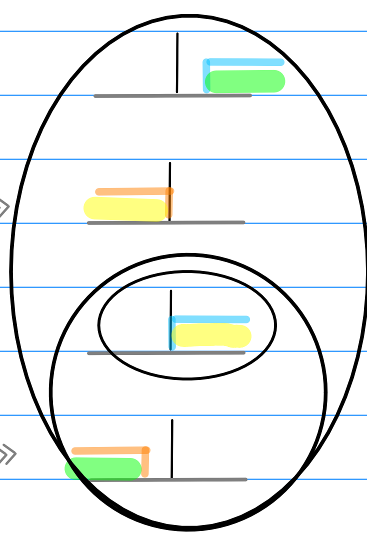


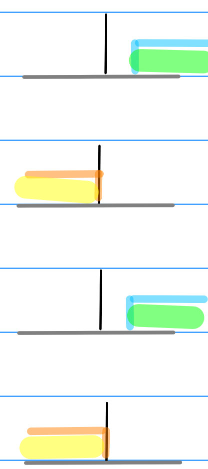
Original Sequence		Shifted Sequence
$(a^0, a^1, a^2, \cdots)$	no shift	$(a^1, a^2, a^3, \cdots)$
$(\cdots, a^{-3}, a^{-2}, a^{-1})$	no shift	$(\cdots, a^{-2}, a^{-1}, a^0)$
$(0, a^1, a^2, \cdots)$	left shift	$(a^1, a^2, a^3, \cdots)$
$(\cdots, a^{-2}, a^{-1}, a^0)$	left shift	$(\cdots, a^{-1}, a^0, 0)$





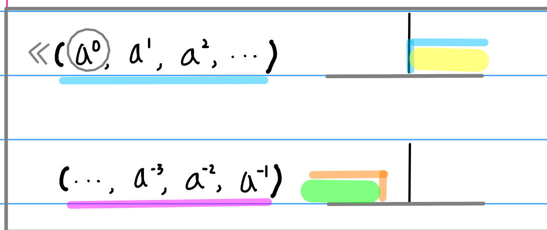
Original Sequence		Shifted Sequence
$(a^1, a^2, a^3, \cdots)$	no shift	$(a^0, a^1, a^2, \cdots)$
$(\cdots, a^{-2}, a^{-1}, a^0)$	no shift	$(\cdots, a^{-3}, a^{-2}, a^{-1})$
$(a^0, a^1, a^2, \cdots)$	right shift	$(0, a^0, a^1, \cdots)$
$(\cdots, a^{-2}, a^{-1}, 0)$	right shift	$(\cdots, a^{-3}, a^{-2}, a^{-1})$



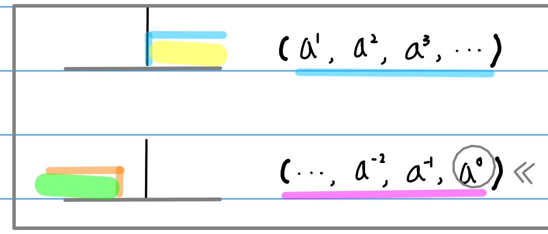


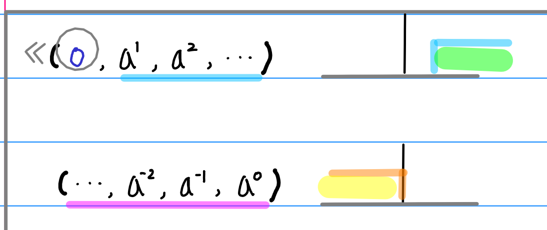
Original Sequence

Shifted Sequence

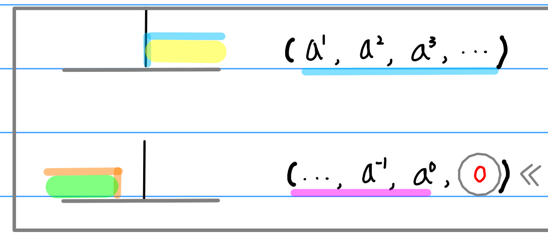


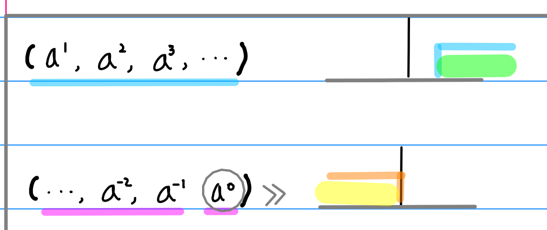
* no shift
* non-zero shift in
* a new value introduced



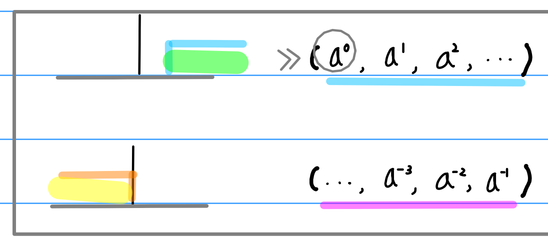


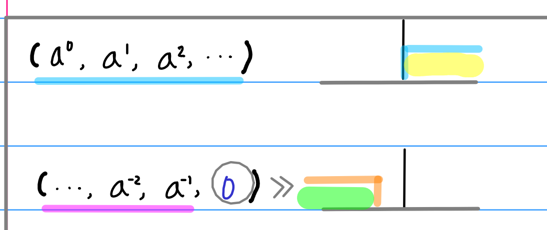
* left shift
* zero shift in
* the same set of values



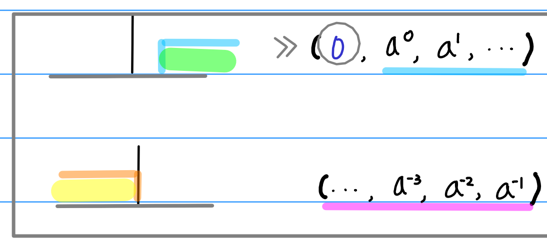


* no shift
* non-zero shift in
* a new value introduced

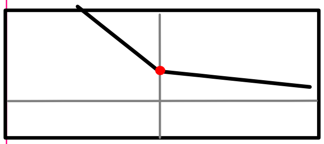




* right shift
* zero shift in
* the same set of values



a^{-n}



scale(1/a) — SHL.Seq

Original Series	Scaled Series	Original Sequence	Shifted Sequence
$-\frac{1}{1-a^{-1}z}$ $\lvert z\rvert<a$	$-\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert<a$	$-(\frac{1}{a})^n$ $(n\geqslant 0)$ $-(\frac{1}{a^0},\frac{1}{a^1},\frac{1}{a^2},\cdots)$	$-(\frac{1}{a})^{n+1}$ $(n\geqslant 0)$ $-(\frac{1}{a^1},\frac{1}{a^2},\frac{1}{a^3},\cdots)$
$\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert>a$	$\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert>a$	$(\frac{1}{a})^n$ $(n<0)$ $(\cdots,a^3,a^2,a^1)$	$(\frac{1}{a})^{n+1}$ $(n<0)$ $(\cdots,a^2,a^1,a^0)$

scale(1/z) — SHL.Seq, SHL.Rng

Original Series	Scaled Series	Original Sequence	Shifted Sequence
$\frac{1}{1-az^{-1}}$ $\lvert z\rvert>a$	$\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert>a$	$(\frac{1}{a})^n$ $(n<1)$ $(\cdots,a^2,a^1,a^0)$	$(\frac{1}{a})^{n+1}$ $(n<0)$ $(\cdots,a^2,a^1,a^0)$
$-\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert<a$	$-\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert<a$	$-(\frac{1}{a})^n$ $(n\geqslant 1)$ $-(\frac{1}{a^1},\frac{1}{a^2},\frac{1}{a^3},\cdots)$	$-(\frac{1}{a})^{n+1}$ $(n\geqslant 0)$ $-(\frac{1}{a^1},\frac{1}{a^2},\frac{1}{a^3},\cdots)$

scale(a) — SHR.Seq

Original Series	Scaled Series	Original Sequence	Shifted Sequence
$-\frac{1}{1-az^{-1}}$ $\lvert z\rvert>a$	$-\frac{a}{1-az^{-1}}$ $\lvert z\rvert>a$	$-(\frac{1}{a})^n$ $(n<1)$ $-(\cdots,a^2,a^1,a^0)$	$-(\frac{1}{a})^{n-1}$ $(n<1)$ $-(\cdots,a^3,a^2,a^1)$
$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert<a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert<a$	$(\frac{1}{a})^n$ $(n\geqslant 1)$ $(\frac{1}{a^1},\frac{1}{a^2},\frac{1}{a^3},\cdots)$	$(\frac{1}{a})^{n-1}$ $(n\geqslant 1)$ $(\frac{1}{a^0},\frac{1}{a^1},\frac{1}{a^2},\cdots)$

scale(z) — SHR.Seq, SHR.Rng

Original Series	Scaled Series	Original Sequence	Shifted Sequence
$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert<a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert<a$	$(\frac{1}{a})^n$ $(n\geqslant 0)$ $(\frac{1}{a^0},\frac{1}{a^1},\frac{1}{a^2},\cdots)$	$(\frac{1}{a})^{n-1}$ $(n\geqslant 1)$ $(\frac{1}{a^0},\frac{1}{a^1},\frac{1}{a^2},\cdots)$
$\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert>a$	$\frac{a}{1-az^{-1}}$ $\lvert z\rvert>a$	$-(\frac{1}{a})^n$ $(n<0)$ $(\cdots,a^3,a^2,a^1)$	$-(\frac{1}{a})^{n-1}$ $(n<1)$ $(\cdots,a^3,a^2,a^1)$

2 formulas   Simple Pole Form

$$\boxed{\frac{1}{z-p}} \qquad \boxed{\frac{1}{z^{-1}-p}}$$

2 representations each   Geometric Series Form

causal   anti- causal

$$\frac{1}{z-p} \begin{cases} \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \triangleq f(z) = X(z^{-1}) \\[2ex] \boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \triangleq Y(z) = g(z^{-1}) \end{cases}$$

$f(z) = Y(z)$, $X(z^{-1}) = g(z^{-1})$

causal   anti- causal

causal   anti- causal

$$\frac{1}{z^{-1}-p} \begin{cases} \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \triangleq X(z) = f(z^{-1}) \\[2ex] \boxed{\dfrac{z}{1-pz}} \triangleq g(z) = Y(z^{-1}) \end{cases}$$

$X(z) = g(z)$, $f(z^{-1}) = Y(z^{-1})$

causal   anti- causal

Simple Pole Form | Geometric Series Form

Geometric Series Form Combinations with a unit start term

$$-\frac{1}{1-\boxed{az}}$$

$$-\frac{1}{1-\boxed{az^{-1}}}$$

$$+\frac{1}{1-\boxed{a^{-1}z^{-1}}}$$

$$+\frac{1}{1-\boxed{a^{-1}z}}$$

$$-\frac{1}{1-\boxed{a^{-1}z}}$$

$$-\frac{1}{1-\boxed{a^{-1}z^{-1}}}$$

$$+\frac{1}{1-\boxed{az^{-1}}}$$

$$+\frac{1}{1-\boxed{az}}$$

Geometric Series Form Combinations with non-unit start term

$$+\frac{a^{-1}z^{-1}}{1-\boxed{a^{-1}z^{-1}}}$$

$$+\frac{a^{-1}z}{1-\boxed{a^{-1}z}}$$

$$-\frac{az}{1-\boxed{az}}$$

$$-\frac{az^{-1}}{1-\boxed{az^{-1}}}$$

$$+\frac{az^{-1}}{1-\boxed{az^{-1}}}$$

$$+\frac{az}{1-\boxed{az}}$$

$$-\frac{a^{-1}z}{1-\boxed{a^{-1}z}}$$

$$-\frac{a^{-1}z^{-1}}{1-\boxed{a^{-1}z^{-1}}}$$

Geometric Series with a unit start term

Laurent Series

$a\,z$ $\quad |z| < a^{-1}$

$$-\left(a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots\right)$$

$$a_n = -a^n \quad (n \geq 0)$$

$a\,z^{-1}$ $\quad |z| > a$

$$-\left(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \cdots\right)$$

$$-\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{-1} z^{-1} + \left(\tfrac{1}{a}\right)^{-2} z^{-2} + \cdots\right)$$

$$a_n = -\left(\tfrac{1}{a}\right)^n \quad (n < 1)$$

$a^{-1} z^{-1}$ $\quad |z| > a^{-1}$

$$\left(a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \cdots\right)$$

$$a_n = a^n \quad (n < 1)$$

$a^{-1} z$ $\quad |z| < a$

$$\left(a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \cdots\right)$$

$$\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^1 z^1 + \left(\tfrac{1}{a}\right)^2 z^2 + \cdots\right)$$

$$a_n = \left(\tfrac{1}{a}\right)^n \quad (n \geq 0)$$

$a^{-1} z$ $\quad |z| < a$

$$-\left(a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \cdots\right)$$

$$-\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^1 z^1 + \left(\tfrac{1}{a}\right)^2 z^2 + \cdots\right)$$

$$a_n = -\left(\tfrac{1}{a}\right)^n \quad (n \geq 0)$$

$a^{-1} z^{-1}$ $\quad |z| > a^{-1}$

$$-\left(a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \cdots\right)$$

$$a_n = -a^n \quad (n < 1)$$

$a\,z^{-1}$ $\quad |z| > a$

$$\left(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \cdots\right)$$

$$\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{-1} z^{-1} + \left(\tfrac{1}{a}\right)^{-2} z^{-2} + \cdots\right)$$

$$a_n = \left(\tfrac{1}{a}\right)^n \quad (n < 1)$$

$a\,z$ $\quad |z| < a^{-1}$

$$\left(a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots\right)$$

$$a_n = a^n \quad (n \geq 0)$$

Geometric Series with a unit start term

z-Transform

$a\,z$	$\lvert z\rvert < a^{-1}$

$-\left(a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots\right)$

$-\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^{-1} z^1 + \left(\frac{1}{a}\right)^{-2} z^2 + \cdots\right)$

$a_n = -a^{-n}$	$(-n \geq 0)$
$a_n = -\left(\frac{1}{a}\right)^n$	$(n < 1)$

$a\,z^{\boxed{-1}}$	$\lvert z\rvert > a$

$-\left(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \cdots\right)$

$-\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^{-1} z^{-1} + \left(\frac{1}{a}\right)^{-2} z^{-2} + \cdots\right)$

$a_n = -\left(\frac{1}{a}\right)^{-n}$	$(-n < 1)$
$a_n = -a^n$	$(n \geq 0)$

$a^{\boxed{-1}}z^{\boxed{-1}}$	$\lvert z\rvert > a^{-1}$

$\left(a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \cdots\right)$

$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^{-1} + \left(\frac{1}{a}\right)^2 z^{-2} + \cdots\right)$

$a_n = a^{-n}$	$(-n < 1)$
$a_n = \left(\frac{1}{a}\right)^n$	$(n \geq 0)$

$a^{\boxed{-1}}z$	$\lvert z\rvert < a$

$\left(a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \cdots\right)$

$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \cdots\right)$

$a_n = \left(\frac{1}{a}\right)^{-n}$	$(-n \geq 0)$
$a_n = a^n$	$(n < 1)$

$a^{\boxed{-1}}z$	$\lvert z\rvert < a$

$-\left(a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \cdots\right)$

$-\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \cdots\right)$

$a_n = -\left(\frac{1}{a}\right)^{-n}$	$(-n \geq 0)$
$a_n = -a^n$	$(n < 1)$

$a^{\boxed{-1}}z^{\boxed{-1}}$	$\lvert z\rvert > a^{-1}$

$-\left(a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \cdots\right)$

$-\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^{-1} + \left(\frac{1}{a}\right)^2 z^{-2} + \cdots\right)$

$a_n = -a^{-n}$	$(-n < 1)$
$a_n = -\left(\frac{1}{a}\right)^n$	$(n \geq 0)$

$a\,z^{\boxed{-1}}$	$\lvert z\rvert > a$

$\left(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \cdots\right)$

$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^{-1} z^{-1} + \left(\frac{1}{a}\right)^{-2} z^{-2} + \cdots\right)$

$a_n = \left(\frac{1}{a}\right)^{-n}$	$(-n < 1)$
$a_n = a^n$	$(n \geq 0)$

$a\,z$	$\lvert z\rvert < a^{-1}$

$\left(a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots\right)$

$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^{-1} z^1 + \left(\frac{1}{a}\right)^{-2} z^2 + \cdots\right)$

$a_n = a^{-n}$	$(-n \geq 0)$
$a_n = \left(\frac{1}{a}\right)^n$	$(n < 1)$

Geometric Series with a unit start term

Laurent Series vs. z-Transform

$a\,z$	$\lvert z\rvert < a^{-1}$

$-\left(a^0z^0 + a^1z^1 + a^2z^2 + \cdots\right)$

$-\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{-1} z^1 + \left(\tfrac{1}{a}\right)^{-2} z^2 + \cdots\right)$

nt Series	$a_n = -a^n$	$(n \geq 0)$
sform	$a_n = -\left(\tfrac{1}{a}\right)^n$	$(n < 1)$

$a\,z^{\boxed{-1}}$	$\lvert z\rvert > a$

$-\left(a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots\right)$

$-\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{-1} z^{-1} + \left(\tfrac{1}{a}\right)^{-2} z^{-2} + \cdots\right)$

$a_n = -\left(\tfrac{1}{a}\right)^n$	$(n < 1)$
$a_n = -a^n$	$(n \geq 0)$

$a^{\boxed{-1}}\,z^{\boxed{-1}}$	$\lvert z\rvert > a^{-1}$

$\left(a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots\right)$

$\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{1} z^{-1} + \left(\tfrac{1}{a}\right)^{2} z^{-2} + \cdots\right)$

nt Series	$a_n = a^n$	$(n < 1)$
nsform	$a_n = \left(\tfrac{1}{a}\right)^n$	$(n \geq 0)$

$a^{\boxed{-1}}\,z$	$\lvert z\rvert < a$

$\left(a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots\right)$

$\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{1} z^1 + \left(\tfrac{1}{a}\right)^{2} z^2 + \cdots\right)$

$a_n = \left(\tfrac{1}{a}\right)^n$	$(n \geq 0)$
$a_n = a^n$	$(n < 1)$

$a^{\boxed{-1}}\,z$	$\lvert z\rvert < a$

$-\left(a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots\right)$

$-\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{1} z^1 + \left(\tfrac{1}{a}\right)^{2} z^2 + \cdots\right)$

ent Series	$a_n = -\left(\tfrac{1}{a}\right)^n$	$(n \geq 0)$
nsform	$a_n = -a^n$	$(n < 1)$

$a^{\boxed{-1}}\,z^{\boxed{-1}}$	$\lvert z\rvert > a^{-1}$

$-\left(a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots\right)$

$-\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{1} z^{-1} + \left(\tfrac{1}{a}\right)^{2} z^{-2} + \cdots\right)$

$a_n = -a^n$	$(n < 1)$
$a_n = -\left(\tfrac{1}{a}\right)^n$	$(n \geq 0)$

$a\,z^{\boxed{-1}}$	$\lvert z\rvert > a$

$\left(a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots\right)$

$\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{-1} z^{-1} + \left(\tfrac{1}{a}\right)^{-2} z^{-2} + \cdots\right)$

ent Series	$a_n = \left(\tfrac{1}{a}\right)^n$	$(n < 1)$
nsform	$a_n = a^n$	$(n \geq 0)$

$a\,z$	$\lvert z\rvert < a^{-1}$

$\left(a^0z^0 + a^1z^1 + a^2z^2 + \cdots\right)$

$\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{-1} z^1 + \left(\tfrac{1}{a}\right)^{-2} z^2 + \cdots\right)$

$a_n = a^n$	$(n \geq 0)$
$a_n = \left(\tfrac{1}{a}\right)^n$	$(n < 1)$

Geometric Series with a non-unit start term

Laurent Series

$a^{\boxed{-1}} z^{\boxed{-1}}$ $\quad |z| > a^{-1}$

$(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$

$a_n = a^n \quad (n < 0)$

$a^{\boxed{-1}} z$ $\quad |z| < a$

$(a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots)$

$((\frac{1}{a})^{1}z^{1} + (\frac{1}{a})^{2}z^{2} + (\frac{1}{a})^{3}z^{3} + \cdots)$

$a_n = (\frac{1}{a})^n \quad (n \geq 1)$

$a\, z$ $\quad |z| < a^{-1}$

$-(a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots)$

$a_n = -a^n \quad (n \geq 1)$

$a\, z^{\boxed{-1}}$ $\quad |z| > a$

$-(a^{1}z^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \cdots)$

$-((\frac{1}{a})^{-1}z^{-1} + (\frac{1}{a})^{-2}z^{-2} + (\frac{1}{a})^{-3}z^{-3} + \cdots)$

$a_n = -(\frac{1}{a})^n \quad (n < 0)$

$a\, z^{\boxed{-1}}$ $\quad |z| > a$

$(a^{1}z^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \cdots)$

$((\frac{1}{a})^{-1}z^{-1} + (\frac{1}{a})^{-2}z^{-2} + (\frac{1}{a})^{-3}z^{-3} + \cdots)$

$a_n = (\frac{1}{a})^n \quad (n < 0)$

$a\, z$ $\quad |z| < a^{-1}$

$(a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots)$

$a_n = a^n \quad (n \geq 1)$

$a^{\boxed{-1}} z$ $\quad |z| < a$

$-(a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots)$

$-((\frac{1}{a})^{1}z^{1} + (\frac{1}{a})^{2}z^{2} + (\frac{1}{a})^{3}z^{3} + \cdots)$

$a_n = -(\frac{1}{a})^n \quad (n \geq 1)$

$a^{\boxed{-1}} z^{\boxed{-1}}$ $\quad |z| > a^{-1}$

$-(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$

$a_n = -a^n \quad (n < 0)$

Geometric Series with a non-unit start term

z-Transform

$a^{\boxed{-1}} z^{\boxed{-1}}$	$\lvert z\rvert > a^{-1}$

$(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$

$((\frac{1}{a})^{1}z^{-1} + (\frac{1}{a})^{2}z^{-2} + (\frac{1}{a})^{3}z^{-3} + \cdots)$

$a_n = a^{-n}$	$(-n < 0)$
$a_n = (\frac{1}{a})^{n}$	$(n \geq 1)$

$a^{\boxed{-1}} z$	$\lvert z\rvert < a$

$(a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots)$

$((\frac{1}{a})^{1}z^{1} + (\frac{1}{a})^{2}z^{2} + (\frac{1}{a})^{3}z^{3} + \cdots)$

$a_n = (\frac{1}{a})^{-n}$	$(-n \geq 1)$
$a_n = a^{n}$	$(n < 0)$

$a\, z$	$\lvert z\rvert < a^{-1}$

$-(a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots)$

$-((\frac{1}{a})^{-1}z^{1} + (\frac{1}{a})^{-2}z^{2} + (\frac{1}{a})^{-3}z^{3} + \cdots)$

$a_n = -a^{-n}$	$(-n \geq 1)$
$a_n = -(\frac{1}{a})^{n}$	$(n < 0)$

$a\, z^{\boxed{-1}}$	$\lvert z\rvert > a$

$-(a^{1}z^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \cdots)$

$-((\frac{1}{a})^{-1}z^{-1} + (\frac{1}{a})^{-2}z^{-2} + (\frac{1}{a})^{-3}z^{-3} + \cdots)$

$a_n = -(\frac{1}{a})^{-n}$	$(-n < 0)$
$a_n = -a^{n}$	$(n \geq 1)$

$a\, z^{\boxed{-1}}$	$\lvert z\rvert > a$

$(a^{1}z^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \cdots)$

$((\frac{1}{a})^{-1}z^{-1} + (\frac{1}{a})^{-2}z^{-2} + (\frac{1}{a})^{-3}z^{-3} + \cdots)$

$a_n = (\frac{1}{a})^{-n}$	$(-n < 0)$
$a_n = a^{n}$	$(n \geq 1)$

$a\, z$	$\lvert z\rvert < a^{-1}$

$(a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots)$

$((\frac{1}{a})^{-1}z^{1} + (\frac{1}{a})^{-2}z^{2} + (\frac{1}{a})^{-3}z^{3} + \cdots)$

$a_n = a^{-n}$	$(-n \geq 1)$
$a_n = (\frac{1}{a})^{n}$	$(n < 0)$

$a^{\boxed{-1}} z$	$\lvert z\rvert < a$

$-(a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots)$

$-((\frac{1}{a})^{1}z^{1} + (\frac{1}{a})^{2}z^{2} + (\frac{1}{a})^{3}z^{3} + \cdots)$

$a_n = -(\frac{1}{a})^{-n}$	$(-n \geq 1)$
$a_n = -a^{n}$	$(n < 0)$

$a^{\boxed{-1}} z^{\boxed{-1}}$	$\lvert z\rvert > a^{-1}$

$-(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$

$-((\frac{1}{a})^{1}z^{-1} + (\frac{1}{a})^{2}z^{-2} + (\frac{1}{a})^{3}z^{-3} + \cdots)$

$a_n = -a^{-n}$	$(-n < 0)$
$a_n = -(\frac{1}{a})^{n}$	$(n \geq 1)$

Geometric Series with a non-unit start term

Laurent Series vs. z-Transform

$a^{-1} z^{-1}$ | $|z| > a^{-1}$

$(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$

$((\frac{1}{a})^{1}z^{-1} + (\frac{1}{a})^{2}z^{-2} + (\frac{1}{a})^{3}z^{-3} + \cdots)$

nt Series	$a_n = a^n$	$(n < 0)$
nsform	$a_n = (\frac{1}{a})^n$	$(n \geq 1)$

$a^{-1} z$ | $|z| < a$

$(a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots)$

$((\frac{1}{a})^{1}z^{1} + (\frac{1}{a})^{2}z^{2} + (\frac{1}{a})^{3}z^{3} + \cdots)$

$a_n = (\frac{1}{a})^n$	$(n \geq 1)$
$a_n = a^n$	$(n < 0)$

$a z$ | $|z| < a^{-1}$

$-(a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots)$

$-((\frac{1}{a})^{-1}z^{1} + (\frac{1}{a})^{-2}z^{2} + (\frac{1}{a})^{-3}z^{3} + \cdots)$

nt Series	$a_n = -a^n$	$(n \geq 1)$
nsform	$a_n = -(\frac{1}{a})^n$	$(n < 0)$

$a z^{-1}$ | $|z| > a$

$-(a^{1}z^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \cdots)$

$-((\frac{1}{a})^{-1}z^{-1} + (\frac{1}{a})^{-2}z^{-2} + (\frac{1}{a})^{-3}z^{-3} + \cdots)$

$a_n = -(\frac{1}{a})^n$	$(n < 0)$
$a_n = -a^n$	$(n \geq 1)$

$a z^{-1}$ | $|z| > a$

$(a^{1}z^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \cdots)$

$((\frac{1}{a})^{-1}z^{-1} + (\frac{1}{a})^{-2}z^{-2} + (\frac{1}{a})^{-3}z^{-3} + \cdots)$

nt Series	$a_n = (\frac{1}{a})^n$	$(n < 0)$
nsform	$a_n = a^n$	$(n \geq 1)$

$a z$ | $|z| < a^{-1}$

$(a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots)$

$((\frac{1}{a})^{-1}z^{1} + (\frac{1}{a})^{-2}z^{2} + (\frac{1}{a})^{-3}z^{3} + \cdots)$

$a_n = a^n$	$(n \geq 1)$
$a_n = (\frac{1}{a})^n$	$(n < 0)$

$a^{-1} z$ | $|z| < a$

$-(a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots)$

$-((\frac{1}{a})^{1}z^{1} + (\frac{1}{a})^{2}z^{2} + (\frac{1}{a})^{3}z^{3} + \cdots)$

nt Series	$a_n = -(\frac{1}{a})^n$	$(n \geq 1)$
sform	$a_n = -a^n$	$(n < 0)$

$a^{-1} z^{-1}$ | $|z| > a^{-1}$

$-(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$

$-((\frac{1}{a})^{1}z^{-1} + (\frac{1}{a})^{2}z^{-2} + (\frac{1}{a})^{3}z^{-3} + \cdots)$

$a_n = -a^n$	$(n < 0)$
$a_n = -(\frac{1}{a})^n$	$(n \geq 1)$

Complemnt ROC Pairs -
Original Geometric Series Form Combinations

unit	$-\frac{1}{1-az}$ $\ \|z\|<a^{-1}$	$-a^n \quad (n \geq 0)$	$-\frac{1}{1-az^{-1}}$ $\ \|z\|>a$	$-(\frac{1}{a})^n \quad (n<1)$
non-unit	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\ \|z\|>a^{-1}$	$a^n \quad (n<0)$	$\frac{a^{-1}z}{1-a^{-1}z}$ $\ \|z\|<a$	$(\frac{1}{a})^n \quad (n\geq 1)$

unit	$\frac{1}{1-a^{-1}z^{-1}}$ $\ \|z\|>a^{-1}$	$a^n \quad (n<1)$	$\frac{1}{1-a^{-1}z}$ $\ \|z\|<a$	$(\frac{1}{a})^n \quad (n\geq 0)$
non-unit	$-\frac{az}{1-az}$ $\ \|z\|<a^{-1}$	$-a^n \quad (n\geq 1)$	$\frac{az^{-1}}{1-az^{-1}}$ $\ \|z\|>a$	$-(\frac{1}{a})^n \quad (n<0)$

unit	$-\frac{1}{1-a^{-1}z}$ $\ \|z\|<a$	$-(\frac{1}{a})^n \quad (n\geq 0)$	$-\frac{1}{1-a^{-1}z^{-1}}$ $\ \|z\|>a^{-1}$	$-a^n \quad (n<1)$
non-unit	$\frac{az^{-1}}{1-az^{-1}}$ $\ \|z\|>a$	$(\frac{1}{a})^n \quad (n<0)$	$\frac{az}{1-az}$ $\ \|z\|<a^{-1}$	$a^n \quad (n\geq 1)$

unit	$\frac{1}{1-az^{-1}}$ $\ \|z\|>a$	$(\frac{1}{a})^n \quad (n<1)$	$\frac{1}{1-az}$ $\ \|z\|<a^{-1}$	$a^n \quad (n\geq 0)$
non-unit	$-\frac{a^{-1}z}{1-a^{-1}z}$ $\ \|z\|<a$	$-(\frac{1}{a})^n \quad (n\geq 1)$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\ \|z\|>a^{-1}$	$-a^n \quad (n<0)$

start term

Complemnt ROC Pairs -
Shifted Geometric Series Form Combinations

$-\frac{a}{1-az}$ $\|z\|<a^{-1}$	$-a^{n+1}$ $(n\geq 0)$	$-\frac{a}{1-az^{-1}}$ $\|z\|>a$	$-(\frac{1}{a})^{n-1}$ $(n<1)$
$\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\|z\|>a^{-1}$	a^{n+1} $(n<0)$	$\frac{z}{1-a^{-1}z}$ $\|z\|<a$	$(\frac{1}{a})^{n-1}$ $(n\geq 1)$

$\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\|z\|>a^{-1}$	a^{n+1} $(n<0)$	$\frac{z}{1-a^{-1}z}$ $\|z\|<a$	$(\frac{1}{a})^{n-1}$ $(n\geq 1)$
$-\frac{a}{1-az}$ $\|z\|<a^{-1}$	$-a^{n+1}$ $(n\geq 0)$	$\frac{a}{1-az^{-1}}$ $\|z\|>a$	$-(\frac{1}{a})^{n-1}$ $(n<1)$

$-\frac{a^{-1}}{1-a^{-1}z}$ $\|z\|<a$	$-(\frac{1}{a})^{n+1}$ $(n\geq 0)$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\|z\|>a^{-1}$	$-a^{n-1}$ $(n<1)$
$\frac{z^{-1}}{1-az^{-1}}$ $\|z\|>a$	$(\frac{1}{a})^{n+1}$ $(n<0)$	$\frac{z}{1-az}$ $\|z\|<a^{-1}$	a^{n-1} $(n\geq 1)$

$\frac{z^{-1}}{1-az^{-1}}$ $\|z\|>a$	$(\frac{1}{a})^{n+1}$ $(n<0)$	$\frac{z}{1-az}$ $\|z\|<a^{-1}$	a^{n-1} $(n\geq 1)$
$-\frac{a^{-1}}{1-a^{-1}z}$ $\|z\|<a$	$-(\frac{1}{a})^{n+1}$ $(n\geq 0)$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\|z\|>a^{-1}$	$-a^{n-1}$ $(n<1)$

Complemnt ROC Pairs - Reduced

Shifted Geometric Series Form Combinations

$-\frac{a}{1-az}$ $\lvert z\rvert<a^{-1}$	$-a^{n+1}$ $(n \geq 0)$
$\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert>a^{-1}$	a^{n+1} $(n<0)$

$-\frac{a}{1-az^{-1}}$ $\lvert z\rvert>a$	$-\left(\frac{1}{a}\right)^{n-1}$ $(n<1)$
$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert<a$	$\left(\frac{1}{a}\right)^{n-1}$ $(n \geq 1)$

$-\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert<a$	$-\left(\frac{1}{a}\right)^{n+1}$ $(n \geq 0)$
$\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert>a$	$\left(\frac{1}{a}\right)^{n+1}$ $(n<0)$

$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert>a^{-1}$	$-a^{n-1}$ $(n<1)$
$\frac{z}{1-az}$ $\lvert z\rvert<a^{-1}$	a^{n-1} $(n \geq 1)$